ALMOST SELF-COMPLEMENTARY UNIFORM HYPERGRAPHS

ADAM PAVEL WOJDA

AGH University of Science and Technology
Faculty of Applied Mathematics
Al. Mickiewicza 30, 30-059 Kraków, Poland
E-mail: wojda@uci.agh.edu.pl

Abstract

A k-uniform hypergraph (k-hypergraph) is almost self-complementary if it is isomorphic with its complement in the complete k-uniform hypergraph minus one edge. We prove that an almost self-complementary k-hypergraph of order n exists if and only if \( \binom{n}{k} \) is odd.

Keywords: uniform hypergraph.

2010 Mathematics Subject Classification: 05C.

1. Introduction

Let k be a positive integer and let V be a set of order \( n > k \). By \( \binom{V}{k} \) we denote the set of all k-subsets of V. A k-hypergraph with vertex set V and edge set \( E \subseteq \binom{V}{k} \) is a pair \( H = (V; E) \). The k-hypergraph \( K_n^{(k)} = (V; \binom{V}{k}) \) is called complete k-hypergraph. The complement of \( H \) in \( K_n^{(k)} \) is then the k-hypergraph \( \overline{H} = (V; \overline{E}) \), where \( \overline{E} = \left( \binom{V}{k} \right) - E \).

For a permutation \( \sigma \) of the set V we define the mapping \( \sigma^\ast \) on the set of edges by the formula \( \sigma^\ast(e) = \{\sigma(x) : x \in e\} \), for every \( e \in \binom{V}{k} \). If \( \sigma^\ast \) restricted to \( E \) induces a bijection onto \( E' \subseteq \binom{V}{k} \) then we say that \( \sigma \) is an isomorphism of the k-hypergraph \( H = (V; E) \) on \( H' = (V; E') \). \( H \) and \( H' \) are then said to be isomorphic.

A k-hypergraph \( H \) is said to be self-complementary if it is isomorphic with \( \overline{H} \). Clearly, if a k-hypergraph of order \( n \) is self-complementary then \( \binom{n}{k} \) is even. It turns out that also a converse, in a sense, is true.

\[\text{\footnotesize{\textsuperscript{1}The research was partially sponsored by the Polish Ministry of Science and Higher Education.}}\]
**Theorem 1.1** [8]. Let $n$ and $k$ be positive integers, $k \leq n$. There is a $k$-uniform self-complementary hypergraph of order $n$ if and only if $\binom{n}{k}$ is even.

A $k$-uniform hypergraph $H = (V; E)$ is called almost self-complementary if it is isomorphic with $H' = \left( V; \left( \binom{V}{k} \right) - E - \{e_0\} \right)$ where $e_0$ is an element of the set $\left( \binom{V}{k} \right) - E$. Almost self-complementary $k$-hypergraphs of order $n$ may be also called self-complementary in $K_n^{(k)} - e_0$. For $k = 2$ the almost self-complementary 2-hypergraphs are almost self-complementary graphs in $K_n - e_0$ defined by Clapham in [1]. The almost self-complementary 3-hypergraphs are considered in [3]. It is clear that an almost self-complementary $k$-hypergraph of order $n$ may exist only when $\binom{n}{k}$ is odd. In Section 3 of the paper we prove the theorem generalizing the corresponding results of [1] for $k = 2$ and of [3] for $k = 3$.

**Theorem 1.2.** Let $n$ and $k$ be positive integers, $k < n$. An almost self-complementary $k$-hypergraph of order $n$ exists if and only if $\binom{n}{k}$ is odd.

2. **Auxiliary Results**

In this section we give some results which are useful while proving Theorem 1.2.

The following is in fact a corollary of a celebrated theorem of Kummer [5] (see also [7] and [4]).

**Theorem 2.1** [5, 4]. Let $n$ and $k$ be positive integers, $k < n$, $n = \sum_{i=0}^{\infty} a_i 2^i$, $k = \sum_{i=0}^{\infty} b_i 2^i$, where $a_i, b_i \in \{0, 1\}$ for all $i$. $\binom{n}{k}$ is odd if and only if $b_i \leq a_i$ for every $i$.

