DEScribing NeighBorhoods of 5-vErtecEs in 3-pOLytoPES with MinimUrn Degree 5 aND wITHOUT VertEcEs of DegreeS from 7 to 11\textsuperscript{1}

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Abstract

In 1940, Lebesgue proved that every 3-polytope contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:

\((6,6,7,7,7), (6,6,6,7,9), (6,6,6,6,11), (5,6,7,7,8), (5,6,6,7,12), (5,6,6,8,10), (5,5,6,6,17), (5,5,7,7,13), (5,5,7,8,10), (5,5,6,7,27), (5,5,6,6,\infty), (5,5,6,8,15), (5,5,6,9,11), (5,5,5,7,41), (5,5,5,8,23), (5,5,5,9,17), (5,5,5,10,14), (5,5,5,11,13)\).

In this paper we prove that every 3-polytope without vertices of degree from 7 to 11 contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences: \((5,5,6,6,\infty), (5,6,6,6,15), (6,6,6,6,6)\), where all parameters are tight.

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1. Introduction

By a 3-polytope we mean a finite 3-dimensional convex polytope. As proved by Steinitz [31], the 3-polytopes are in one to one correspondence with the 3-connected planar graphs.

The degree $d(v)$ of a vertex $v$ ($r(f)$ of a face $f$) in a 3-polytope $P$ is the number of edges incident with it. By $\Delta$ and $\delta$ we denote the maximum and minimum vertex degrees of $P$, respectively. A $k$-vertex ($k$-face) is a vertex (face) with degree $k$; a $k^+$-vertex has degree at least $k$, etc.

The weight of a face in $P$ is the degree sum of its boundary vertices, and $w(P)$, or simply $w$, denotes the minimum weight of 5-faces in $P$.

In 1904, Wernicke [32] proved that every 3-polytope with $\delta = 5$ has a 5-vertex adjacent with a 6-vertex, which was strengthened by Franklin [15] in 1922, who proved that every 3-polytope with $\delta = 5$ has a 5-vertex adjacent with two 6-vertices. Recently, Borodin and Ivanova [11] proved that every such 3-polytope has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which is tight.

We say that $v$ is a vertex of type $(k_1, k_2, \ldots)$ or simply a $(k_1, k_2, \ldots)$-vertex if the set of degrees of the vertices adjacent to $v$ is majorized by the vector $(k_1, k_2, \ldots)$. If the order of neighbors in the type is not important, then we put a line over the corresponding degrees. The following description of the neighborhoods of 5-vertices in a 3-polytope with $\delta = 5$ was given by Lebesgue [28, p. 36] in 1940, which includes the results of Wernicke [32] and Franklin [15].

**Theorem 1** (Lebesgue [28]). Every triangulated 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

\[
(6, 6, 7, 7, 7), \ (6, 6, 6, 7, 9), \ (6, 6, 6, 6, 11), \\
(5, 6, 7, 7, 8), \ (5, 6, 6, 7, 11), \ (5, 6, 6, 8, 8), \\
(5, 6, 6, 9, 7), \ (5, 7, 6, 6, 12), \ (5, 8, 6, 6, 10), \ (5, 6, 6, 6, 17), \\
(5, 5, 7, 7, 8), \ (5, 13, 5, 7, 7), \ (5, 10, 5, 7, 8), \\
(5, 8, 5, 7, 9), \ (5, 7, 5, 7, 10), \ (5, 7, 5, 8, 8), \\
(5, 5, 7, 6, 12), \ (5, 5, 8, 6, 10), \ (5, 6, 5, 7, 12), \\
(5, 6, 5, 8, 10), \ (5, 17, 5, 6, 7), \ (5, 11, 5, 6, 8), \\
(5, 11, 5, 6, 9), \ (5, 7, 5, 6, 13), \ (5, 8, 5, 6, 11), \ (5, 9, 5, 6, 10), \ (5, 6, 6, 5, \infty), \\
(5, 5, 7, 5, 41), \ (5, 5, 8, 5, 23), \ (5, 5, 9, 5, 17), \ (5, 5, 10, 5, 14), \ (5, 5, 11, 5, 13).
\]

Theorem 1, along with other ideas in Lebesgue [28], has many applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [7, 30]). Some parameters of Lebesgue’s Theorem were improved for narrow classes of plane graphs. For example, in 1963, Kotzig [27] proved that every plane triangulation with $\delta = 5$ satisfies $w \leq 18$ and conjectured
that \( w \leq 17 \). In 1989, Kotzig’s conjecture was confirmed by Borodin [3] in a more general form.

