P-APEX GRAPHS

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Dedicated to the memory
of Professor Horst Sachs (1927 – 2017)

Abstract

Let \( \mathcal{P} \) be an arbitrary class of graphs that is closed under taking induced subgraphs and let \( C(\mathcal{P}) \) be the family of forbidden subgraphs for \( \mathcal{P} \). We investigate the class \( \mathcal{P}(k) \) consisting of all the graphs \( G \) for which the removal of no more than \( k \) vertices results in graphs that belong to \( \mathcal{P} \). This approach provides an analogy to apex graphs and apex-outerplanar graphs studied previously. We give a sharp upper bound on the number of vertices of graphs in \( C(\mathcal{P}(1)) \) and we give a construction of graphs in \( C(\mathcal{P}(k)) \) of relatively large order for \( k \geq 2 \). This construction implies a lower bound on the maximum order of graphs in \( C(\mathcal{P}(k)) \). Especially, we investigate \( C(W_r(1)) \), where \( W_r \) denotes the class of \( P_r \)-free graphs. We determine some forbidden subgraphs for the class \( W_r(1) \) with the minimum and maximum number of vertices. Moreover, we give sufficient conditions for graphs belonging to \( C(\mathcal{P}(k)) \), where \( \mathcal{P} \) is an additive class, and a characterisation of all forests in \( C(\mathcal{P}(k)) \). Particularly we deal with \( C(\mathcal{P}(1)) \), where \( \mathcal{P} \) is a class closed under substitution and obtain a characterisation of all graphs in the corresponding \( C(\mathcal{P}(1)) \). In order to obtain desired results we exploit some hypergraph tools and this technique gives a new result in the hypergraph theory.

Keywords: induced hereditary classes of graphs, forbidden subgraphs, hypergraphs, transversal number.

2010 Mathematics Subject Classification: 05C75, 05C15.
1. Introduction

We only consider finite and simple graphs and follow [1] for graph-theoretical terminology and notation not defined here. A graph $G$ is an apex graph if it contains a vertex $w$ such that $G - w$ is planar. Although apex graphs seem to be close to planar graphs, some of their properties are far from corresponding properties of planar graphs (for example, see [18]).

A result of Robertson and Seymour (see [19]) says that every proper minor-closed class of graphs $P$ can be characterized by a finite family of forbidden minors (minor-minimal graphs not in $P$). Evidently, the class of apex graphs is minor-closed but the long-standing problem of finding the complete family of forbidden minors for this class is still open.

However, Dziobak in [9] introduced an apex-outerplanar graph that is a conceptual analogue to an apex graph. Namely, a graph $G$ is apex-outerplanar if there exists $w \in V(G)$ such that $G - w$ is outerplanar. Moreover, Dziobak provided the complete list of 57 forbidden minors for this class.

Another attempt to extend the concept of an apex graph is presented in [20] where an $l$-apex graph is defined. A graph $G$ is an $l$-apex graph if it can be made planar by removing at most $l$ vertices.

This paper concerns classes of graphs that generalize the aforementioned. Formally, by a class of graphs we mean an arbitrary family of non-isomorphic graphs. The empty class of graphs and the class of all graphs are called trivial. A class of graphs $P$ is induced hereditary if it is closed with respect to taking induced subgraphs. Such a class $P$ can be uniquely characterized by the family of forbidden subgraphs $C(P)$ that is defined as the set

$$\{G : G \notin P \text{ and } H \in P \text{ for each proper induced subgraph } H \text{ of } G\}.$$

By $L_\leq$ we denote the class of all non-trivial induced hereditary classes of graphs. Each class $P \in L_\leq$ has a non-empty family of forbidden subgraphs, consisting of graphs with at least two vertices. Moreover, $C(P)$ contains only connected graphs when $P$ is additive, i.e., closed under taking the union of disjoint graphs. By $L^a_\leq$ we denote the family of all non-trivial induced hereditary and additive classes of graphs.

Let $P \in L_\leq$ and let $k$ be a non-negative integer. A graph $G$ is a $P$-$(k)$-apex graph if there is $W \subseteq V(G)$, $|W| \leq k$ (with $W$ allowed to be the empty set), such that $G - W$ belongs to $P$. We denote the set of all $P$-$(k)$-apex graphs by $P(k)$ for short.

We can see immediately that if $k$ is a non-negative integer and $P \in L_\leq$, then $P(k) \in L_\leq$ too. On the other hand, the additivity of $P \in L_\leq$ implies the additivity of $P(k)$ if and only if $k = 0$. Indeed, $P(0) = P$. Moreover, if $P \in L^a_\leq$, then $C(P) \neq \emptyset$ and assuming that $F \in C(P)$ we can easily see that the union of
$k + 1$ disjoint copies of $F$ is in $C(P(k))$. Thus, for $k \geq 1$, it yields the existence of at least one disconnected graph that is forbidden for $P(k)$. Hence, for $k \geq 1$, the class $P(k)$ is not additive.

Lewis and Yannakakis in [17] have shown that for any non-trivial induced hereditary class $P$ containing infinitely many graphs and for a given positive integer $k$, the decision problem: "does $G$ belong to $P(k)$?" is NP-complete.

In this paper, we investigate the classes $P(k)$, in particular we focus on forbidden subgraphs for the classes $P(k)$ (i.e., we study graphs in $C(P(k))$). Additionally, we use hypergraphs as an effective tool in the research on $P(k)$.

Let $H$ be a hypergraph with vertex set $V(H)$ and edge set $E(H)$ and let $W \subseteq V(H)$. The hypergraph $H[W]$ induced in $H$ by $W$ has vertex set $W$ and edge set $\{E \in E(H) : E \subseteq W\}$. To simplify the notation we write $H - W$ instead of $H[V(H) \setminus W]$ and, moreover, $H - v$ instead of $H - \{v\}$ when $v$ is a vertex of $H$. Analogously, we write $H - E$ to denote the hypergraph obtained from $H$ by the deletion of the edge $E$ from $E(H)$.

By $H_1 \cup H_2$ we mean the union of disjoint hypergraphs $H_1$ and $H_2$, i.e., the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Moreover, notations $2H_1$, $H_1 \cup H_1$, and their generalization are used interchangeably.

The symbol $H_1 \leq H_2$ denotes that the hypergraph $H_1$ is isomorphic to a sub-hypergraph of $H_2$ induced by some of its vertex subset. Let $r$ be a non-negative integer. A hypergraph $H$ is $r$-uniform if each edge in $E(H)$ has exactly $r$ vertices.

A set $T \subseteq V(H)$ is called a transversal of the hypergraph $H$ if $T \cap E \neq \emptyset$ for each $E \in E(H)$. By $\tau(H)$ we denote the cardinality of the minimum transversal of $H$, i.e.,

$$\tau(H) = \min\{|T| : T \text{ is a transversal of } H\}.$$ 

A hypergraph $H$ is $\tau$-vertex critical if for any $v \in V(H)$ the inequality $\tau(H - v) \leq \tau(H) - 1$ holds. If a $\tau$-vertex critical hypergraph $H$ satisfies $\tau(H) = l$ for some positive integer $l$, then we call it $\tau$-vertex $l$-critical.

Recall that each graph is a hypergraph, which allows us to use these notations also for graphs. The symbols $K_n$, $P_n$, $C_n$ are used only for graphs and denote the complete graph, the path and the cycle with $n$ vertices, respectively.

This paper is organized as follows. We start with $\tau$-vertex $l$-critical hypergraphs in Section 2. We prove an upper bound on the order of a $\tau$-vertex 2-critical hypergraph and describe the construction of $\tau$-vertex $l$-critical hypergraphs with large number of vertices. Next, in Section 3, we prove some results on relations between $\tau$-vertex $(k + 1)$-critical hypergraphs and graphs in $C(P(k))$ for $P \in L_2$.

In Section 4, for $P \in L_2$ we show some sufficient conditions that have to be satisfied by a graph to be in $C(P(k))$ and we characterize all forests in $C(P(k))$.

Section 5 deals with the class $P$ of graphs that does not contain $P_r$ as an induced subgraph. We determine some forbidden subgraphs for $P(1)$ with minimum and maximum order in this case. In Section 6 we characterize all graphs in $C(P(1))$. 


where \( \mathcal{P} \) is a class of graphs that is induced hereditary and closed under substitution (for the definition see Section 6).

2. \( \tau \)-Vertex Critical Hypergraphs

A hypergraph \( \mathcal{H} \) is \( \tau \)-edge \( l \)-critical if \( \tau(\mathcal{H}) = l \) and the deletion of an edge decreases the transversal number of the resulting hypergraph. It is clear that the class of \( \tau \)-edge \( l \)-critical hypergraphs without isolated vertices forms a subclass of the class of \( \tau \)-vertex \( l \)-critical hypergraphs. On the other hand, it is easy to prove that the maximum order of hypergraphs in both classes is the same. In this section we prove that an \( r \)-uniform \( \tau \)-vertex 2-critical hypergraph has at most \( \left\lfloor \frac{(r+2)^2}{4} \right\rfloor \) vertices. Our proof is different than Tuza’s proof in [21] concerning a corresponding theorem for \( r \)-uniform \( \tau \)-edge 2-critical hypergraphs.

Next, for \( l \geq 3 \) we give the construction of an \( r \)-uniform \( \tau \)-vertex \( l \)-critical hypergraph with a large order. Gyárfás et al. [15] proved that each \( r \)-uniform \( \tau \)-vertex \( l \)-critical hypergraph has order bounded from above by \( \left( \frac{l+r-2}{r-2} \right)^{l+1} \). This bound is probably far from the exact value of the maximum number of vertices in a hypergraph that is \( r \)-uniform \( \tau \)-vertex \( l \)-critical. Our construction gives a large lower bound on the maximum order of a hypergraph that is \( r \)-uniform \( \tau \)-vertex \( l \)-critical.

**Theorem 1.** Let \( r \) be an integer, \( r \geq 2 \), and let \( \mathcal{H} \) be a \( \tau \)-vertex 2-critical hypergraph. If for each \( E \in \mathcal{E}(\mathcal{H}) \) we have \( |E| \leq r \), then

\[
|V(\mathcal{H})| \leq \left\lfloor \frac{(r+2)^2}{4} \right\rfloor.
\]

Moreover, the bound is sharp.

**Proof.** Denote by \( \mathcal{H}' \) a hypergraph obtained from \( \mathcal{H} \) by the optional deletion of some edges in such a way that \( \tau(\mathcal{H}) = \tau(\mathcal{H}') = 2 \) and \( \tau(\mathcal{H}' - E') \leq 1 \) for each edge \( E' \) of \( \mathcal{H}' \). Let \( \mathcal{E}' = \mathcal{E}(\mathcal{H}') \) and assume \( \mathcal{E}' = \{E'_1, \ldots, E'_m\} \). Observe that each vertex of \( \mathcal{H}' \) is contained in at least one of the edges in \( \mathcal{E}(\mathcal{H}') \). Otherwise, if there is \( x \in V(\mathcal{H}') \) such that \( x \) belongs to no edge in \( \mathcal{E}(\mathcal{H}') \), then \( \tau(\mathcal{H} - x) = 2 \) giving a contradiction to the \( \tau \)-vertex criticality of \( \mathcal{H} \).