For an integer $l = 2^p (2q + 1)$ we write $C_2(l) = p$ ($C_2(l)$ is the maximum integer $p$ such that $2^p$ divides $l$). We shall use the following lemma which an immediate consequence of Lemmas 4 and 5 proved in [9].

**Lemma 2.2.** Let $k$ and $m$ be positive integers, $k < m$, and let $\sigma = (1, 2, \ldots, m)$ be a cyclic permutation. If $C_2(k) < C_2(m)$, then for every $k$-subset $e \subset \{1, 2, \ldots, m\}$ and for every $q \in \mathbb{Z}$ we have $(\sigma^q)^{2q+1}(e) \neq e$.

3. **Proof of Theorem 1.2**

Let us suppose that $\binom{n}{k}$ is odd. We shall construct an almost self-complementary $k$-hypergraph of order $n$.

---

2Note that, very probably, neither the authors of [4], nor Glaisher [2] and Lucas [6] were aware of the result of Kummer while writing about the divisibility of Newton coefficients.
Write $k$ and $n$ in base-2 numeral system:

$$k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s},$$

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t},$$

where $k_1 < k_2 < \cdots < k_s$ and $n_1 < n_2 < \cdots < n_t$.

By Theorem 2.1, for every $i \in \{1, 2, \ldots, s\}$ there exists $j_i \in \{1, 2, \ldots, t\}$ such that $n_{j_i} = k_i$ (the sequence $(k_i)_{i=1}^s$ is a subsequence of $(n_j)_{j=1}^t$).

Let $V$ be a set of cardinality $n$ and let $V = V_1 \cup V_2 \cup \cdots \cup V_t$ be such a partition of $V$ that $|V_j| = 2^{n_j}$, $V_j = \{a_{j,1}^1, a_{j,2}^2, \ldots, a_{j,m_j}^j\}$ say, for $j = 1, 2, \ldots, t$.

Set $e_0 = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_s}$. We see at once that $|e_0| = k$. Write

$$\sigma_j = (a_{j,1}^1, a_{j,2}^2, \ldots, a_{j,m_j}^j)$$

for every $j = 1, 2, \ldots, t$ and

$$\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_t.$$

Let $e$ be any $k$-subset of $V$. If $e = e_0$ then $\sigma^*(e) = e$. If $e \neq e_0$ then there is an index $l_0 \in \{1, 2, \ldots, t\}$ such that $e \cap V_{l_0} \neq \emptyset$ and $e \neq V_{l_0}$. It is easily seen that

$$C_2(|e \cap V_{l_0}|) < C_2(|V_{l_0}|).$$

By Lemma 2.2, for every integer $q$

$$(\sigma^*)^{2q+1}(|e \cap V_{l_0}|) \neq e \cap V_{l_0}$$

and by consequence

$$(\sigma^*)^{2q+1}(e) \neq e.$$

We color now all the $k$-subsets of $V$, except $e_0$, with colors blue and red by applying the following algorithm.

If $e$ is a $k$-subset of $V$ which is not yet colored and $e \neq e_0$, then color all the edges of the form $(\sigma^*)^{2m+1}(e)$ red, and all the edges of the form $(\sigma^*)^{2m}(e)$ blue.

It is clear that we may color in this way all the $k$-subsets of $V$ except $e_0$ and that the hypergraph $H_b$ with the vertex set $V$ and the set $E_b$ of edges consisting of all the blue $k$-subsets of $V$ is isomorphic with the $k$-hypergraph $H_r$ with vertex set $V$ and edge set $E_r$ consisting of all the red $k$-subsets $V$. Moreover, $\sigma$ is an isomorphism of $H_b = (V; E_b)$ with $H_r = (V; E_r)$, $E_b \cap E_r = \emptyset$, $e_0 \notin E_b \cup E_r$ and $E_b \cup E_r \cup \{e_0\} = \binom{V}{k}$. This finishes the proof.

It is easy to adopt the proof of Theorem 1.2 given above to obtain a proof of Theorem 1.1. This proof is much simpler than the one presented in [8].
References


Received 28 July 2017
Accepted 16 January 2017