**Theorem 2** (Borodin [3]). Every 3-polytope with \( \delta = 5 \) has a \((5,5,7)\)-face or a \((5,6,6)\)-face, where all parameters are tight.

By a minor \( k \)-star \( S_k^{(m)} \) we mean a star with \( k \) rays centered at a \( 5^- \)-vertex. The Lebesgue’s description [28, p. 36] of the neighborhoods of 5-vertices in 3-polytopes with minimum degree 5, \( P_5 \), shows that there is a 5-vertex with three \( 8^- \)-neighbors. Another corollary of Lebesgue’s description [28] is that \( \nu(S_2^{(m)}) \leq 24 \), which was improved in 1996 by Jendrol’ and Madaras [23] to the sharp bound \( \nu(S_2^{(m)}) \leq 23 \). Furthermore, Jendrol’ and Madaras [23] gave a precise description of minor 3-stars in \( P_5 \): there is a \((6,6,6)\)- or \((5,6,7)\)-star.

Also, Lebesgue [28] proved that \( \nu(S_4^{(m)}) \leq 31 \), which was strengthened by Borodin and Woodall [13] to the sharp bound \( \nu(S_4^{(m)}) \leq 30 \). Note that \( \nu(S_3^{(m)}) \leq 23 \) easily implies \( \nu(S_2^{(m)}) \leq 17 \) and immediately follows from \( \nu(S_4^{(m)}) \leq 30 \) (in both cases, it suffices to delete a vertex of maximum degree from a minor star of minimum weight). In [9], Borodin and Ivanova obtained a tight description of minor 4-stars in \( P_5 \).

As for minor 5-stars in \( P_5 \), it follows from Lebesgue [28, p. 36] that if there are no minor \((5,5,6,6)\)-stars, then \( \nu(S_5^{(m)}) \leq 68 \) and \( h(S_5^{(m)}) \leq 41 \). Borodin, Ivanova, and Jensen [10] showed that the presence of minor \((5,5,6,6)\)-stars can make \( \nu(S_5^{(m)}) \) arbitrarily large and otherwise lowered Lebesgue’s bounds to \( \nu(S_5^{(m)}) \leq 55 \) and \( h(S_5^{(m)}) \leq 28 \). On the other hand, a construction in [10] shows that \( \nu(S_5^{(m)}) \geq 48 \) and \( h(S_5^{(m)}) \geq 20 \). Recently, Borodin and Ivanova [12] proved that \( \nu(S_5^{(m)}) \leq 51 \) and \( h(S_5^{(m)}) \leq 23 \).

More results on the structure of edges and higher stars in various classes of 3-polytopes can be found in [1, 2, 4–6, 8, 9, 14, 16, 19–22, 24–26], with a detailed summary in [12].

In [28] Lebesgue did not give a proof of Theorem 1 and only gave its idea. In 2013, Ivanova and Nikiforov [17] gave a full proof of Theorem 1 and corrected the following imprecisions in its statement:

1. in the type \((5,11,5,6,8)\) there should be 15 instead of 11;
2. in the type \((5,17,5,6,7)\) there should be 27 instead of 17;
3. in the type \((6,6,6,6,11)\) the line is not needed;
4. instead of type \((5,6,7,7,8)\) there should be \((5,8,6,7,7)\) and \((5,7,6,8,7)\);
Corollary 4. Every one of the following types: (5, 6, 6, 9, 7) is redundant;
(6) instead of (5, 5, 7, 7, 8) it suffices to write (5, 5, 7, 7, 8).

Later on, Ivanova and Nikiforov [18, 29] improved the corrected version of Theorem 1 by replacing 41 and 23 in the types (5, 5, 7, 5, 41) and (5, 5, 8, 5, 23) to 31 and 22, respectively.

