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HOMOMORPHIC PREIMAGES OF GEOMETRIC PATHS

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Abstract

A graph G is a homomorphic preimage of another graph H, or equivalently G is H-colorable, if there exists a graph homomorphism $f : G \to H$. A geometric graph $\overline{G}$ is a simple graph G together with a straight line drawing of G in the plane with the vertices in general position. A geometric homomorphism (respectively, isomorphism) $\overline{G} \to \overline{H}$ is a graph homomorphism (respectively, isomorphism) that preserves edge crossings (respectively, and non-crossings). The homomorphism poset $\mathcal{G}$ of a graph G is the set of isomorphism classes of geometric realizations of G partially ordered by the existence of injective geometric homomorphisms. A geometric graph $\overline{G}$ is $\mathcal{H}$-colorable if $\overline{G} \to \overline{H}$ for some $\overline{H} \in \mathcal{H}$. In this paper, we provide necessary and sufficient conditions for $\overline{G}$ to be $\mathcal{P}_n$-colorable for $n \geq 2$. Along the way, we also provide necessary and sufficient conditions for $\overline{G}$ to be $\mathcal{K}_{2,3}$-colorable.

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1. Basic Definitions

A *graph homomorphism* $f : G \to H$ is a vertex function such that for all $u, v \in V(G)$, $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If such a function exists, we write $G \to H$ and say that G is *homomorphic* to H, or equivalently, that G is a *homomorphic preimage* of H. A proper n-coloring of a graph G is a homomorphism $G \to K_n$. Thus, G is n-colorable if and only if G is a homomorphic preimage of K_n. In 1981, Maurer, Salomaa and Wood [13] generalized this notion by defining G to be H-colorable if and only if $G \to H$, or equivalently, G is a homomorphic preimage of H. For a given graph H, the H-coloring problem is the decision problem, "Is a given graph H-colorable?" In 1990, Hell and Nešetřil showed that if $\chi(H) \leq 2$, then this problem can be solved in polynomial time and if $\chi(H) \geq 3$, then it is NP-complete [8].

The concept of H-colorability can be extended to directed graphs. Work has been done by Hell, Zhu and Zhou in characterizing homomorphic preimages of certain families of directed graphs, including oriented cycles [9, 12, 14], oriented paths [11] and local acyclic tournaments [10].

In [1], Boutin and Cockburn generalized the notion of graph homomorphisms to geometric graphs. A *geometric graph* $\overline{G}$ is a simple graph G together with a straight-line drawing of G in the plane with vertices in general position (no three vertices are collinear and no three edges intersect except at a common endpoint). Two edges are said to *cross* if the interiors of the line segments representing them have nonempty intersection; in particular, an edge cannot cross itself. A geometric graph $\overline{G}$ with underlying abstract graph G is called a *geometric realization* of G. The definition below formalizes when two geometric realizations of G are considered the same.

Definition [1]**.** A *geometric isomorphism* $f : \overline{G} \to \overline{H}$ is a graph isomorphism such that for all $u, v, x, y \in V(\overline{G})$ with $xy, uv \in E(\overline{G})$, xy crosses uv in $\overline{G}$ if and only if $f(x)f(y)$ crosses $f(u)f(v)$ in $\overline{H}$.

(Note that this definition is weaker than one introduced by Harborth in [6], requiring that geometric isomorphisms also preserve regions and parts of edges.) Relaxing the biconditional to an implication gives the following.

Definition [1]**.** A *geometric homomorphism* $f : \overline{G} \to \overline{H}$ is a graph homomorphism such that for all $u, v, x, y \in V(\overline{G})$ with $xy, uv \in E(\overline{G})$, if xy crosses uv in $\overline{G}$, then $f(x)f(y)$ crosses $f(u)f(v)$ in $\overline{H}$. If such a function exists, we write $\overline{G} \to \overline{H}$ and say that $\overline{G}$ is homomorphic to $\overline{H}$, or that $\overline{G}$ is a homomorphic preimage of $\overline{H}$. If $\overline{G} \to \overline{H}$ and $\overline{H} \to \overline{G}$, then we say $\overline{G}$ and $\overline{H}$ are homomorphically equivalent.

An obvious consequence of this definition is that no two vertices that are adjacent or *co-crossing* (incident to distinct edges that cross each other) in a geometric graph can have the same image under a geometric homomorphism.