Let a bipartite graph \( B \) be the incidence graph of the hypergraph \( \mathcal{H}' \). Thus \( B = (V(\mathcal{H}), \mathcal{E}; E(B)) \), where \( vE' \in E(B) \) if and only if \( v \in E' \). The previous consideration says that \( d_B(v) \geq 1 \) for all \( v \in V(\mathcal{H}) \) and \( d_B(E'_i) \leq r \) for all \( i \in \{1, \ldots, m\} \). The last condition implies \( |E(B)| \leq mr \).

**Claim 2.** For every \( E'_i \) there is a vertex, say \( v_i \in V(\mathcal{H}) \subseteq V(B) \), such that \( v_i \notin E'_i \) but \( v_i \in E'_j \in \mathcal{E}' \) for all \( j \neq i \).
Proof. Delete a vertex $E'_i$ from the graph $B$. The graph \( B - E'_i \) is an incidence graph of the hypergraph $H' - E'_i$, so $\tau(H' - E'_i) = 1$, i.e., there is a vertex, say $x$, which is adjacent in $B$ to every $E'_j$, $j \neq i$. Obviously the vertex $x$ is not adjacent to $E'_i$, otherwise in the hypergraph $H'$ there would be a 1-element transversal $\{x\}$, which is impossible. Thus $x$ can play the role of $v_i$ from the statement. □

By Claim 2, in the graph $B$ there is a set of $m$ vertices $\{v_1, \ldots, v_m\}$ with $d_B(v_i) = m - 1$, for $i \in \{1, \ldots, m\}$. Since $d_B(v) \geq 1$ for each $v \in V(H)$ we have $m(m-1) + (n-m) \leq |E(B)| \leq mr$, where $n = |V(H)|$. It leads to the inequality $n \leq -m^2 + (r+2)m$. Thus for fixed $r$, the maximum $n$ is $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$ and it is achieved at $m = \left\lceil \frac{r+2}{2} \right\rceil + 1$.

Finally, we prove that the bound is sharp. All the previous arguments imply that the structure of the $\tau$-vertex 2-critical hypergraph with maximum number of vertices must be defined in the following way. For $m = \left\lfloor \frac{r+2}{2} \right\rceil + 1$ or $\left\lceil \frac{r+2}{2} \right\rceil + 1$ let $U = \{1, \ldots, m\}$ and let $A_i = \{a_1^i, \ldots, a_{r+1-m}^i\}$ with $i \in U$. The $r$-uniform hypergraph $H$ such that $V(H) = U \cup \bigcup_{i=1}^{m} A_i$ and $E(H) = \{E_1, \ldots, E_m\}$ where \( E_i = (U \setminus \{i\}) \cup A_i \) for $i \in \{1, \ldots, m\}$, confirms the sharpness of the inequality given in the assertion.

The construction from the proof of Theorem 1 can be generalized in an easy way resulting in the following $r$-uniform $\tau$-vertex $\ell$-critical hypergraph with a large number of vertices.

Construction 1. Let $k, r, x$ be integers, $k \geq 1$, $r \geq 3$ and $r \leq x \geq 1$ and let $U = \{1, \ldots, k, k+1, \ldots, k+x\}$. Next let $m = \left(\frac{k+x}{r}\right)$ and let $\{U_1, \ldots, U_m\}$ be the family of all $x$-element subsets of $U$. Additionally, let $A_i = \{a_1^i, \ldots, a_{r+1-m}^i\}$ with $i \in \{1, \ldots, m\}$ be $m$ pairwise disjoint sets each of which is also disjoint with $U$.

We define an $r$-uniform hypergraph $H = H^*(k, r, x)$ in the following way:

\[
E(H^*) = \{E_1, \ldots, E_m\}, \quad \text{where } E_i = U_i \cup A_i, \quad i \in \{1, \ldots, m\};
\]

\[
V(H^*) = \bigcup_{i=1}^{m} E_i = U \cup A, \quad \text{where } A = \bigcup_{i=1}^{m} A_i.
\]

Theorem 3. If $k, r, x$ are integers such that $k \geq 1$, $r \geq 3$ and $r \leq x \geq 1$, then $H^*(k, r, x)$ is $\tau$-vertex $(k+1)$-critical.

Proof. Let $H^*(k, r, x) = H^*$. We use the notations connected with $H^*$ given in Construction 1. Observe that an arbitrary $(k+1)$-element subset of $U$ is a transversal of $H^*$. Thus $\tau(H^*) \leq k + 1$. Suppose, for a contradiction, that $T$ is a transversal of $H^*$ and $|T| \leq k$. If $T \subseteq U$, then $U \setminus T$ contains at least one $x$-element subset $U_i$ and consequently $E_i$ is an edge of $H^* - T$. Hence $T$ is not a transversal of $H^*$, a contradiction. Thus $T \setminus U = S \neq \emptyset$. Denote $t = |T \cap U|$ and $s = |S|$. There are at least $\binom{k+x-t}{x}$ edges of $H^*$ each of which has nonempty
intersection with $S$. It follows \( \binom{k+x-t}{x} \leq s \). Recall that \( s + t \leq k \). It means \( \binom{k+x-t}{x} \leq k-t \), which is impossible for any $x \geq 1$.

To observe the $\tau$-vertex criticality of $H^*$ it is enough to show that for each $v \in V(H^*)$ the condition $\tau(H^* - v) \leq k$ holds. If $v \in U$, then the removal of any $k$ vertices of $U$, all different from $v$, results in a hypergraph without edges. If $v \in A_i$ for some $i \in \{1, \ldots, m\}$, then the $k$-element transversal $U \setminus U_i$ realizes the inequality $\tau(H^* - v) \leq k$.

In the next lemma we find the maximum order of $H^*(k, r, x)$. This result gives a lower bound on the maximum number of vertices in an $r$-uniform $\tau$-vertex $(k+1)$-critical hypergraph.

Given $k, r$ we introduce $n(x) = \binom{k+x}{x}(r-x) + k + x = \binom{k+x}{k}(r-x) + k + x$.

**Lemma 4.** If $k, r$ are integers such that $k \geq 1$, $r \geq 3$, then

$$\max_{1 \leq x \leq r} |V(H^*(k, r, x))| = \max_{1 \leq x \leq r} n(x) = n \left( \frac{k(r-1)}{k+1} \right).$$

**Proof.** By Construction 1 we have $\max_{1 \leq x \leq r} |V(H^*(k, r, x))| = \max_{1 \leq x \leq r} n(x)$.

Consider the difference function $D(x) = n(x) - n(x+1) = -1 + \binom{k+x}{k}(r-x) - \frac{k+x+1}{x+1}((r-x)-1) = -1 + \binom{k+x}{k}(r-x)(-x)(k+x+1)(x+1) = -1 + \frac{k+x}{k(x+1)}(x+1)(k+1) - kr$.

Since $x, k$ and $r$ are positive integers, $D(x) \geq 0$ if and only if $(x+1)(k+1) - kr \geq 1$ and therefore the maximum $n(x)$ is reached at the smallest $x$ such that $D(x) \geq 0$, i.e., at $x = \left\lfloor \frac{k(r-1)}{k+1} \right\rfloor$.

3. **Graph Approach**

In this section we formulate some results on relations between $\tau$-vertex $(k+1)$-critical hypergraphs and forbidden subgraphs for $P(k)$. They are preceded by the helpful lemmas.

**Lemma 5.** Let $k$ be a non-negative integer and $P \in L_\leq$. If $F \in C(P(k))$, then $F \in P(k+1) \setminus P(k)$.

**Proof.** By the definition of $C(P(k))$ it follows that $F \notin P(k)$. Moreover, for an arbitrary $v \in V(F)$ we have $F - v \in P(k)$. It means that there exists a set $W$, contained in $V(F - v)$, such that $|W| \leq k$ and $(F - v) - W \in P$. Because $|W \cup \{v\}| \leq k + 1$ it leads to $F \in P(k+1)$.

Let $P \in L_\leq$ and $G$ be a graph. By $\mathcal{H}_P(G)$ we denote a hypergraph whose vertex set is $V(G)$ and whose edge set is $\{W \subseteq V(G) : G|W| \in C(P)\}$. Note the following facts.
**Remark 1.** Let $k$ be a non-negative integer, $\mathcal{P} \in \mathcal{L}_\leq$ and $G$ be a graph.

(i) $G \in \mathcal{P}(k)$ if and only if $\tau(\mathcal{H}_\mathcal{P}(G)) \leq k$.

(ii) $G \in \mathcal{P}(k + 1) \setminus \mathcal{P}(k)$ if and only if $\tau(\mathcal{H}_\mathcal{P}(G)) = k + 1$.

**Lemma 6.** Let $k$ be a non-negative integer and $\mathcal{P} \in \mathcal{L}_\leq$. A graph $G$ is a forbidden subgraph for $\mathcal{P}(k)$ if and only if $\mathcal{H}_\mathcal{P}(G)$ is $\tau$-vertex $(k + 1)$-critical.

**Proof.** Suppose that $G \in \mathcal{C}(\mathcal{P}(k))$. By Lemma 5 and Remark 1, $\tau(\mathcal{H}_\mathcal{P}(G)) = k + 1$. Moreover, for each $v \in V(G)$ we have $G - v \in \mathcal{P}(k)$, which again by Remark 1 implies $\tau(\mathcal{H}_\mathcal{P}(G - v)) \leq k$. Since $\mathcal{H}_\mathcal{P}(G - v) = \mathcal{H}_\mathcal{P}(G) - v$ we conclude that $\mathcal{H}_\mathcal{P}(G)$ is $\tau$-vertex $(k + 1)$-critical.

Now assume that $\mathcal{H}_\mathcal{P}(G)$ is $\tau$-vertex $(k + 1)$-critical. Remark 1 and the equality $\mathcal{H}_\mathcal{P}(G - v) = \mathcal{H}_\mathcal{P}(G) - v$ yield $G \in \mathcal{P}(k + 1) \setminus \mathcal{P}(k)$ and $G - v \in \mathcal{P}(k)$ for each $v \in V(G)$. Hence $G \in \mathcal{C}(\mathcal{P}(k))$.

Lemma 6 and Theorem 3 make it easy to formulate one more observation.

**Corollary 1.** Let $k, r, x$ be integers such that $k \geq 1$, $r \geq 3$, $r \geq x \geq 1$ and let $\mathcal{P} \in \mathcal{L}_\leq$. If $G$ is a graph such that $\mathcal{H}_\mathcal{P}(G)$ is isomorphic to $\mathcal{H}^*(k, r, x)$ defined in Construction 1, then $G$ is a forbidden subgraph for $\mathcal{P}(k)$.