Theorem 3 (Ivanova, Nikiforov [17, 18, 29]). Every 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

\[
(6, 6, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11), \\
(5, 8, 6, 7, 7), (5, 7, 6, 8, 7), (5, 6, 6, 7, 11), (5, 6, 6, 8, 8), \\
(5, 7, 6, 6, 12), (5, 8, 6, 6, 10), (5, 6, 6, 6, 17), \\
(5, 5, 7, 7, 8), (5, 5, 7, 8, 7), (5, 5, 8, 7, 10), (5, 5, 8, 7, 11), \\
(5, 7, 5, 7, 10), (5, 7, 5, 8, 8), (5, 7, 6, 6, 12), (5, 7, 6, 8, 10), \\
(5, 6, 5, 7, 12), (5, 6, 5, 8, 10), (5, 6, 7, 6, 7), (5, 5, 5, 6, 8), \\
(5, 11, 5, 6, 9), (5, 7, 5, 6, 13), (5, 8, 5, 6, 11), (5, 9, 5, 6, 10), \\
(5, 6, 6, 5, \infty), \\
(5, 5, 7, 5, 31), (5, 5, 8, 5, 22), (5, 5, 9, 5, 17), (5, 5, 10, 5, 14), (5, 5, 11, 5, 13).
\]

Theorem 1 subject to the corrections (1)–(6) implies the following fact.

Corollary 4. Every 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

\[
(6, 6, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11), \\
(5, 6, 7, 7, 8), (5, 6, 6, 7, 12), (5, 6, 6, 8, 10), (5, 6, 6, 6, 17), \\
(5, 5, 7, 7, 13), (5, 5, 7, 8, 10), (5, 5, 6, 7, 27), \\
(5, 5, 6, 6, \infty), (5, 5, 6, 8, 15), (5, 5, 6, 9, 11), \\
(5, 5, 5, 7, 41), (5, 5, 5, 8, 23), (5, 5, 5, 9, 17), (5, 5, 5, 10, 14), (5, 5, 5, 11, 13).
\]

We can see already from Theorem 1 that if vertices of degree from 7 to 11 are forbidden, then there is a 5-vertex of one of the following types: (5, 5, 6, 6, \infty), (5, 6, 6, 6, 17), (6, 6, 6, 6, 6).

The purpose of this note is to obtain a precise description of 5-stars in this subclass of \(P_5\).

Theorem 5. Every 3-polytope with minimum degree 5 and without vertices of degree from 7 to 11 contains a 5-vertex of one of the following types: (5, 5, 6, 6, \infty), (5, 6, 6, 6, 15), (6, 6, 6, 6, 6), where all parameters are tight.

2. Proving Theorem 5

All parameters in Theorem 5 are best possible. Indeed, the following construction confirming the tightness of the type (5, 5, 6, 6, \infty) appears in [10]. Take three
Describing Neighborhoods of 5-Vertices in 3-Polytopes ...

concentric $n$-cycles $C^i = v^i_1 \cdots v^i_n$, where $n$ is not limited and $1 \leq i \leq 3$, and join $C^2$ with $C^1$ by edges $v^2_j v^1_j$ and $v^2_j v^1_{j+1}$, where $1 \leq j \leq n$ (addition modulo $n$). Then do the same with $C^2$ and $C^3$. Finally, join all vertices of $C^1$ with a new $n$-vertex, and do the same for $C^3$.

The tightness of $(6, 6, 6, 6, 6)$ is confirmed by putting a 5-vertex in each face of the dodecahedron.

To confirm the tightness of $(5, 6, 6, 6, 15)$, we take the dodecahedron and insert the fragment shown in Figure 1 into each face. As a result, we have a 3-polytope with only $(5, 6, 6, 6, 15)$-vertices.

![Figure 1. The insert in each face of the dodecahedron to produce a 3-polytope with 5-vertices only of type $(5, 6, 6, 6, 15)$.](image)

Now suppose a 3-polytope $P'$ is a counterexample to Theorem 5. Let $P$ be a counterexample on the same number of vertices with maximum possible number of edges.