Geometric homomorphisms can be used to define n-colorability for geometric graphs. One complication, however, is that there are multiple geometric realizations of the n-clique for $n > 3$. This can be resolved by taking advantage of some additional structure.

Definition [1]**.** Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of G. Then set $\overline{G} \preceq \widehat{G}$ if there exists a geometric homomorphism $f : \overline{G} \to \widehat{G}$ whose underlying map $f : G \to G$ is a graph isomorphism. The set of isomorphism classes of geometric realizations of G under this partial order, denoted $\mathcal{G}$, is the *homomorphism poset* of G.

Boutin and Cockburn define $\overline{G}$ to be *n-geocolorable* if $\overline{G} \to \overline{K}_n$ for some $\overline{K}_n \in \mathcal{K}_n$. In [2], it is shown that $\mathcal{K}_3, \mathcal{K}_4$ and $\mathcal{K}_5$ are all chains. Thus for $3 \leq n \leq 5$, $\overline{G}$ is n-geocolorable if $\overline{G}$ is homomorphic to the greatest element of the chain. By contrast, $\mathcal{K}_6$ has three maximal elements, so $\overline{G}$ is 6-geocolorable if it is homomorphic to any one of these three realizations. As with graphs and digraphs, the definition of colorability can be broadened.

Definition. Let $\mathcal{H}$ denote the homomorphism poset of geometric realizations of a simple graph H. Then $\overline{G}$ is $\mathcal{H}$-colorable if and only if $\overline{G} \to \overline{H}$ for some (maximal) $\overline{H} \in \mathcal{H}$.

In this paper, we provide necessary and sufficient conditions for $\overline{G}$ to be $\mathcal{P}_n$-colorable, for all $n \geq 2$. Results on the structure of the homomorphism poset $\mathcal{P}_n$ can be found in [2]. We adopt the following notation from that paper: the vertices of P_n are denoted $1, 2, \ldots, n$ and its edges by $e_i = \{i, i+1\}$ for $1 \leq i \leq n-1$. It is shown in [2] that for all $n \geq 3$, $\mathcal{P}_n$ has a greatest element $\widehat{P}_n$, with $(n-2)(n-3)/2$ edge crossings, where all pairs of edges cross except for the consecutively labeled ones. Figure 1 shows this greatest element for $n = 7$. Thus $\overline{G}$ is $\mathcal{P}_n$-colorable if and only if $\overline{G} \to \widehat{P}_n$.

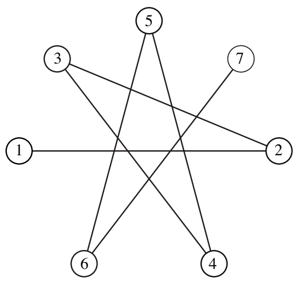

Figure 1. $\widehat{P}_7$, the greatest element of $\mathcal{P}_7$.

The organization of this paper is as follows. In Section 2, we give some tools for investigating the edge crossing structure of a geometric graph. In Section 3, we characterize $\mathcal{P}_n$-colorability for $2 \leq n \leq 4$. In particular, we relate $\mathcal{P}_4$-colorability to a characterization of $\mathcal{C}_4$-colorability (equivalently, $\mathcal{K}_{2,2}$-colorability) from [1]. In Section 4, we characterize $\mathcal{P}_5$-colorability and investigate its relationship to $\mathcal{C}_5$-colorability (characterized in [4]) and $\mathcal{K}_{2,3}$-colorability. A general theorem characterizing $\mathcal{P}_n$-colorability for $n \geq 5$ is given in Section 5. We end with some open questions in Section 6.

2. Edge Crossing Structure

Given a geometric graph $\overline{G}$, there are a number of ways of representing its edge crossing structure.

Definition [2]**.** The *edge-crossing graph* of $\overline{G}$, denoted $EX(\overline{G})$, is the abstract graph whose vertices correspond to the edges of $\overline{G}$, with adjacency when the corresponding edges of $\overline{G}$ cross.

Definition [1]**.** A *thickness edge m-coloring* ϵ of $\overline{G}$ is a coloring of the edges of $\overline{G}$ with m colors such that no two edges of the same color cross. The *thickness* of $\overline{G}$ is the minimum number of colors required for a thickness edge coloring of $\overline{G}$.

From these two definitions, a thickness edge m-coloring ϵ of $\overline{G}$ is a graph homomorphism $\epsilon : EX(\overline{G}) \to K_m$, and the thickness of $\overline{G}$ is the chromatic number of the edge-crossing graph, $\chi(EX(\overline{G}))$. (Note that the thickness of an abstract graph G is the minimum thickness of any geometric realization of G.)