A graph $G$ is a host-graph of a hypergraph $\mathcal{H}$ if $V(G) = V(\mathcal{H})$ and for each edge $e$ of $G$ there is an edge $E$ of $\mathcal{H}$ satisfying $e \subseteq E$. For an arbitrary family $\mathcal{F}$ of graphs, a graph $G$ is an $\mathcal{F}$-host-graph of a hypergraph $\mathcal{H}$ when it is a host-graph of $\mathcal{H}$ such that $G[E] \in \mathcal{F}$ for each edge $E$ of $\mathcal{H}$.

![Figure 1](image-url)  

Figure 1. The example of a host-graph of a hypergraph.

Observe that for a given family of graphs $\mathcal{F}$ and a hypergraph $\mathcal{H}$ an $\mathcal{F}$-host-graph of a hypergraph $\mathcal{H}$ does not necessarily exist. However, we can easily find a family $\mathcal{F}$ and a hypergraph $\mathcal{H}$ having an $\mathcal{F}$-host-graph. As an example, for a
fixed positive integer $r$, take $F = \{K_r\}$ and any $r$-uniform hypergraph $H$ (see Figure 1).

Furthermore, if $G$ is a $C(P)$-host-graph of a hypergraph $H$ then $H_P(G)$ is not necessarily isomorphic to $H$ (see Figure 1 again). We use $C(P)$-host-graphs to describe forbidden subgraphs for $P(k)$ with large number of vertices. In Section 2, we have constructed the family of hypergraphs $H^*(k, r, x)$ that are $r$-uniform $\tau$-vertex $(k + 1)$-critical and have large number of vertices. So, a $C(P)$-host-graph of a hypergraph $H^*(k, r, x)$ could be potentially a forbidden subgraph for $P(k)$. First we give some examples of families $F$ of graphs for which an $F$-host-graph of $H^*$ from Construction 1 exists.

Let $G$ be a graph. The symbols $\omega(G)$ and $\alpha(G)$ denote the order of the maximum clique and the cardinality of the maximum independent set of $G$, respectively.

**Lemma 7.** Let $F$ be a family of graphs. Next let $k, r, x$ be integers, $k \geq 1$, $r \geq 3$, $r > x \geq 1$ and $H^* = H^*(k, r, x)$ be a hypergraph from Construction 1.

(i) If there is $F \in F$ such that $|V(F)| = r$ and $\omega(F) \geq x$, then there exists an $F$-host-graph of the hypergraph $H^*$.

(ii) If there is $F \in F$ such that $|V(F)| = r$ and $\alpha(F) \geq x$, then there exists an $F$-host-graph of the hypergraph $H^*$.

(iii) If there is $F \in F$ such that $|V(F)| = r$ and moreover $r \geq x + k$, then there exists an $F$-host-graph of the hypergraph $H^*$.

**Proof.** Using the notations from Construction 1 we show how to obtain an $F$-host-graph $G$ of the hypergraph $H^*$. First we prove statements (i) and (ii). In the hypergraph $H^*$ we add all the edges between vertices in $U$ to obtain $K_{x+k}$ for (i) and we leave $U$ independent for (ii). Then we choose $F \in F$ such that $|V(F)| = r$ and $\omega(F) \geq x$ (for (i)) or $\alpha(F) \geq x$ (for (ii)). Now in each set $A_i$ from Construction 1 we enter a part of $F$ such that each $E_i$ induces $F$ in $G$. Observe that the assumption $\omega(F) \geq x$ or $\alpha(F) \geq x$ guarantees that all steps of this procedure can be done. To construct an $F$-host-graph $G$ for (iii) we choose an arbitrary vertex subset $W$ of $F$ of the cardinality $k + x$. Such a subset always exists since $r \geq k + x$. Next, we join some of the vertices in $U$ by edges in such a way that the resulting graph is isomorphic to $F[W]$. Then, similarly to above, in each set $A_i$ from Construction 1 we enter a part of the graph $F$ such that each $E_i$ induces $F$ in the graph $G$.

Consider $P \in L$ and a hypergraph $H^* = H^*(k, r, x)$. As we mentioned before if $G$ is a $C(P)$-host-graph of a hypergraph $H$, then $H_P(G)$ may be non-isomorphic to $H$. Hence we do not know whether a $C(P)$-host-graph of $H^*$ is a forbidden subgraph for $P(k)$ or not. In the next theorem, we solve this problem positively for some cases, regardless of whether the hypergraphs $H_P(G)$ and $H^*$ are isomorphic.
A set $S$ is a vertex-cut-set in a connected graph $G$ if $G - S$ has at least two connected components. For a positive integer $x$, a connected graph $G$ is $x$-vertex connected if it does not contain any vertex-cut-set of the cardinality less than $x$. As usual, for a given graph $G$ and $v \in V(G)$, we denote by $N_G(v)$ the set of neighbours of $v$ in $G$.

**Theorem 8.** Let $k, r, x$ be integers, $k \geq 1$, $r \geq 3$, $r > x \geq 1$, and let $H^* = \mathcal{H}^*(k, r, x)$ be the hypergraph from Construction 1. If $\mathcal{P} \in \mathcal{L}_< \mathcal{C}(\mathcal{P})$ is a class of graphs such that $\mathcal{C}(\mathcal{P})$ consists only of $x$-vertex connected graphs of order at least $r$, then each $\mathcal{C}(\mathcal{P})$-host-graph of the hypergraph $H^*$ is a forbidden subgraph for $\mathcal{P}(k)$.

**Proof.** In the proof we refer to the notations from Construction 1. Let $G$ be an arbitrary $\mathcal{C}(\mathcal{P})$-host-graph of the hypergraph $H^*$. Applying Lemma 6, the aim is to show that $\mathcal{H}_p(G)$ is $\tau$-vertex $(k + 1)$-critical.

First we prove that any $(k + 1)$-element subset $W$ of $U$ is a transversal of $\mathcal{H}_p(G)$, i.e., for any $(k + 1)$-element subset $W$ of $U$, the graph $G - W$ does not contain any induced subgraph $F$ satisfying $F \in \mathcal{C}(\mathcal{P})$. Suppose that this is not the case and let $F$ be a subgraph of $G - W$ such that $F \in \mathcal{C}(\mathcal{P})$. Denote by $U'_1, \ldots, U'_m$ the subsets of $V(G - W)$ that correspond to $U_1, \ldots, U_m$ in $G$. Thus, $|U'_i| \leq x - 1$ for each $i \in \{1, \ldots, m\}$. Furthermore, since $r > x$, it follows that $V(F)$ is not contained in $U - W$ and consequently $F$ must contain at least one vertex of some $A_i$ with $i \in \{1, \ldots, m\}$. Because of the symmetry, we may assume that $A' = A_i \cap V(F) \neq \emptyset$. Since $|A' \cup U'_i| < r$, there is a vertex of $F$ that does not belong to $A' \cup U'_i$. Hence, we can divide vertices of $F$ into three parts $V_1 = V(F) \cap A'$, $V_2 = V(F) \cap U'_i$, and $V_3 = V(F) \setminus (V_1 \cup V_2)$. By our earlier observation $V_3 \neq \emptyset$. Since $N_G(A_i) \subseteq U_1$, it follows that $N_F(V_1) \subseteq V_2$. Thus, $V_2$ is a vertex-cut-set of $F$. Furthermore, $|V_2| \leq |U'_i| \leq x - 1$, which contradicts that $F$ is $x$-vertex connected and proves $\tau(\mathcal{H}_p(G)) \leq k + 1$. Recall that, by the construction of $G$, each edge of $H^*$ is an edge of $\mathcal{H}_p(G)$. It means, by Theorem 3, that $\tau(\mathcal{H}_p(G)) \geq k + 1$ and consequently $\tau(\mathcal{H}_p(G)) = k + 1$.

Now, we prove the $\tau$-vertex criticality of $\mathcal{H}_p(G)$. By Remark 1 and the fact that $\mathcal{H}_p(G - v) = \mathcal{H}_p(G) - v$, we have to argue that for any $i \in \{1, \ldots, m\}$ and for any $v \in A_i$ we obtain $G - v \in \mathcal{P}(k)$. Let $W' = U - U_i$. Observe that $|W'| = k$ and $U_j \cap W' \neq \emptyset$ for $j \neq i$. We show that $(G - v) - W' \notin \mathcal{P}$ or equivalently that $(G - v) - W'$ does not contain an induced subgraph isomorphic to any $F \in \mathcal{C}(\mathcal{P})$. Let $U''_j, \ldots, U''_m$ be subsets of $V(G - W')$ that correspond to $U_1, \ldots, U_m$ in $G$. Thus, $|U''_j| \leq x - 1$ for each $j \neq i$ and $|U''_i| = x$. Suppose that there is $F \in \mathcal{C}(\mathcal{P})$ such that $F \subseteq (G - v) - W'$. It is clear that there is $j \neq i$ such that $F$ contains at least one vertex of $A_j$. Therefore, similarly as above, we can divide $V(F)$ into three parts $V_1 = V(F) \cap A_j$, $V_2 = V(F) \cap U''_j$, and $V_3 = V(F) \setminus (V_1 \cup V_2)$ with $V_3 \neq \emptyset$. Since $N_F(V_1) \subseteq V_2$, the set $V_2$ is a vertex cut-set of $F$, contrary to the $x$-vertex connectivity of $F$. □
Theorem 8 gives us a very fruitful tool to construct forbidden subgraphs for \( \mathcal{P}(k) \).

**Corollary 2.** Let \( k, x \) be positive integers and let \( \mathcal{P} \in \mathcal{L}_\leq \) be a class of graphs such that each graph in \( \mathcal{C}(\mathcal{P}) \) is \( x \)-vertex connected of order at least \( r \). If \( r \) is the order of some \( F \in \mathcal{C}(\mathcal{P}) \) and \( r \geq k + x \), then there exists \( G \) that is a forbidden subgraph for \( \mathcal{P}(k) \) and \( |V(G)| = k + x + (k + x)(r - x) \).

**Theorem 9.** Let \( \mathcal{P} \in \mathcal{L}_\leq \). If \( r = \max\{|F| : F \in \mathcal{C}(\mathcal{P})\} \) and \( G \in \mathcal{C}(\mathcal{P}(1)) \), then \( |V(G)| \leq \left\lfloor \frac{(r+2)^2}{4} \right\rfloor \). Moreover, this bound is achieved for infinitely many classes \( \mathcal{P} \in \mathcal{L}_\leq \).

**Proof.** By Lemma 6 and Theorem 1 we only need to show the last sentence of the statement. However, if we put \( k = 1 \) and \( x = \left\lceil \frac{r-1}{2} \right\rceil \) in Corollary 2, then for \( r \geq 3 \) we obtain a forbidden subgraph for \( \mathcal{P}(k) \) with \( \left\lfloor \frac{(r+2)^2}{4} \right\rfloor \) vertices and hence the theorem follows.