**Remark 6.** In $P$, each $4^+$-face $f = v_1 \cdots v_{d(f)}$ with $d(v_1) = 5$ or $d(v_1) \geq 15$ satisfies $d(v_i) \geq 6$ whenever $3 \leq i \leq d(f) - 1$. Otherwise, we could put a diagonal $v_1v_i$, which contradicts the maximality of $P$.

**Corollary 7.** In $P$, each $4^+$-face has at most two vertices with degree 5 and/or at least 15. Moreover, if there are precisely two such vertices, then they are adjacent to each other.
2.1. Discharging

The sets of vertices, edges, and faces of $P$ are denoted by $V$, $E$, and $F$, respectively. Euler’s formula $|V| - |E| + |F| = 2$ for $P$ implies

\[ \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2r(f) - 6) = -12. \]  

(1)

We assign an initial charge $\mu(v) = d(v) - 6$ to every vertex $v$ and $\mu(f) = 2d(f) - 6$ to every face $f$, so that only $5^-$-vertices have negative charge. Using the properties of $P$ as a counterexample, we define a local redistribution of charges, preserving their sum, such that the new charge $\mu'(x)$ is non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to $-12$. The technique of discharging is often used in solving structural and coloring problems on plane graphs.

Let $v_1, \ldots, v_{d(v)}$ denote the neighbors of a vertex $v$ in a cyclic order round $v$, and let $f_1, \ldots, f_{d(v)}$ be the faces incident with $v$ in the same order.

We use the following rules of discharging (see Figure 2).

**R1.** Every $4^+$-face gives 1 to every incident $5$-vertex.

**R2.** Every $12^+$-vertex $v$ gives a simplicial $5$-vertex $v_2$ the following charge through a face $f = v_2vv_3$:

(a) $\frac{1}{4}$ if $d(v_3) = 5$,

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{rules_of_discharging.png}
\caption{Rules of discharging.}
\end{figure}
\end{center}
(b) \( \frac{1}{2} \) if \( d(v_3) \geq 6 \),
with the following exception.

(c) If \( d(v) \geq 16 \), \( d(v_1) = 5 \), \( d(v_3) = d(x) = d(y) = 6 \), where \( v_2 \) is incident to face \( v_2xy \), then \( v \) gives \( \frac{2}{3} \) to \( v_2 \) through face \( v_2vv_3 \) and \( \frac{1}{2} \) through face \( v_1vv_2 \).

R3. Suppose a simplicial 5-vertex \( v \) is adjacent to a 16-vertex \( v_1 \), simplicial 5-vertices \( v_2 \) and \( v_3 \), and \( v_2 \) is surrounded by \( v_1, v, v_3, x, y \), where \( d(v_3) = d(x) = d(y) = 6 \), (consequently \( d(v_4) \geq 12 \)), while \( v_3 \) is surrounded by \( v_1, v, v_4, w, z \), where \( d(z) \geq 6 \). Then \( v \) gives \( \frac{1}{4} \) to \( v_1 \).

2.2. Proving \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \)

First consider a face \( f \) in \( P \). If \( d(f) = 3 \), then \( f \) does not participate in discharging, and so \( \mu'(v) = \mu(f) = 2 \times 3 - 6 = 0 \). Note that every \( 4^+ \)-face is incident with at most two 5-vertices due to Corollary 7, which implies that \( \mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0 \) by R1.

Now let \( v \) be a vertex in \( P \).

Case 1. \( d(v) = 5 \). If \( v \) is incident with a \( 4^+ \)-face, then \( \mu'(v) \geq 5 - 6 + 1 = 0 \) due to R1. In what follows we can assume that \( v \) is simplicial.

Subcase 1.1. \( v \) is incident only with \( 6^+ \)-vertices. Then there is at least one \( v_i \) with \( d(v_i) \geq 12 \) due to the absence of \((6,6,6,6,6)\)-vertices in \( P \). Hence, \( \mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0 \) by R2(b).