Definition. If ϵ is a thickness edge coloring of $\overline{G}$, then the plane subgraph of $\overline{G}$ induced by all edges of a given color is called a *monochromatic* subgraph of $\overline{G}$ under ϵ. The monochromatic subgraph corresponding to edge color i is called the i-subgraph of $\overline{G}$ under ϵ, and is denoted by $\overline{G}^i$.

We assume from now on that $\overline{G}$ has no isolated vertices, which implies that every vertex belongs to at least one monochromatic subgraph under any thickness edge coloring, i.e., $V(\overline{G}) = \bigcup_{i=1}^m V(\overline{G}^i)$.

Example 1. Figure 2 shows the edge-crossing graph of $\widehat{P}_7$. Observe that for all $n \geq 3$, $EX(\widehat{P}_n)$ is the complement of P_{n-1}, which has chromatic number $\lceil (n-1)/2 \rceil = k$. To see this, note that the complete subgraph of $EX(\widehat{P}_n)$ induced by $e_1, e_3, \ldots, e_{2k-1}$ requires k different colors. We can complete a proper coloring of $EX(\widehat{P}_n)$ by giving e_{2i} the same color as e_{2i-1} for all $1 \leq i \leq k$. Throughout the rest of the paper, we will refer to this as the *canonical* thickness edge coloring of $\widehat{P}_n$. For $1 \leq i \leq k$, the i-subgraph is given by

$$E(\widehat{P}_n^i) = \{e_{2i-1}, e_{2i}\} \text{ and } V(\widehat{P}_n^i) = \{2i-1, 2i, 2i+1\}$$

(with the understanding that if n is even, then $V(\widehat{P}_n^k) = \{2k-1, 2k\}$).

Definition [1]**.** The *crossing subgraph* of $\overline{G}$ is the geometric subgraph $\overline{G}_\times$ spanned by the crossing edges of $\overline{G}$. The vertices in this subgraph are called *crossing* vertices of $\overline{G}$; the remaining vertices of $\overline{G}$ are called *non-crossing.*

The following result is obvious.

Lemma 2 [2]**.** *A geometric homomorphism $\overline{G} \to \overline{H}$ induces*

1. *a geometric homomorphism $\overline{G}_\times \to \overline{H}_\times$, and*
2. *graph homomorphisms $G \to H$ and $EX(\overline{G}) \to EX(\overline{H})$.*

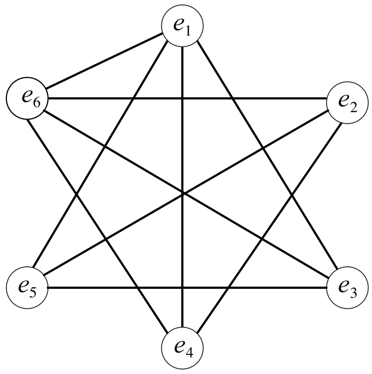

Figure 2. $EX(\widehat{P}_7)$.

Less obvious is the fact that neither of these results can be reversed (see [1]).

If $\overline{H}$ is a plane graph, then $EX(\overline{H})$ is an empty graph and thus homomorphically equivalent to K_1. In this case, we do have a biconditional: $\overline{G} \to \overline{H}$ if and only if $G \to H$ and $\overline{G}$ is also a plane graph.

Applying Lemma 2 to geometric paths yields the following.

Corollary 3. *If $\overline{G}$ is $\mathcal{P}_n$-colorable, $n \geq 2$, then G is bipartite and $\overline{G}$ has a thickness edge k-coloring, where $k = \lceil (n-1)/2 \rceil$.*

If $\overline{G}$ is $\mathcal{P}_n$-colorable, then the fact that G is bipartite means that $V(\overline{G})$ can be partitioned into two independent sets. Throughout this paper, a bipartition of $V(\overline{G})$ into $X \cup Y$ will always refer to a bipartition based on vertex adjacency. The fact that $\overline{G}$ has a thickness edge k-coloring means $\overline{G}$ can be broken down into k monochromatic plane subgraphs, $\overline{G}^1, \overline{G}^2, \ldots, \overline{G}^k$ that have no edges in common, but that may have vertices in common. Thus the vertex adjacency structure and edge crossing structure determine two ways of breaking down the vertex set,

$$V(\overline{G}) = X \cup Y = \bigcup_{i=1}^{k} V(\overline{G}^i).$$

Neither of these decompositions need to be unique. A connected bipartite graph has a unique bipartition of its vertex set, but a bipartite graph with p components will have 2^{p-1} bipartitions. Similarly, different thickness edge k-colorings will generate different subsets $V(\overline{G}^1), \ldots, V(\overline{G}^k)$. The characterization theorems in this paper hinge on finding a bipartition and thickness edge coloring with a particular interrelationship between these two organizations of $V(\overline{G})$.