The next remark is an immediate consequence of Theorem 9 and the fact that \( (\mathcal{P}(k))(1) = \mathcal{P}(k+1) \).

**Remark 2.** Let \( k \) be a non-negative integer and \( \mathcal{P} \in \mathcal{L}_\leq \). If \( \mathcal{C}(\mathcal{P}) \) is finite, then the family \( \mathcal{C}(\mathcal{P}(k)) \) is also finite.

### 4. The Structure of Forbidden Subgraphs

At the beginning of this section we describe connected forbidden subgraphs for \( \mathcal{P}(k) \) in terms of connected forbidden subgraphs for \( \mathcal{P}(l) \), where \( l < k \). To do it we use the following hypergraph tool.

**Remark 3.** If \( \mathcal{H}_1 \cup \mathcal{H}_2 \) is the union of disjoint hypergraphs \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), then

\[
\tau(\mathcal{H}_1 \cup \mathcal{H}_2) = \tau(\mathcal{H}_1) + \tau(\mathcal{H}_2).
\]

Note that the definition of the \( \tau \)-vertex criticality of a hypergraph and Remark 3 imply the following observation.

**Remark 4.** Let \( s \) be an integer, \( s \geq 2 \). The union \( \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_s \) of disjoint hypergraphs \( \mathcal{H}_1, \ldots, \mathcal{H}_s \) is \( \tau \)-vertex critical if and only if for each \( i \in \{1, \ldots, s\} \) the hypergraph \( \mathcal{H}_i \) is \( \tau \)-vertex critical.

The next result is the consequence of Remark 4.
Theorem 10. Let $k, s$ be integers, $k \geq 0$, $s \geq 1$ and $P \in L_0^\leq$. The union $F_1 \cup \cdots \cup F_s$ of disjoint connected graphs $F_1, \ldots, F_s$ is a forbidden subgraph for $P(k)$ if and only if there exist non-negative integers $k_1, \ldots, k_s$ such that $\sum_{i=1}^s k_i = k + 1 - s$ and for each $i \in \{1, \ldots, s\}$ the graph $F_i$ is a forbidden subgraph for $P(k_i)$.

Proof. From Lemma 6 we have $F_1 \cup \cdots \cup F_s \in C(P(k))$ if and only if $H_P(F_1 \cup \cdots \cup F_s)$ is $\tau$-vertex $(k+1)$-critical. Since $H_P(F_1 \cup \cdots \cup F_s) = H_P(F_1) \cup \cdots \cup H_P(F_s)$ and because of Remarks 3, 4 we know that it is equivalent to the conditions $\tau(H_P(F_1)) + \cdots + \tau(H_P(F_s)) = k + 1$ and for each $i \in \{1, \ldots, s\}$ the hypergraph $H_P(F_i)$ is $\tau$-vertex critical. It means that there exist non-negative integers $k_1, \ldots, k_s$ such that for each $i \in \{1, \ldots, s\}$ the hypergraph $H_P(F_i)$ is $\tau$-vertex $(k_i + 1)$-critical and moreover $\sum_{i=1}^s (k_i + 1) = k + 1$. From Lemma 6 these conditions are equivalent to the statement $F_i \in C(P(k_i))$ for each $i \in \{1, \ldots, s\}$ and $\sum_{i=1}^s k_i = k + 1 - s$. ■

Corollary 3. Let $k$ be a non-negative integer and $P \in L_0^\leq$. If $F$ is the union of disjoint connected graphs $F_1, \ldots, F_s$ and $F \in C(P(k))$, then $s \leq k + 1$.

Corollary 4. Let $k$ be a non-negative integer and $P \in L_0^\leq$ and let $|C(P)| = p$. The number of forbidden subgraphs for $P(k)$ that have exactly $k + 1$ connected components is equal to $\binom{k+p}{k+1}$.

Proof. From Theorem 10 we know that forbidden subgraphs for $P(k)$ with exactly $k + 1$ connected components have the form $F_1 \cup \cdots \cup F_{k+1}$, where for each $i \in \{1, \ldots, k+1\}$ the condition $F_i \in C(P(k))$ holds. Let $C(P) = \{H_1, \ldots, H_p\}$. Thus, if $m_i$ denotes $|\{l : F_l = H_i\}|$, then we actually are interested in the number of sequences $(m_1, \ldots, m_p)$ whose elements are non-negative integers and for which the equality $m_1 + \cdots + m_p = k + 1$ holds, which leads to the assertion. ■

The remaining part of this section is devoted to other constructions of forbidden subgraphs for $P(k)$ in terms of forbidden subgraphs for $P$. In this consideration the structure of $H_P(G)$ is unknown. It means that our results are based only on the analysis of graph structures.

Construction 2. Let $s$ be a positive integer, $G_1, \ldots, G_s$ be graphs and $T$ be a forest with the vertex set $\{x_1, \ldots, x_s\}$. By $T(G_1, \ldots, G_s)$ we denote the family of all graphs obtained from disjoint $G_1, \ldots, G_s$ by the addition of exactly $|E(T)|$ new edges, such that a new edge joins an arbitrary vertex of $G_i$ with an arbitrary vertex of $G_j$ when $x_ix_j$ is an edge of $T$. Next we use a symbol $(G_1, \ldots, G_s)$ to denote the family of all graphs $T(G_1, \ldots, G_s)$ taken over all $s$-vertex forests $T$ and all possible orderings of their vertices.

Theorem 11. If $k$ is a non-negative integer and $P \in L_0^\leq$ and $G_1, \ldots, G_{k+1} \in C(P)$, then each graph $G$ in $(G_1, \ldots, G_{k+1})$ is a forbidden subgraph for $P(k)$. 
Proof. Suppose that $G \in (G_1, \ldots, G_{k+1})$. It follows that there exists a forest $T$ with $k + 1$ vertices $x_1, \ldots, x_{k+1}$ such that $G \in T(G_1, \ldots, G_{k+1})$. Observe that $G \not\in \mathcal{P}(k)$ since it contains $k + 1$ disjoint induced subgraphs that are forbidden subgraphs for $\mathcal{P}$.

Next, let $v \in V(G)$. We show that there exist $k$ vertices $u_2, \ldots, u_{k+1}$ in $V(G) \setminus \{v\}$ such that the graph resulting from $G$ by the removal of $v, u_2, \ldots, u_{k+1}$ is in $\mathcal{P}$.

The construction of $G$ implies the existence of the unique index $i$ such that $v \in V(G_i)$. Let $x_{j_1}, \ldots, x_{j_{k+1}}$ be a new ordering of vertices of $T$ such that $x_{j_l} = x_i$ and for $l \geq 2$ each vertex $x_{j_l}$ has at most one neighbour in $\{x_{j_1}, \ldots, x_{j_{l-1}}\}$. Such an ordering can be done by brute-force search algorithm. Suppose, without loss of generality, that $x_{j_l} = x_l$ for each $l \in \{1, \ldots, k + 1\}$. Consequently, $G_{j_l} = G_l$ for each $l \in \{1, \ldots, k + 1\}$ and especially $G_i = G_1$.

Now we describe how to choose vertices $u_2, \ldots, u_{k+1}$. For each $j \in \{2, \ldots, k + 1\}$ there is at most one edge $x_l x_j$ with $l < j$. Thus when such an edge exist we take as $u_j$ the vertex of $G_j$ that is the end of the unique edge joining $G_l$ with $G_l$ (see the construction of $G$), otherwise $u_j$ is an arbitrary vertex of $G_j$. Observe that $G \setminus \{v, u_2, \ldots, u_{k+1}\}$ is the union of $k + 1$ disjoint graphs $G_1 - v$ and $G_j - u_j$ for $j \in \{2, \ldots, k + 1\}$. The assertion follows by the additivity of $\mathcal{P}$ and properties of all $G_j$.

Theorem 12. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathcal{L}_\leq$. A forest $G$ is a forbidden subgraph for $\mathcal{P}(k)$ if and only if $G \in (G_1, \ldots, G_{k+1})$, where $G_1, \ldots, G_{k+1}$ are trees that are forbidden subgraphs for $\mathcal{P}$.

Proof. By Theorem 11, it is enough to prove that if $G$ is simultaneously a forest and a forbidden subgraph for $\mathcal{P}(k)$, then there are graphs $G_1, \ldots, G_{k+1}$ belonging to $\mathcal{C}(\mathcal{P})$ and there exists a $(k+1)$-vertex forest $T$ such that $G \in T(G_1, \ldots, G_{k+1})$.

To do it we use the induction on $k$.

By the additivity of $\mathcal{P}$, each forest that is a forbidden subgraph for $\mathcal{P}(0) = \mathcal{P}$ is a tree. The conclusion follows from the fact that there is only one 1-vertex forest $T = K_1$ and each graph $G$ can be represented as $K_1(G)$, which means as $T(G)$.

Assume that the implication is true for parameters less than $k$ and $k \geq 1$. First suppose that $G$ has at least two connected components $H_1, \ldots, H_s$. Obviously, each of them is a tree. By Theorem 10, $H_i \in \mathcal{C}(\mathcal{P}(k_i))$, where $\sum_{i=1}^s k_i = k + 1 - s$. Because all $k_i$ are non-negative integers and $s \geq 2$ we obtain $0 \leq k_i \leq k - 1$ for each $i \in \{1, \ldots, s\}$. By the induction hypothesis, $H_i \in T_i(G_1, \ldots, G_{k_{i+1}})$, which implies

$$G \in T(G_1^1, \ldots, G_{k_{i+1}}^1, \ldots, G_{k_{i+1}}^s, \ldots, G_{k_{s+1}}^s),$$

where $T$ is the union of disjoint $T_1, \ldots, T_s$ and $G_j^i \in \mathcal{C}(\mathcal{P})$ for each $l \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, k_l + 1\}$. Since each $T_i$ has exactly $k_i + 1$ vertices, the forest $T$ has
\[ \sum_{i=1}^{s}(k_i + 1) \] vertices, which means \( T \) has \( k + 1 \) vertices. Thus \( G \) has a required form.

Now suppose that \( G \) is connected, which means \( G \) is a tree.

**Claim 13.** There is \( x \in V(G) \) such that \( G - x \) has at least one connected component in \( P \) and if \( H_1, \ldots, H_p \) are all connected components of \( G - x \) belonging to \( P \), then the graph induced in \( G \) by \( V(H_1) \cup \cdots \cup V(H_p) \cup \{x\} \) is not in \( P \).

**Proof.** We describe the procedure which finds the required \( x \) in a finite number of steps.

Let \( v_0 \) be an arbitrary vertex of \( G \) that is not a leaf (such a vertex always exists because \( k \geq 1 \), which implies \( |V(G)| \geq 3 \)). Next let \( G_1 \) be an arbitrary connected component of \( G - v_0 \) such that \( G_1 \notin P \) (since \( G \) is in \( C(P(k)) \) and \( k \geq 1 \) such a connected component exists).

