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Note

ALMOST SELF-COMPLEMENTARY UNIFORM HYPERGRAPHS [1]

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Abstract

A k-uniform hypergraph (k-hypergraph) is *almost self-complementary* if it is isomorphic with its complement in the complete k-uniform hypergraph minus one edge. We prove that an almost self-complementary k-hypergraph of order n exists if and only if $\binom{n}{k}$ is odd.

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1. Introduction

Let k be a positive integer and let V be a set of order $n > k$. By $\binom{V}{k}$ we denote the set of all k-subsets of V. A *k-hypergraph* with *vertex set* V and *edge set* $E \subset \binom{V}{k}$ is a pair $H = (V; E)$. The k-hypergraph $K_n^{(k)} = \left(V; \binom{V}{k}\right)$ is called *complete k-hypergraph*. The *complement of H in $K_n^{(k)}$* is then the k-hypergraph $\overline{H} = (V; \overline{E})$, where $\overline{E} = \binom{V}{k} - E$.

For a permutation σ of the set V we define the mapping σ^* on the set of edges by the formula $\sigma^*(e) = \{\sigma(x) : x \in e\}$, for every $e \in \binom{V}{k}$. If σ^* restricted to E induces a bijection onto $E' \subset \binom{V}{k}$ then we say that σ is an *isomorphism* of the k-hypergraph $H = (V; E)$ on $H' = (V; E')$. H and H' are then said to be *isomorphic*.

A k-hypergraph H is said to be *self-complementary* if it is isomorphic with $\overline{H}$. Clearly, if a k-hypergraph of order n is self-complementary then $\binom{n}{k}$ is even. It turns out that also a converse, in a sense, is true.

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Theorem 1.1 [8]**.** *Let n and k be positive integers, $k \leq n$. There is a k-uniform self-complementary hypergraph of order n if and only if $\binom{n}{k}$ is even.*

A k-uniform hypergraph $H = (V; E)$ is called *almost self-complementary* if it is isomorphic with $H' = \Big(V; \binom{V}{k} - E - \{e_0\}\Big)$ where e_0 is an element of the set $\binom{V}{k} - E$. Almost self-complementary k-hypergraphs of order n may be also called *self-complementary in* $K_n^{(k)} - e_0$. For $k = 2$ the almost self-complementary 2-hypergraphs are almost self-complementary graphs in $K_n - e_0$ defined by Clapham in [1]. The almost self-complementary 3-hypergraphs are considered in [3].

It is clear that an almost self-complementary k-hypergraph of order n may exist only when $\binom{n}{k}$ is odd. In Section 3 of the paper we prove the theorem generalizing the corresponding results of [1] for $k = 2$ and of [3] for $k = 3$.

Theorem 1.2. *Let n and k be positive integers, $k < n$. An almost self-complementary k-hypergraph of order n exists if and only if $\binom{n}{k}$ is odd.*

2. Auxiliary Results

In this section we give some results which are useful while proving Theorem 1.2.

The following is in fact a corollary of a celebrated theorem of Kummer [5] (see also [7] and [4][2]).

Theorem 2.1 [5, 4]**.** *Let n and k be positive integers, $k < n$, $n = \sum_{i=0} a_i 2^i, k = \sum_{i=0} b_i 2^i$, where $a_i, b_i \in \{0, 1\}$ for all i. $\binom{n}{k}$ is odd if and only if $b_i \leq a_i$ for every i.*

For an integer $l = 2^p(2q + 1)$ we write $C_2(l) = p$ ($C_2(l)$ is the maximum integer p such that 2^p divides l). We shall use the following lemma which an immediate consequence of Lemmas 4 and 5 proved in [9].

Lemma 2.2. *Let k and m be positive integers, $k < m$, and let $\sigma = (1, 2, \ldots, m)$ be a cyclic permutation. If $C_2(k) < C_2(m)$, then for every k-subset $e \subset \{1, 2, \ldots, m\}$ and for every $q \in \mathbb{Z}$ we have $(\sigma^*)^{2q+1}(e) \neq e$.*

3. Proof of Theorem 1.2

Let us suppose that $\binom{n}{k}$ is odd. We shall construct an almost self-complementary k-hypergraph of order n.

[2]Note that, very probably, neither the authors of [4], nor Glaisher [2] and Lucas [6] were aware of the result of Kummer while writing about the divisibility of Newton coefficients.

Write k and n in base-2 numeral system:

$$k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s},$$

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t},$$

where $k_1 < k_2 < \cdots < k_s$ and $n_1 < n_2 < \cdots < n_t$.

By Theorem 2.1, for every $i \in \{1, 2, \ldots, s\}$ there exists $j_i \in \{1, 2, \ldots, t\}$ such that $n_{j_i} = k_i$ (the sequence $(k_i)_{i=1}^{s}$ is a subsequence of $(n_j)_{j=1}^{t}$).

Let V be a set of cardinality n and let $V = V_1 \cup V_2 \cup \cdots \cup V_t$ be such a partition of V that $|V_j| = 2^{n_j}$, $V_j = \left\{a_1^j, a_2^j, \ldots, a_{2^{n_j}}^j\right\}$ say, for $j = 1, 2, \ldots, t$.

Set $e_0 = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_s}$. We see at once that $|e_0| = k$. Write

$$\sigma_j = \left(a_1^j, a_2^j, \ldots, a_{2^{n_j}}^j\right)$$

for every $j = 1, 2, \ldots, t$ and

$$\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_t \,.$$

Let e be any k-subset of V. If $e = e_0$ then $\sigma^*(e) = e$. If $e \neq e_0$ then there is an index $l_0 \in \{1, 2, \ldots, t\}$ such that $e \cap V_{l_0} \neq \emptyset$ and $e \neq V_{l_0}$. It is easily seen that

$$C_2(|e \cap V_{l_0}|) < C_2(|V_{l_0}|).$$

By Lemma 2.2, for every integer q

$$(\sigma^*)^{2q+1}(|e \cap V_{l_0}|) \neq e \cap V_{l_0}$$

and by consequence

$$(\sigma^*)^{2q+1}(e) \neq e.$$

We color now all the k-subsets of V, except e_0, with colors blue and red by applying the following algorithm.

If e is a k-subset of V which is not yet colored and $e \neq e_0$, then color all the edges of the form $(\sigma^*)^{2m+1}(e)$ red, and all the edges of the form $(\sigma^*)^{2m}(e)$ blue.

It is clear that we may color in this way all the k-subsets of V except e_0 and that the hypergraph H_b with the vertex set V and the set E_b of edges consisting of all the blue k-subsets of V is ismorphic with the k-hypergraph H_r with vertex set V and edge set E_r consisting of all the red k-subsets V. Moreover, σ is an isomorphism of $H_b = (V; E_b)$ with $H_r = (V; E_r)$, $E_b \cap E_r = \emptyset$, $e_0 \notin E_b \cup E_r$ and $E_b \cup E_r \cup \{e_0\} = \binom{V}{k}$. This finishes the proof.

It is easy to adopt the proof of Theorem 1.2 given above to obtain a proof of Theorem 1.1. This proof is much simpler than the one presented in [8].

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