3. EASY CASES: $2 \leq n \leq 4$

Both posets $\mathcal{P}_2$ and $\mathcal{P}_3$ consist only of a plane realization, and moreover P_3 is homomorphically equivalent to $P_2 = K_2$. Hence,

$$\overline{G} \text{ is } \mathcal{P}_3\text{-colorable} \iff \overline{G} \text{ is } \mathcal{P}_2\text{-colorable} \iff \overline{G} \text{ is bipartite and plane.}$$

To characterize $\mathcal{P}_4$-colorability, we take advantage of the close relationship between the posets $\mathcal{P}_4$ and $\mathcal{C}_4$, depicted in Figure 3. A thickness edge 2-coloring of the crossing subgraph of the greatest element, denoted $\widehat{P}_4$ and $\widehat{C}_4$, respectively, is shown, with non-crossing edges depicted in grey. Necessary and sufficient conditions for $\mathcal{C}_4$-colorability were originally given by Boutin and Cockburn in [1]; the rephrasing below by Cockburn appears in [4].

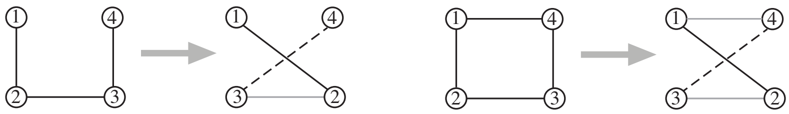

Figure 3. The homomorphism posets $\mathcal{P}_4$ and $\mathcal{C}_4$.

Theorem 4. *A geometric graph $\overline{G}$ is $\mathcal{C}_4$-colorable if and only if G is bipartite and there exists a thickness edge 2-coloring of $\overline{G}_\times$ such that the monochromatic subgraphs ${\overline{G}_\times}^1$ and ${\overline{G}_\times}^2$ are vertex-disjoint, i.e., $V({\overline{G}_\times}^1) \cap V({\overline{G}_\times}^2) = \emptyset$.*

The necessary and sufficient condition in this theorem requires a partition of the *crossing* vertices into *disjoint* subsets; a third part consists of the non-crossing vertices. This part and the bipartition corresponding to the adjacency structure are represented in the Venn diagram in Figure 4, where the rightmost ellipse contains the non-crossing vertices of $\overline{G}$. Only the shaded areas of the diagram contain vertices. (Note that this is a set-theoretic representation and does not reflect the geometric position of the vertices in $\overline{G}$.)

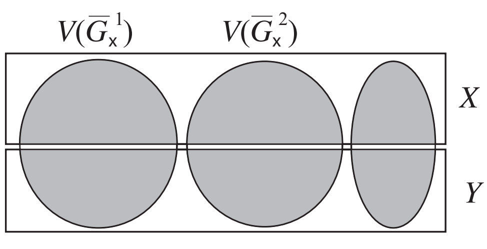

Figure 4. Two partitions of $V(\overline{G})$ for $\mathcal{C}_4$-colorability.

Determining whether a geometric graph $\overline{H}$ can be decomposed into two plane subgraphs that are vertex-disjoint can be accomplished as follows. The line graph $L(\overline{H})$ has the same vertex set as the edge-crossing graph $EX(\overline{H})$, namely the edges of $\overline{H}$, with adjacency when two edges share a common endvertex. The decomposition of $L(\overline{H})$ into connected components yields a partition θ on the common set of vertices,

$$E(\overline{H}) = V(EX(\overline{H})) = T_1 \cup \cdots \cup T_m.$$

The quotient of $EX(\overline{H})$ by θ is the abstract graph $EX(\overline{H})/\theta$ on the vertex set $\{1, \ldots, m\}$, with a loop ii if and only if there are two vertices in T_i that are adjacent in $EX(\overline{H})$, and an edge ij if and only if a vertex in T_i is adjacent to a vertex in T_j in $EX(\overline{H})$. (Quotient graphs are discussed in [7].)