Let \( v_1 \) be the unique neighbour of \( v_0 \) in \( G_1 \). If \( G_1 - v_1 \in P \), then \( x = v_1 \).

Otherwise, let \( G_2 \) be an arbitrary connected component of \( G_1 - v_1 \) such that \( G_2 \notin P \) and let \( v_2 \) be the unique neighbour of \( v_1 \) in \( G_2 \). If \( G_2 - v_2 \in P \), then \( x = v_2 \). Otherwise, since \( G \) is finite, we find the finite sequence of vertices \( v_0, \ldots, v_q \) and the sequence of graphs \( G = G_0, G_1, \ldots, G_q \) such that \( G_i - v_i \notin P \) for \( i \in \{0, \ldots, q-1\} \), \( G_q \notin P \) and \( G_q - v_q \notin P \). Moreover for \( i \in \{1, \ldots, q\} \) the graph \( G_i \) is a connected component of \( G_{i-1} - v_{i-1} \) and \( v_i \) is the unique neighbour of \( v_{i-1} \) in \( G_i \).

Observe that \( v_q \) can play the role of \( x \). Indeed, the procedure implies that the connected components of \( G_q - v_q \) are simultaneously the connected components of \( G - v_q \).

Let \( x \) be a vertex that satisfies the assumptions of Claim 13. Recall that \( G \) is a tree, which means that \( G - x \) is a forest. Since \( G \) is a forbidden subgraph for \( P(k) \) we obtain \( G - x \notin P(k - 1) \). It follows that \( G - x \) contains an induced subgraph \( G' \in C(P(k-1)) \) that is a forest. By the induction hypothesis \( V(G') \) can be partitioned into \( k \) sets \( V_1, \ldots, V_k \) such that for each \( i \in \{1, \ldots, k\} \) the graph \( G_i' \) induced by \( V_i \) in \( G - x \) is forbidden for \( P \). Because \( P \) is additive, all of the graphs \( G_i' \) are connected and as subgraphs of \( G - x \) they are trees. Additionally, \( V(G_1') \cup \cdots \cup V(G_k') \cup \{x\} \cap V(H_i) = \emptyset \) for \( i \in \{1, \ldots, p\} \) (keep in mind that \( H_1, \ldots, H_p \in P \), see Claim 13).

Recall that, by Claim 13, \( V(H_1) \cup \cdots \cup V(H_p) \cup \{x\} \) contains at least one subset that induces a graph, say \( G_{k+1}' \), forbidden for \( P \). Hence \( G_1', \ldots, G_{k+1}' \) are disjoint induced subgraphs of \( G \), each of which is in \( C(P) \). Suppose, for a contradiction, that there is a vertex \( u \in V(G) \setminus \bigcup_{i=1}^{k+1} V(G_i') \). Since \( G \in C(P(k)) \) we can find at most \( k \) different vertices of \( G - u \) such that the removal of all of them from \( G - u \) results in a graph in \( P \). Because \( G \) contains disjoint induced subgraphs \( G_1', \ldots, G_{k+1}' \) that are forbidden for \( P \), it is impossible, giving a contradiction.
It means $V(G) = \bigcup_{i=1}^{k+1} V(G'_i)$ and, since $G$ is a tree, there is a tree $T$ with $k + 1$ vertices such that $G \in T \left(G'_1, \ldots, G'_{k+1}\right).$ \hfill\(\blacksquare\)

Below we present one more construction of graphs that are forbidden for $\mathcal{P}(k)$.

**Construction 3.** Let $G_1, \ldots, G_s$ be rooted graphs, which means that for each $i \in \{1, \ldots, s\}$ the graph $G_i$ has a marked vertex $v_i$, called its root. Next let $H$ be a graph with $V(H) = \{x_1, \ldots, x_s\}$. We take disjoint $H, G_1, \ldots, G_s$ and identify vertices $v_i$ with $x_i$ for all $i \in \{1, \ldots, s\}$. By $H|G_1, \ldots, G_s$ we denote the family of all graphs of this type taken over all possible choices of roots $v_1, \ldots, v_s$. More precisely, for each graph $G$ in $H|G_1, \ldots, G_s$ we have $V(G) = \bigcup_{i=1}^{s} V(G_i)$ and $E(G) = \bigcup_{i=1}^{s} E(G_i) \cup \{v_iv_j : x_ix_j \in E(H)\}$ with a choice of roots $v_1, \ldots, v_s$. Now we use a symbol $|G_1, \ldots, G_s|$ to denote the union of sets $H|G_1, \ldots, G_s|$ taken over all $s$-vertex graphs $H$.

**Theorem 14.** If $k$ is a non-negative integer and $\mathcal{P} \in \mathbb{L}_\leq$ and $G_1, \ldots, G_{k+1} \in \mathcal{C}(\mathcal{P})$, then each graph $G$ in $|G_1, \ldots, G_{k+1}|$ is a forbidden subgraph for $\mathcal{P}(k)$.

**Proof.** By the assumption $G \in |G_1, \ldots, G_{k+1}|$, we have that $G \in H|G_1, \ldots, G_{k+1}$ for some $(k+1)$-vertex graph $H$. Let $x_i = v_i$ be a common vertex of $H$ and $G_i$, described in Construction 3.

Because $G$ contains disjoint induced subgraphs $G_1, \ldots, G_{k+1}$ it follows that $G \notin \mathcal{P}(k)$. If $v \in V(G)$, then $v \in V(G_j)$ for exactly one index $j \in \{1, \ldots, k+1\}$. The graph obtained from $G - v$ by the removal of the vertex set $S$, where $S = \{x_l : l \neq j\}$, has at least $k + 1$ connected components each of which is in $\mathcal{P}$. The additivity of $\mathcal{P}$ implies $G - v \notin \mathcal{P}(k)$. \hfill\(\blacksquare\)

## 5. $P_r$-Free Graphs

In this section we focus our attention on the class $\mathcal{W}_r$ of graphs not containing $P_r$ as an induced subgraph. We determine the minimum and maximum number of vertices of a graph in $\mathcal{C}(\mathcal{W}_r(1))$. First we consider $\mathcal{C}(\mathcal{W}_3(1))$. Because of Theorem 9, each graph that is forbidden for $\mathcal{W}_3(1)$ has at most six vertices. Searching all non-isomorphic graphs of this type we can derive that $\mathcal{C}(\mathcal{W}_3(1))$ has 14 elements: $C_4, C_5, C_6, P_6, 2P_3, F_1, \ldots, F_9$, where the graphs $F_i$ for $i \in \{1, \ldots, 9\}$ are depicted in Figure 2. Similar arguments we apply to the classes $\mathcal{O}$ of edgeless graphs and $\mathcal{K}$ of complete graphs. In this case, the facts $\mathcal{C}(\mathcal{O}) = \{K_2\}$ and $\mathcal{C}(\mathcal{K}) = \{\bar{K}_2\}$ yield $\mathcal{C}(\mathcal{O}(1)) = \{K_3, P_4, C_4, 2K_2\}$ and $\mathcal{C}(\mathcal{K}(1)) = \{\bar{K}_3, P_3, C_4, 2K_2\}$.

Of course the brute searching method is not too effective if forbidden subgraphs have big orders. Thus for $r \geq 4$ we start with determining forbidden subgraphs for $\mathcal{W}_r(1)$ with the minimum number of vertices. If $G \in \mathcal{C}(\mathcal{W}_r(1))$,
then $G$ must contain an induced subgraph $P_r$ after deletion of any vertex. Thus $r + 1$ is the lower bound on the number of vertices of a graph in $C(W_r(1))$. We conclude the following fact.

**Proposition 1.** If $r$ is an integer, $r \geq 3$, then $C_{r+1}$ is a forbidden subgraph for $W_r(1)$ with the minimum number of vertices.

By Theorem 9 we have that the upper bound on the number of vertices of a graph in $C(W_r(1))$ is $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$. However, for $r = 4$ we find no graph that realizes this bound. For any $r \geq 5$ there exists a graph in $C(W_r(1))$ of order $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$.

To prove this fact we use the class of graphs that contains all the complements of graphs in $W_r$.

For a given class of graphs $P \in L_{\leq}$ let us define $\overline{P} = \{\overline{G} : G \in P\}$. It is a known fact that if $P \in L_{\leq}$, then $P$ is also in $L_{\leq}$. Moreover, there is a coincidence between forbidden subgraphs for $P$ and $\overline{P}$ given by the equality $C(P) = \{\overline{F} : F \in C(\overline{P})\}$ [2]. Let $P_1, P_2$ be classes of graphs. By $P_1 \circ P_2$ we denote the class of all graphs $G$ whose vertex set can be partitioned into two parts $V_1, V_2$ (possible empty) such that, for all $i \in \{1, 2\}$, if $V_i$ is non-empty, then $G[V_i] \in P_i$. In that case $P_1 \circ P_2$ is called a product of $P_1$ and $P_2$. In [4] it is proved that $F \in C(P_1 \circ P_2)$ if and only if $\overline{F} \in C(\overline{P_1} \circ \overline{P_2})$. It is easy to observe that for each class of graphs $P$ and a positive integer $k$, the class $P(k)$ is identical with $P \circ Q$, where $Q$ consists of all the graphs of order at most $k$. Moreover, for such $Q$ we have $\overline{Q} = Q$. Hence, taking into account the previous consideration, we have the following observation.

**Proposition 2.** If $P \in L_{\leq}$, then

(i) $G \in P(k)$ if and only if $\overline{G} \in P(k)$, and

(ii) $F \in C(P(k))$ if and only if $\overline{F} \in C(\overline{P}(k))$, and
(iii) \( G \in \mathcal{P}(k) \) if and only if \( \overline{G} \in \mathcal{P}(k) \).

Let us consider \( \overline{W_r} \). Thus, \( \mathcal{C}(\overline{W_r}) = \{\overline{P_r}\} \) and, by Proposition 2, it follows that \( G \in \mathcal{C}(\overline{W_r}(1)) \) if and only if \( \overline{G} \in \mathcal{C}(W_r(1)) \). As a consequence, the complement of a forbidden subgraph for \( \overline{W_r}(1) \) with the maximum number of vertices is a forbidden subgraph for \( W_r(1) \) with the maximum number of vertices. Since the vertex connectivity of \( P_r \) is relatively big we will be able to apply Theorem 8. First we give the supporting observation.