Subcase 1.2. \( v \) is incident with precisely one 5-vertex. Since there is no \((5,6,6,6,15)\)-vertex in \( P \), we can assume that \( v \) has either at least two \( 12^+ \)-neighbors, or precisely one \( 16^+ \)-neighbor. So we have either \( \mu'(v) \geq -1 + 2 \times \frac{1}{2} + 2 \times \frac{1}{4} > 0 \) by R2(a),(b), or \( \mu'(v) = -1 + \frac{3}{4} = 0 \) by R2(e), respectively.

Subcase 1.3. \( v \) is incident with at least two 5-vertices. Note that now R2(e) is not applicable to \( v \). Also note that \( v \) cannot be incident with more than three 5-vertices due to the absence of \((5,5,6,6,\infty)\)-vertices in \( P \), which implies that \( v \) has at least two \( 12^+ \)-neighbors. If \( v \) is incident with precisely three 5-vertices, then we have \( \mu'(v) \geq -1 + 4 \times \frac{1}{4} = 0 \) by R2(a),(b).

Suppose \( v \) is incident with precisely two 5-vertices. If \( v \) does not participate in R3, then \( \mu'(v) \geq -1 + 3 \times \frac{1}{4} + \frac{1}{2} > 0 \) by R2(a),(b). Note that if \( v \) participates in R3, then it gives \( \frac{1}{2} \) only to one 16-neighbor, hence \( \mu'(v) \geq -1 + 3 \times \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0 \).

Case 2. \( d(v) = 6 \). Since \( v \) does not participate in discharging, we have \( \mu'(v) = \mu'(v) = 6 - 6 = 0 \).

Case 3. \( 12 \leq d(v) \leq 15 \). Now R2(e) is not applicable to \( v \), so \( v \) sends at most \( \frac{1}{2} \) through each face by R2(a),(b), which implies that \( \mu'(v) \geq d(v) - 6 - d(v) \times \frac{1}{2} = \frac{d(v) - 12}{2} \geq 0 \).
Case 4. 16 ≤ d(v) ≤ 17. Note that v gives at most \( \frac{2}{3} \) through each 3-face and only to a simplicial 5-vertex. If v gives nothing through at least one incident face, then \( \mu'(v) ≥ 16 − 6 − 15 \times \frac{2}{3} = 0 \) by R1, R2. Further, we can assume that v is simplicial and each face takes away some positive charge from v, which implies that each face at v is incident with a 5-vertex, and all 5-vertices adjacent to v are simplicial. Thus, \( \mu'(v) ≥ d(v) − 6 − d(v) \times \frac{2}{3} = \frac{d(v)−18}{3} \), and we have the deficiency \( \frac{1}{2} \) for a 17-vertex and \( \frac{2}{3} \) for a 16-vertex with respect to donating \( \frac{2}{3} \) per face.

Suppose \( S_k = v_1, \ldots, v_k \) is a sequence of neighbors of v with \( d(v_1) ≥ 6, d(v_k) ≥ 6 \), while \( d(v_i) = 5 \) whenever \( 2 ≤ i ≤ k−1 \) and \( k ≥ 3 \), and \( f_1, \ldots, f_{k−1} \) are the corresponding faces. (It is not excluded that \( S_k = S_{d(v)} \), which happens when v has precisely one 6\(^2\)-neighbor.) We say that the sequence of faces \( f_1, \ldots, f_{k−1} \) saves \( ε \) with respect to the level of \( \frac{2}{3} \) if these faces take away the total of \( (k−1) \times \frac{2}{3} − ε \) from v.

Remark 8. Only \( v_2 \) and \( v_{k−1} \) in \( S_k \) can receive the charge \( \frac{2}{3} \) from v by R2(e), while each of the other 5-vertices \( v_i \) receives precisely \( \frac{1}{2} \) from v through each incident face. So, if \( k ≥ 5 \), then \( v_2 \) receives at most 1, and \( v_3 \) receives \( \frac{1}{2} \) from v through incident faces.

Remark 9. If v is completely surrounded by 5-vertices, then \( \mu'(v) ≥ \frac{d(v)−12}{3} > 0 \), and hence we can assume from now on that the neighborhood of v is partitioned into \( S_k \)s.