Lemma 5. *A geometric graph $\overline{H}$ has a thickness edge 2-coloring such that $\overline{H}^1$ and $\overline{H}^2$ are vertex-disjoint if and only if $EX(\overline{H})/\theta$ is bipartite.*

Proof. Assume $\overline{H}$ has a thickness edge 2-coloring such that $\overline{H}^1$ and $\overline{H}^2$ are vertex-disjoint. Hence no $e_1 \in E(\overline{H}^1)$ is adjacent to any $e_2 \in E(\overline{H}^2)$ in $L(\overline{H})$. This implies that each T_i lies either entirely in $E(\overline{H}^1)$ or entirely in $E(\overline{H}^2)$. In particular, each T_i is an independent set in $EX(\overline{H})$ and so the quotient graph $EX(\overline{H})/\theta$ has no loops. Assigning color j to vertex i if and only if $T_i \subseteq E(\overline{H}^j)$ gives a proper 2-coloring of $EX(\overline{H})/\theta$.

Conversely, assume $EX(\overline{H})/\theta$ is bipartite. Then there exists a homomorphism $f : EX(\overline{H})/\theta \to K_2$. Since all quotients are homomorphic images, there is another homomorphism $g : EX(\overline{H}) \to EX(\overline{H})/\theta$. Define a thickness edge 2-coloring of $\overline{H}$ by

$$E\left(\overline{H}^1\right) = g^{-1}\left(f^{-1}(1)\right) \text{ and } E\left(\overline{H}^2\right) = g^{-1}\left(f^{-1}(2)\right).$$

If $\overline{H}^1$ and $\overline{H}^2$ are not vertex-disjoint, then there exist $e_1 \in E(\overline{H}^1), e_2 \in E(\overline{H}^2)$ with a common endvertex in $\overline{H}$. By definition, e_1 and e_2 belong to the same component of $L(\overline{H})$, so $g(e_1) = g(e_2)$. This contradicts the assumption that $f(g(e_1)) = 1$ and $f(g(e_2)) = 2$. ∎

Corollary 6. *Deciding whether a geometric graph is $\mathcal{C}_4$-colorable can be done in polynomial time.*

We can characterize $\mathcal{P}_4$-colorability by adding an extra condition linking the vertex adjacency and edge crossing structures that forbids (non-crossing) edges between vertices in $X \cap V({\overline{G}_\times}^1)$ and vertices in $Y \cap V({\overline{G}_\times}^2)$.

Theorem 7. *A geometric graph $\overline{G}$ is $\mathcal{P}_4$-colorable if and only if there exists a bipartition of $V(\overline{G}) = X \cup Y$ and a thickness edge 2-coloring of $\overline{G}_\times$ such that*

1. *$V({\overline{G}_\times}^1) \cap V({\overline{G}_\times}^2) = \emptyset$, and*

2. $[X \cap (V(\overline{G}_\times{}^1)] \cup [Y \cap (V(\overline{G}_\times{}^2)]$ *is an independent set.*

Proof. First note that $\widehat{P}_4$ satisfies both conditions 1 and 2, with bipartition $\{1,3\} \cup \{2,4\}$ and the canonical thickness edge 2-coloring given in section 2. Assume $f : \overline{G} \to \widehat{P}_4$. By Lemma 2, this restricts to $f : \overline{G}_\times \to (\widehat{P}_4)_\times$. We can pull back this bipartition on $V(\widehat{P}_4)$ and thickness edge 2-coloring on $(\widehat{P}_4)_\times$ to obtain a bipartition on $V(\overline{G})$ and a thickness edge 2-coloring on $\overline{G}_\times$. More precisely,

$$X = f^{-1}(\{1,3\}) \text{ and } Y = f^{-1}(\{2,4\})$$

and

$$E\left(\overline{G}_\times{}^1\right) = f^{-1}(e_1) \text{ and } E\left(\overline{G}_\times{}^2\right) = f^{-1}(e_3),$$

meaning

$$V\left(\overline{G}_\times{}^1\right) = f^{-1}(\{1,2\}) \text{ and } V\left(\overline{G}_\times{}^2\right) = f^{-1}(\{3,4\}).$$

Clearly, $V(\overline{G}_\times{}^1) \cap V(\overline{G}_\times{}^2) = \emptyset$, so condition 1 is satisfied. Next,

$$\left[X \cap V\left(\overline{G}_\times{}^1\right)\right] \cup \left[Y \cap V\left(\overline{G}_\times{}^2\right)\right] = f^{-1}\{1\} \cup f^{-1}\{4\} = f^{-1}\{1,4\}.$$

Since the preimage of an independent set under any graph homomorphism is independent, condition 2 is also satisfied.