**Lemma 15.** If \( r \) is an integer, \( r \geq 5 \), then \( P_r \) is \( \lceil \frac{r-1}{2} \rceil \)-connected.

**Proof.** Let \( G = \overline{P_r} \). Observe that the vertices of \( G \) can be divided into two sets \( W_1, W_2 \) such that subgraphs induced by \( W_i \) for \( i \in \{1, 2\} \) are complete graphs and \( |W_1| = \left\lceil \frac{r}{2} \right\rceil \), \( |W_2| = \left\lfloor \frac{r}{2} \right\rfloor = \left\lceil \frac{r-1}{2} \right\rceil \). Suppose that there is a vertex-cut-set \( S \) of \( G \) such that \( |S| < \left\lfloor \frac{r}{2} \right\rfloor \). Thus \( G - S \) has two disjoint subgraphs \( G_1 \) and \( G_2 \) such that there is no edge joining a vertex of \( G_1 \) with a vertex of \( G_2 \). Furthermore, observe that \( V(G_1) = W_1 \setminus S \) and \( V(G_2) = W_2 \setminus S \) and moreover, \( V(G_1) \neq \emptyset \) and \( V(G_2) \neq \emptyset \). Let us denote \( W'_1 = W_1 \setminus S \) and \( W'_2 = W_2 \setminus S \). So, by our assumptions, there is no edge joining a vertex of \( W'_1 \) with a vertex of \( W'_2 \) in \( G \). This implies that in \( G \) each vertex of \( W'_1 \) is adjacent to each vertex of \( W'_2 \). If \( |W'_1| \geq 2 \) and \( |W'_2| \geq 2 \), then \( G \) contains \( C_4 \), which contradicts that \( G = \overline{P_r} \). If one of the sets \( W'_1, W'_2 \) contains exactly one vertex, then since \( |S| < \left\lfloor \frac{r}{2} \right\rfloor \), there are at least three vertices in the second set. Thus \( G \) has a vertex of degree three, which again gives a contradiction with the assumption that \( G \) is a path. \( \blacksquare \)

By Lemma 7 we have the additional fact.

**Lemma 16.** Let \( r \) be an integer, \( r \geq 5 \). There exists a \( \{\overline{P_r}\}\)-host-graph of a hypergraph \( \mathcal{H}^*(1, r, \left\lceil \frac{r-1}{2} \right\rceil) \) given in Construction 1.

Finally, by Theorem 8, Lemma 16 and Proposition 2, we obtain the conclusion.

**Theorem 17.** Let \( r \) be an integer, \( r \geq 5 \). The complement of a \( \{\overline{P_r}\}\)-host-graph of the hypergraph \( \mathcal{H}^*(1, r, \left\lceil \frac{r-1}{2} \right\rceil) \), given in Construction 1, is a forbidden subgraph for \( W_r(1) \) with the maximum number of vertices.

In Figure 3 we present the complement of a forbidden subgraph for \( W_5(1) \). Theorem 17 says that this graph has the maximum number of vertices among all the graphs in \( \mathcal{C}(W_5(1)) \). Moreover, by Proposition 2, the graph in Figure 3 is in \( \mathcal{C}(\overline{W_5}(1)) \) and also in \( \mathcal{C}(\overline{W_5}(1)) \) and realizes the maximum order among all the graphs in both these families.
6. Classes of Graphs That Are Closed Under Substitution

Let $H, G_1, \ldots, G_n$ be graphs and $v_1, \ldots, v_n$ be an arbitrary ordering of the set $V(H)$. By $H[G_1, \ldots, G_n]$ we denote the graph resulting from $H$ by the simultaneous substitution of each vertex $v_i$ with the graph $G_i$. Here the substitution of the vertex $v$ with the graph $G$ in the graph $H$ means the removal of $v$ and joining all the vertices of $G$ with all the neighbours of $v$ in $H$. A class $\mathcal{P}$ of graphs is closed under substitution if for any graphs $H, G_1, \ldots, G_n \in \mathcal{P}$ and every ordering of $V(H)$, the graph $H[G_1, \ldots, G_n]$, called a substitution graph, is also in $\mathcal{P}$. By $\mathbf{L}_\leq$ we denote the class of all non-trivial induced hereditary classes of graphs that are closed under substitution. The smallest of such ones (in the sense of the number of elements) is $\{K_1\}$, among most notable we should list the classes $\mathcal{O}$ of edgeless graphs, $\mathcal{K}$ of complete graphs, the class of perfect graphs and the classes $W_r$, where $r = 2$ or $r \geq 4$. Observe that $P_4$-free graphs are just cographs. In this section we characterize all forbidden subgraphs for $\mathcal{P}(1)$ where $\mathcal{P} \in \mathbf{L}_\leq$.

A set $W \subseteq V(G)$ is a module in a graph $G$ if for each two vertices $x, y \in W$, $N_G(x) \setminus W = N_G(y) \setminus W$. The trivial modules in $G$ are $V(G)$, $\emptyset$ and singletons. A graph having only trivial modules is called prime. By $\mathbf{PRIME}$ we denote the class of all prime graphs that have at least two vertices.

In 1997 Giakoumakis [14] proved that for each class of graphs $\mathcal{P} \in \mathbf{L}_\leq$ its closure under substitution $\mathcal{P}^*$ consisting of all the graphs in $\mathcal{P}$ and all their substitution graphs can be characterized by $\mathcal{C}(\mathcal{P}^*)$ that consists of all minimal prime extensions of all the graphs in $\mathcal{C}(\mathcal{P})$. It has to be said that $G'$ is a minimal prime extension of $G$ if it is a prime induced supergraph of $G$ and it does not contain as a proper induced subgraph any other prime induced supergraph of $G$.

Since for each class $\mathcal{P} \in \mathbf{L}_\leq$ we have $\mathcal{P} = \mathcal{P}^*$ (by the definition of $\mathbf{L}_\leq$), the Giakoumakis consideration leads to the following conclusion.

**Remark 5.** If $\mathcal{P} \in \mathbf{L}_\leq$, then $\mathcal{P} \in \mathbf{L}_\leq^*$ if and only if $\mathcal{C}(\mathcal{P}) \subseteq \mathbf{PRIME}$. 
In [4] the following two theorems concerning \( C(P_1 \circ P_2) \) when both \( P_1, P_2 \) are in \( L^*_\leq \) have been proven.

**Theorem 18** [4]. Let \( P_1, P_2 \in L^*_\leq \) and let \( H \in \text{PRIME} \) with \( V(H) = \{v_1, \ldots, v_n\} \). If \( G = H[G_1, \ldots, G_n] \) and \( G \in C(P_1 \circ P_2) \), then \( H \notin P_1 \) or \( H \notin P_2 \) and there exists a partition \((A, B, C, D)\) of \( \{1, \ldots, n\} \) (empty parts are allowed), such that

(i) \( G_i = K_1 \) for \( i \in A \), and  
(ii) \( G_i \in C(P_2) \cap P_1 \) for \( i \in B \), and  
(iii) \( G_i \in C(P_1) \cap P_2 \) for \( i \in C \), and  
(iv) \( G_i \in C(P_1 \cup P_2) \) for \( i \in D \).

A graph \( G \), different from \( K_1 \), is **strongly decomposable** if in its description \( G = H[G_1, \ldots, G_n] \) with \( H \in \text{PRIME} \), all the graphs \( G_i \) satisfy \( |V(G_i)| \geq 2 \).

In the next theorem we will restrict our attention to graphs that are strongly decomposable and are forbidden subgraphs for a product of classes of graphs.

**Theorem 19** [4]. Let \( P \in L^*_\leq \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\} \). A graph \( G \) is a forbidden subgraph for \( P_1 \circ P_2 \) and it is strongly decomposable if and only if there exists a representation \( H[G_1, \ldots, G_n] \) of \( G \), with \( H \in \text{PRIME} \), \( V(H) = \{v_1, \ldots, v_n\} \), such that either for \( j = 1 \) and \( l = 2 \) or for \( j = 2 \) and \( l = 1 \) the following three conditions hold:

(i) \( H \in C(P_j) \), and  
(ii) for each \( i \in \{1, \ldots, n\} \), \( G_i \in C(P_i) \), and  
(iii) for \( M = \{i \in \{1, \ldots, n\} : G_i \notin P_j\} \) and for each \( s \in \{1, \ldots, n\} \setminus M \) the subgraph of \( H \) induced by \( \{v_i : i \in M \cup \{s\}\} \) is in \( P_i \); moreover, if \( M = \{1, \ldots, n\} \), then \( H \in P_i \).

Observe that \( \text{PRIME} \) includes only two graphs, \( K_2, \overline{K_2} \), with two vertices, no graph on three vertices and only one graph, \( P_4 \), with four vertices. Next \( C(\mathcal{O}) = \{K_2\}, C(\mathcal{K}) = \{\overline{K_2}\}, C(\{K_1\}) = \{K_2, \overline{K_2}\} \). Thus if \( P \in L^*_\leq \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\} \), then the family \( C(P) \) has to contain at least one graph in \( \text{PRIME} \setminus \{K_2, \overline{K_2}\} \). Since each graph on at least 4 vertices contains an induced subgraph \( K_2 \) or \( \overline{K_2} \) and graphs in \( C(P) \) are not comparable with respect to induced subgraph relation, we conclude that \( C(P) \cap \{K_2, \overline{K_2}\} = \emptyset \). Hence we have the following fact.

**Remark 6.** If \( P \in L^*_\leq \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\} \), then \( \{K_2, \overline{K_2}\} \subseteq P \).

Recall that \( P(1) = P \circ \{K_1\} \) and \( \{K_1\} \in L^*_\leq \). Hence, from Theorem 19, we obtain the following immediate consequence.
Corollary 5. If $P \in \mathcal{L}_\leq \backslash \{O, K, \{K_1\}\}$, then $G$ is a forbidden subgraph for $P(1)$ that is strongly decomposable if and only if $G = K_2[H_1, H_2]$ or $G = \overline{K_2}[H_1, H_2] = H_1 \cup H_2$ or $G = H_1|G_1, \ldots, G_n$, where $H_1, H_2 \in \mathcal{C}(P)$ and $G_1, \ldots, G_n \in \{K_2, \overline{K_2}\}$.

Proof. We apply Theorem 19 together with the notations. If $\mathcal{P} = \mathcal{P}_j$ and $\{K_1\} = \mathcal{P}_i$, then, by Remark 6, $M = \emptyset$ and the graph induced in $H$ by $\{v_i : i \in M \cup \{s\}\}$ is $K_1$. Consequently we obtain that $H_1[G_1, \ldots, G_n]$ is forbidden for $\mathcal{P} \circ \{K_1\} = \mathcal{P}(1)$. If $\mathcal{P} = \mathcal{P}_i$ and $\{K_1\} = \mathcal{P}_j$, then $H$ is one of the graphs $K_2, \overline{K_2}$. By Remark 6 we have $M = \{1, 2\}$ and we obtain that $K_2[H_1, H_2]$ and $H_1 \cup H_2$ are graphs in $\mathcal{C}(\mathcal{P}(1))$. Theorem 19 guarantees no other strongly decomposable graphs in $\mathcal{C}(\mathcal{P}(1))$.

In [5] the author explained that an arbitrary graph can be obtained from a prime graph by the iterative substitution of some of its vertices by prime graphs. This procedure corresponds to the well-known construction (which has been discovered many times and is based on the Gallai Theorem [13]) called a tree decomposition of a graph. For a given graph $G$, all prime graphs applied in this tree-like iterative procedure and all their prime induced subgraphs create the unique family denoted by $Z^*(G)$. In the next investigation we use the following fact from this field.