(P1) If \( k = 3 \), then \( ε = \frac{1}{3} \). Indeed, here \( v_2 \) receives \( \frac{1}{2} \) through each of the faces \( v_1v_2 \) and \( v_2v_3 \) by R2(b), whence \( ε = 2 \times \frac{2}{3} − 2 \times \frac{1}{2} = \frac{1}{3} \).

(P2) If \( k = 4 \), then \( ε = 0 \). Now each of \( v_2 \) and \( v_3 \) receives at most 1 from v by Remark 8, so \( ε = 3 \times \frac{2}{3} − 2 = 0 \).

(P3) If \( k = 5 \), then \( ε = \frac{2}{3} \). Suppose \( w_1, \ldots, w_4 \) are the neighbors of \( v_1, \ldots, v_5 \) such that there are the faces \( v_1w_1v_{i+1} \), where \( 1 ≤ i ≤ 4 \).

If \( v_2 \) receives 1 by R2(e), then \( d(w_1) = d(w_2) = 6 \). Hence, \( d(w_3) ≥ 12 \) due to the absence of a \((5, 5, 6, 6, \infty)\)-vertex in \( P \), which implies that \( v_4 \) is adjacent to two 12\(^\ast\)-vertices, whence it receives \( \frac{1}{2} \) from v through \( f_4 \) and \( \frac{1}{4} \) through \( f_3 \). Moreover, \( v_3 \) gives \( \frac{1}{1} \) to v by R3. Hence, \( ε = 4 \times \frac{2}{3} − 1 − \frac{1}{2} − \frac{3}{3} + \frac{1}{4} = \frac{2}{3} \).

If R2(e) is not applicable to v, then \( ε = 4 \times \frac{2}{3} − 4 \times \frac{1}{2} = \frac{2}{3} \).

(P4) If \( k = 6 \), then \( ε = \frac{1}{3} \). Here, each of \( v_2 \) and \( v_5 \) receives at most 1, while each of \( v_3 \) and \( v_4 \) receives \( \frac{1}{2} \) from v by Remark 8, so \( ε = 5 \times \frac{2}{3} − 2 \times 1 − 2 \times \frac{1}{2} = \frac{1}{3} \).

(P5) If \( k = 7 \), then \( ε = \frac{1}{2} \). Now we have \( ε = 6 \times \frac{2}{3} − 2 \times 1 − 3 \times \frac{1}{2} = \frac{1}{2} \) by Remark 8.
(P6) If \( k \geq 8 \), then \( \varepsilon \geq \frac{2}{3} \). Now we have \( \varepsilon = (k - 1) \times \frac{2}{3} - 2 \times 1 - (k - 4) \times \frac{1}{2} = \frac{k - 4}{6} \geq \frac{2}{3} \).

If \( d(v) = 17 \), then it suffices to assume that the neighborhood of \( v \) consists of pairs of 5-vertices separated from each other by 6-vertices by (P1)–(P6) (since otherwise we pay off the deficiency), which is impossible due to the fact that 17 is not divisible by 3.

Suppose that \( d(v) = 16 \) and \( \mu'(v) < 0 \). As follows from (P1)–(P6), the neighborhood of \( v \) can have at most one of the paths \( S_{t+2} \) of \( t \) vertices of degree 5, where \( t \in \{1, 4, 5\} \), while all other vertices are partitioned into pairs of 5-vertices separated from each other by 6-vertices. Indeed, if there are either two paths with \( t \in \{1, 4, 5\} \), or at least one path with \( t = 3 \) or \( t \geq 6 \), then we can pay off the deficiency \( \frac{2}{3} \), a contradiction. But none of these cases is possible due to the divisibility by 3. Namely, if \( t = 1 \) we have \( 16 - 2 = 14 \) faces to be divided into triplets of faces with a sequence \( S_4 \) of neighbors of \( v \) as in (P2), or \( 16 - 5 = 11 \) and \( 16 - 6 = 10 \) faces for \( t = 4 \) and \( t = 5 \), respectively; a contradiction.

Case 6. \( d(v) \geq 18 \). Now \( \mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v) - 18}{3} \geq 0 \) by R2.

Thus we have proved \( \mu'(x) \geq 0 \) for every \( x \in V \cup F \), which contradicts (1) and completes the proof of Theorem 5.

References


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