Lemma 20 [5]. Let $G, G'$ be graphs. If $G' \in \text{PRIME}$, then $G' \leq G$ if and only if $G' \in Z^*(G)$.

Consequently we have the following observation.

Lemma 21. If $\mathcal{P} \in \mathcal{L}_\leq^\ast$ and $G$ is a graph, then $G \in \mathcal{P}$ if and only if $Z^*(G) \subseteq \mathcal{P}$.

Proof. If $G \in \mathcal{P}$, then all induced subgraphs of $G$ are in $\mathcal{P}$, which means $Z^*(G) \subseteq \mathcal{P}$.

Suppose that $Z^*(G) \subseteq \mathcal{P}$ and, for a contradiction, $G \notin \mathcal{P}$. Hence there is an induced subgraph of $G$, say $F$, such that $F \in \mathcal{C}(\mathcal{P})$ (obviously $F \notin \mathcal{P}$). Remark 5 implies that $F$ is prime, which by Lemma 20 leads to $F \in Z^*(G)$, and gives a contradiction.

We use Lemma 21 in proofs of forthcoming results.

Lemma 22. Let $\mathcal{P} \in \mathcal{L}_\leq^\ast$ and $H_1, H_2 \in \mathcal{C}(\mathcal{P})$. If $v_1, \ldots, v_n$ is an arbitrary ordering of the set $V(H_1)$, then $H_1[H_2, K_1, \ldots, K_1]$ is a forbidden subgraph for $\mathcal{P}(1)$.

Proof. Let $G = H_1[H_2, K_1, \ldots, K_1]$ and let $V(G) = \{u_1, \ldots, u_l, v_2, \ldots, v_n\}$, where $v_1$ is substituted with vertices $u_1, \ldots, u_l$ of $H_2$. Hence for each $i \in \{1, \ldots, l\}$ the vertices $u_i, v_2, \ldots, v_n$ induce $H_1$ in $G$. 

\begin{flushright} \text{ \hfill \textcircled{\textbullet}} \end{flushright}
First we observe that $G - v \notin \mathcal{P}$ for any vertex $v \in V(G)$. Indeed, if $v = v_i$ for some $i \in \{2, \ldots, n\}$, then $H_2$ is an induced subgraph of $G - v$. If $v = u_i$ for some $i \in \{1, \ldots, l\}$, then $H_1$ is an induced subgraph of $G - v$.

Now we argue that for each $v \in V(G)$ there is $x \in V(G) \setminus \{v\}$ such that $G - \{v, x\} \in \mathcal{P}$. If $v \in \{v_2, \ldots, v_n\}$, then we choose as $x$ one of the vertices $u_1, \ldots, u_l$. If $v \in \{u_1, \ldots, u_l\}$, then we choose as $x$ one of the vertices $v_2, \ldots, v_n$. In both cases $Z^*(G - \{v, x\})$ contains only proper prime induced subgraphs of $H_1$ and $H_2$, which means $Z^*(G - \{v, x\}) \subseteq \mathcal{P}$ and, by Lemma 21, implies $G - \{v, x\} \in \mathcal{P}$.

**Lemma 23.** Let $\mathcal{P} \in \mathbf{L}^*_c$, $H_1, H_2 \in \mathcal{C}(\mathcal{P})$ and $X \in \text{PRIME}$. If $v_1, \ldots, v_n$ is an ordering of the set $V(X)$ such that $X[\{v_2, \ldots, v_n\}] = H_1$ and $X - v_i \in \mathcal{P}$ for each $i \in \{2, \ldots, n\}$, then $X[H_2, K_1, \ldots, K_1]$ is a forbidden subgraph for $\mathcal{P}(1)$.

**Proof.** Let $G = X[H_2, K_1, \ldots, K_1]$ and let $V(G) = \{u_1, \ldots, u_l, v_2, \ldots, v_n\}$, where $v_1$ is substituted with vertices $u_1, \ldots, u_l$ of $H_2$. Thus $G$ contains two disjoint subgraphs $H_1, H_2$ induced by vertices $v_2, \ldots, v_n$ and $u_1, \ldots, u_l$, respectively. Hence $G \notin \mathcal{P}(1)$.

Now we argue that each pair of vertices $u_i, v_j$, with $i \in \{1, \ldots, l\}$ and $j \in \{2, \ldots, n\}$ satisfies the condition $G - \{u_i, v_j\} \in \mathcal{P}$. Indeed, $Z^*(G - \{u_i, v_j\})$ contains only prime graphs that are induced subgraphs of $H_2 - u_i$ and $X - v_j$. Both these graphs are in $\mathcal{P}$, which implies $Z^*(G - \{u_i, v_j\}) \subseteq \mathcal{P}$. Lemma 21 yields $G - \{u_i, v_j\} \in \mathcal{P}$, as we desired.

Now we are ready to prove that $G - v \in \mathcal{P}(1)$ for each $v \in V(G)$, which means that for each vertex $v \in V(G)$ there is $x \in V(G) \setminus \{v\}$ such that $G - \{x, v\} \in \mathcal{P}$. If $v = u_i$ for some $i \in \{1, \ldots, l\}$, then we put $x = v_j$ for an arbitrary $j \in \{2, \ldots, n\}$, and if $v = v_j$ for some $j \in \{2, \ldots, n\}$, then we put $x = u_i$ for an arbitrary $i \in \{1, \ldots, l\}$. The earlier consideration confirms that $G - \{x, v\} \in \mathcal{P}$ in both cases.

**Theorem 24.** Let $\mathcal{P} \in \mathbf{L}^*_c \setminus \{O, K_3, \{K_1\}\}$. A graph $G$ is a forbidden subgraph for $\mathcal{P}(1)$ if and only if $G$ has one of the following forms:

(i) $G = G_1[H_1, H_2]$, or  
(ii) $G = H_1[G_1, \ldots, G_{\mathcal{C}(H_1)}]$, or  
(iii) $G = H_1[H_2, K_1, \ldots, K_1]$, or  
(iv) $G = X[H_2, K_1, \ldots, K_1]$, or  
(v) $G = Y[G_1, \ldots, G_s, K_1, \ldots, K_1]$,  

where $H_1, H_2 \in \mathcal{C}(\mathcal{P})$ and $G_i \in \{K_2, \overline{K}_2\}$ for all permissible $i$; further $X, Y \in \text{PRIME}$ and, assuming that $V(X) = \{v_1, \ldots, v_n\}$ and $V(Y) = \{u_1, \ldots, u_n\}$, the following conditions are fulfilled:

- $X[\{v_2, \ldots, v_n\}] \in \mathcal{C}(\mathcal{P})$, and
for each \( i \in \{2, \ldots, n_1\} \), \( X - v_i \in \mathcal{P} \), and

\( n_2 \geq s + 2 \), and

for each \( i \in \{1, \ldots, s\} \), \( Y - u_i \in \mathcal{P} \), and

for each \( i \in \{s+1, \ldots, n_2\} \), \( Y - u_i \notin \mathcal{P} \) and there exists \( j \in \{s+1, \ldots, n_2\} \backslash \{i\} \) satisfying \( Y - \{u_i, u_j\} \in \mathcal{P} \).

**Proof.** Lemmas 22, 23 and Corollary 5 show that graphs having forms (i), (ii), (iii) or (iv) are forbidden subgraphs for \( \mathcal{P}(1) \). Recall that a graph \( G \) belongs to \( \mathcal{C}(\mathcal{P}(1)) \) if the graph resulting by the removal of any vertex of \( G \) does not belong to \( \mathcal{P} \) and for each vertex \( v \in V(G) \) there exists another vertex \( x \in V(G) \) such that \( G - \{v, x\} \in \mathcal{P} \). Observe that if a graph has the form (v), then it satisfies these conditions. Namely, if \( v \) is one of the vertices of \( G_i \) with \( i \in \{1, \ldots, s\} \), then we choose another vertex of \( G_i \) as \( x \). If \( v \) is one of the vertices \( u_2 \) with \( i \geq s + 1 \), then the role of \( x \) is played by \( u_j \) given by the assumptions of the theorem. In both cases the conclusion follows by the construction of \( G \).

Corollary 5 characterizes all strongly decomposable graphs in \( \mathcal{C}(\mathcal{P}(1)) \). It means that to finish the proof it is enough to show that if \( G \) is not strongly decomposable and forbidden for \( \mathcal{P}(1) \), then \( G \) has either the form (iii) or (iv) or (v). The mentioned earlier observation that graphs in \( \mathcal{C}(\mathcal{P}(1)) \) are pairwise incomparable with respect to the induced subgraph relation allows us to simplify analysis. Namely, it is enough to show that such \( G \) contains as an induced subgraph a graph of one of the forms (i), (ii), (iii), (iv), (v). As a consequence, we observe that \( G \) has to be of the corresponding form.

Assume that \( G \) is not strongly decomposable. By Theorem 18, Remark 5 and the iterative construction of graphs via prime graphs, we can assume that \( G \) has a form \( W[U_1, \ldots, U_l, K_1, \ldots, K_l] \), where \( W, U_1, \ldots, U_l \in \mathbf{PRIME} \) and \( V(W) = \{w_1, \ldots, w_l, w_{l+1}, \ldots, w_n\} \) with \( n \geq l + 1 \) (we adopt the convention that \( l = 0 \) is equivalent to \( G = W[K_1, \ldots, K_1] = W \)). Moreover, graphs \( U_1, \ldots, U_l \) are forbidden subgraphs for \( \mathcal{P} \) or are elements of the set \( \{K_2, \overline{K_2}\} \).

Suppose that two of the graphs \( U_1, \ldots, U_l \), say \( U_i, U_j \), are forbidden subgraphs for \( \mathcal{P} \). Hence \( K_2[U_i, U_j] \) or \( \overline{K_2}[U_i, U_j] \) is an induced subgraph of \( G \) depending on whether or not \( w_i, w_j \) are adjacent in \( W \). In both cases it leads to the conclusion that \( G \) contains an induced subgraph of the form (i).

In the next part of the proof we assume that at most one among graphs \( U_1, \ldots, U_l \) is in \( \mathcal{C}(\mathcal{P}) \) and, without loss of generality, only \( U_1 \) can be such a graph. Following this assumption \( W \notin \mathcal{P} \). If not, then \( Z^*(G - v) \subseteq \mathcal{P} \), where \( v \) is an arbitrary vertex of \( U_1 \) and next, by Remark 6, \( G - v \in \mathcal{P} \) giving \( G \in \mathcal{P}(1) \), which is impossible. Thus \( W \notin \mathcal{P} \).

Now we consider the case \( U_1 \in \mathcal{C}(\mathcal{P}) \). It means that if \( l \geq 2 \), then \( U_2, \ldots, U_l \in \{K_2, \overline{K_2}\} \). If there is \( W' \leq W \) such that \( W' \in \mathcal{C}(\mathcal{P}) \) with \( w_1 \in V(W') \), then \( G \) contains an induced subgraph of the form (iii). Otherwise, since \( W \notin \mathcal{P} \)
there is \( W' \leq W \) such that \( W' \in \mathcal{C}(\mathcal{P}) \) but \( w_1 \notin V(W') \) and more over, for \( W'' = W'[\{w_1\} \cup V(W')] \) we have \( W'' - x \in \mathcal{P} \) for each \( x \in V(W') \). Observe that \( W''[U_1, K_1, \ldots, K_1] \leq G \) and \( W''[U_1, K_1, \ldots, K_1] \) is of the form (iv), which completes the proof in this case.

Suppose that \( U_1 \notin \mathcal{C}(\mathcal{P}) \). Hence \( G = W[U_1, \ldots, U_l, K_1, \ldots, K_1] \), where \( U_1, \ldots, U_l \in \{K_2, \overline{K}_2\} \). Assume that \( V(G) = \{w_1, w_0, \ldots, w_1, w_0, w_{l+1}, \ldots, w_n\} \), where for \( i \in \{1, \ldots, l\} \) \( w_i \) is substituted with vertices \( w_1, w_0 \) of either \( K_2 \) or \( \overline{K}_2 \). Next we show that \( W - w_i \notin \mathcal{P} \) for \( i \in \{l + 1, \ldots, n\} \). For a contradiction, let \( W - w_i \in \mathcal{P} \) for some \( i \) from the range. Hence, because \( K_2, \overline{K}_2 \in \mathcal{P} \), by Remark 6, we have \( Z^+(G - w_i) \subseteq \mathcal{P} \). It implies, by Lemma 21, that \( G \in \mathcal{P}(1) \) and gives a contradiction. Therefore \( W - w_i \notin \mathcal{P} \) for \( i \in \{l + 1, \ldots, n\} \). By the definition of \( \mathcal{C}(\mathcal{P}(1)) \) we know that there exists a vertex \( v \in V(G) \setminus \{w_j\} \) such that \( G - \{w_j, v\} \in \mathcal{P} \). We ask whether or not \( v \) could be \( w_t \) for some \( t \in \{1, \ldots, l\} \) and \( j \in \{1, 2\} \). Without loss of generality, let \( v = w_t \) for some \( t \) from the range. Thus \( G[\{w_1, \ldots, w_t, w_{t+1}, \ldots, w_{l-1}, w_{l+1}, \ldots, w_n\}] = W - w_i \). We observed previously that \( W - w_i \notin \mathcal{P} \), which means that \( G - \{w_t\} \notin \mathcal{P} \) and excludes this possibility. Thus \( v \) must be \( w_j \) for some \( j \in \{l + 1, \ldots, n\} \setminus \{i\} \) and moreover, it implies \( n \geq l + 2 \). Finally, we show that if \( l \geq 1 \), then \( W - w_i \notin \mathcal{P} \) for each \( i \in \{1, \ldots, l\} \). If not, then \( W - w_i \notin \mathcal{P} \) for some \( i \in \{1, \ldots, l\} \). It implies \( G - \{w_1, w_t\} \notin \mathcal{P} \). By the definition of graphs in \( \mathcal{C}(\mathcal{P}(1)) \) we know that there exists \( v \in V(G) \setminus \{w^2_2\} \) such that \( G - \{v, w^2_2\} \in \mathcal{P} \). Obviously \( v \neq w^1_1 \). Moreover, \( W - w_1 \leq G - \{w_1, w^2_1\} \) for each \( t \in \{l + 1, \ldots, n\} \) and \( W \leq G - \{w^1_1, w^2_1\} \) for each \( t \in \{1, \ldots, l\} \setminus \{i\} \) and \( j \in \{1, 2\} \). Because \( W - w_i \notin \mathcal{P} \) for \( t \in \{l + 1, \ldots, n\} \) and \( W \notin \mathcal{P} \), we obtain a contradiction. Hence we conclude that \( W - w_i \in \mathcal{P} \) for each \( i \in \{1, \ldots, l\} \). Thus, adopting \( l = s \) and \( n = n_2 \), \( G \) satisfies all the conditions that define the form (v) in this case.

In Figures 4, 5(d), 5(e), and 6 we present all possible graphs in \( \mathcal{C}(W_4(1)) \) that have forms pointed out in Theorem 24(i), 24(iii) and Theorem 24(iv). Some examples of graphs in \( \mathcal{C}(W_4(1)) \) having the construction given by Theorem 24(ii) are shown in Figures 5(a), 5(b), 5(c). Figure 7 illustrates Theorem 24(v). It refers to cases \( s = 0, s = 1, s = 2 \), represented by \( Y \) being \( C_5, \overline{P}_5, P_6 \), respectively. It should be mentioned here that the graph in Figure 3 has the form given by Theorem 24(v) with \( s = 0 \).

![Graphs](image.png)

Figure 4. All the graphs in \( \mathcal{C}(W_4(1)) \) of the form given in Theorem 24(i).
Figure 5. Some examples of graphs in $C(W_4(1))$ of the form given in Theorem 24(ii) ((a), (b), (c)) and all the graphs in $C(W_4(1))$ of the form given in Theorem 24(iii) ((d), (e)).

Figure 6. The unique graph in $C(W_4(1))$ of the form given in Theorem 24(iv).

7. Concluding Remarks

In this final section we would like to present relations between the concept of a $P(k)$-apex graph and a concept of an $(H, k)$-stable graph. According to [12, 16], let $H$ be a fixed graph, a graph $G$ is $(H, k)$-stable whenever the deletion of any set of $k$ edges of $G$ results in a graph that still contains a subgraph isomorphic to $H$.

An $(H, k)$-stable graph $G$ is minimal if for every $A \subseteq E(G)$, $|A| = k$, there is $e \in E(G) \setminus A$ such that $(G - A) - e$ does not contain a subgraph isomorphic to $H$. Let us denote by $Stab(H, k)$ the set of all minimal $(H, k)$-stable graphs.

**Proposition 3.** Let $k$ be an integer and $H$ be a graph such that $|V(H)| \geq 4$. Next let $Q$ be the class of all graphs that do not contain $L(H)$ (the line graph of $H$) as an induced subgraph. If $G \in Stab(H, k)$, then $L(G) \in C(Q(k))$.

**Proof.** On the contrary, suppose that $L(G) \notin C(Q(k))$. Consider now two cases.
Case 1. $L(G) \in Q(k)$. It follows that there is a set $B \subseteq V(L(G)), |B| \leq k$ such that $L(G) - B \in Q$. The graph $L(G) - B$ is a line graph of some graph $G'$. Thus $L(G) - B = L(G') \not\cong L(H)$. From Whitney’s Theorem [22] and assumptions it follows that $G' \not\cong H$. The graph $G'$ is obtained by removing at most $k$ edges from the graph $G$ which correspond in a unique way to the vertices of the set $B$. This contradicts our assumption that $G \in Stab(H, k)$.

Case 2. $L(G) \geq F \in C(Q(k))$. If $L(G) = F$, then the conclusion is obvious. Suppose that $L(G) \neq F$. Thus $F$ is a line graph of some graph $G'$ which is a proper spanning subgraph of $G$. Let $e \in E(G) \setminus E(G')$. From the assumption $G \in Stab(H, k)$ it follows that for the edge $e$ there is a set $B' \subseteq E(G) \setminus \{e\}$, $|B'| = k$ such that $(G - e) - B'$ has no subgraph $H$. Obviously, $|B' \cap E(G')| \leq k$. Since $G' \subseteq G - e$, then $G' - B'$ has no subgraph $H$. This fact implies that there is
a set $A' \subseteq V(F)$, $|A'| = k$ such that $F - A' \in \mathcal{Q}$. This contradicts our assumption that $F \in \mathcal{C}(Q(k))$ and the proof is complete.

In [16] the minimum size of $(P_4, k)$-stable graphs was determined. In Section 5 of this paper we deal with the minimum and maximum order of graphs in $\mathcal{C}(W_r(k))$. Since $L(P_{r+1}) = P_r$ we have the following observation.

**Corollary 6.** Let $k, r$ be integers, $r \geq 3$. If $G \in \text{Stab}(P_{r+1}, k)$, then $L(G) \in \mathcal{C}(W_r(k))$.

Let us define a vertex version of the $H$-stability. Let $H$ be a graph and $k$ be a positive integer. A graph $G$ of order at least $k$ is said to be $(H, k)$-vertex stable if for any set $S$ of $k$ vertices the subgraph $G - S$ contains an induced subgraph isomorphic to $H$. An $(H, k)$-vertex stable graph $G$ is minimal if for every $W \subseteq V(G)$, $|W| = k$, there is $v \in V(G) \setminus W$ such that $(G - W) - v$ does not contain $H$. Let us denote by $\text{Stab}_V(H, k)$ the set of all minimal $(H, k)$-vertex stable graphs. Observe the following fact.

**Proposition 4.** If $k$ is an integer and $H$ is a connected graph, then $\text{Stab}_V(H, k) = \mathcal{C}(\mathcal{P}(k))$, where $\mathcal{P}$ is the class of all graphs that do not contain $H$ as an induced subgraph.

**Proof.** If $G \in \mathcal{C}(\mathcal{P}(k))$, then $G - v \in \mathcal{P}(k)$ and $G - v \notin \mathcal{P}(k-1)$ for every $v \in V(G)$. In the case when $G - v \in \mathcal{P}(k-1)$ for an vertex $v$, then there is a set $A \subseteq V(G)$, $|A| = k - 1$ such that $(G - v) - A \in \mathcal{P}$. This contradicts our assumption that $G \in \mathcal{C}(\mathcal{P}(k))$. It implies that for every set $A \subseteq V(G)$, $|A| = k$ we have $G - A \geq H$, i.e., $G \in \text{Stab}_V(H, k)$. Thus, $\mathcal{C}(\mathcal{P}(k)) \subseteq \text{Stab}_V(H, k)$.

Now let $G \in \text{Stab}_V(H, k)$. Then for every $A \subseteq V(G)$, $|A| = k$, there is $v \in V(G) \setminus A$ such that $(G - A) - v$ does not contain $H$ as an induced subgraph. It follows that for every $v \in V(G)$ there is a set $A \subseteq V(G)$, $|A| = k$ such that $(G - v) - A \in \mathcal{P}$, i.e., $G \in \mathcal{C}(\mathcal{P}(k))$. Hence $\text{Stab}_V(H, k) \subseteq \mathcal{C}(\mathcal{P}(k))$.

Yet another version of an $(H, k)$-stable graph was studied in a series of papers [3,6–8,10,11] where the $(H, k)$-vertex stability was considered taking into account, instead of induced subgraphs, subgraphs of $G$ isomorphic to $H$. In case of $H = K_q$, both concepts coincide.

**References**


Received 17 January 2018
Accepted 5 February 2018