Conversely, suppose there is a bipartition of $V(\overline{G}) = X \cup Y$ and a thickness edge 2-coloring of $\overline{G}_\times$ satisfying conditions 1 and 2. Define $f : \overline{G} \to \widehat{P}_4$ by

$$f(v) = \begin{cases} 1, & v \in X \cap V(\overline{G}_\times{}^1), \\ 3, & v \in X \backslash V(\overline{G}_\times{}^1), \\ 2, & v \in Y \backslash V(\overline{G}_\times{}^2), \\ 4, & v \in Y \cap V(\overline{G}_\times{}^2). \end{cases}$$

The Venn diagram in Figure 5 shows the two partitions of $V(\overline{G})$; the dashed line indicates that no vertex in $X \cap V(\overline{G}_\times{}^1)$ can be adjacent to a vertex in $Y \cap V(\overline{G}_\times{}^2)$. The numbers in each subset indicate the scheme used to define f. By condition 2, no vertex sent to vertex 1 in $\widehat{P}_4$ is adjacent to a vertex sent to 4 in $\widehat{P}_4$. Thus f is a graph homomorphism. Moreover, if edge xy crosses edge uv in $\overline{G}$, then without loss of generality $xy \in E(\overline{G}_\times{}^1)$ and $uv \in E(\overline{G}_\times{}^2)$. Assuming $x, u \in X$ and $y, v \in Y$, $f(x) = 1$, $f(y) = 2$, $f(u) = 3$ and $f(v) = 4$. Since e_1 crosses e_3 in $\widehat{P}_4$, f is a geometric homomorphism $\overline{G} \to \widehat{P}_4$. ■

By Lemma 5, we can determine in polynomial time whether a given geometric graph $\overline{G}$ is both bipartite and has a thickness edge 2-coloring satisfying condition 1 of Theorem 7. However, determining whether $\overline{G}$ has a bipartition and

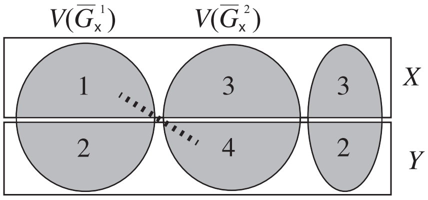

Figure 5. Two partitions of $V(\overline{G})$ for $\mathcal{P}_4$-colorability.

a thickness edge 2-coloring that together satisfy condition 2 could take significantly longer. If G has p components and $EX(\overline{G})$ has q components, then there are 2^{p-1} bipartitions and 2^{q-1} thickness edge 2-colorings. We could therefore have to check 2^{p+q-2} different combinations to find a pair satisfying condition 2. This makes the existence of a polynomial algorithm to determine $\mathcal{C}_4$-colorability unlikely. The same can be said for all subsequent characterization theorems in this section and the next section.

Example 8. Three geometric realizations of C_6 are shown in Figure 6. In each, a thickness edge 2-coloring of the crossing subgraph is shown, with non-crossing edges in grey. The vertex labeling on the leftmost one indicates that it is $\mathcal{P}_4$-colorable. The middle one is not, because two adjacent vertices in one monochromatic subgraph of the crossing subgraph (namely, the one indicated by solid line edges) have neighbors in the other; condition (2) of Theorem 7 requires that in each monochromatic subgraph, only vertices in one bipartition (either X or Y) can be adjacent to vertices in the other monochromatic subgraph. However, since the monochromatic subgraphs are vertex-disjoint, this realization satisfies Theorem 4 and so is $\mathcal{C}_4$-colorable, as the vertex labeling indicates. The rightmost one is neither $\mathcal{P}_4$-colorable nor $\mathcal{C}_4$-colorable, because under the only thickness edge 2-coloring of the crossing subgraph (which is the whole graph), the two monochromatic subgraphs are not vertex-disjoint. In fact, since any two vertices of this realization are either adjacent or co-crossing, it is not homomorphic to a geometric graph of smaller order. This realization is a maximal element in the poset $\mathcal{C}_6$ (see [2]).

4. INTERMEDIATE CASE: $n = 5$

The greatest element $\widehat{P}_5$ of the poset $\mathcal{P}_5$ and its edge-crossing graph are shown in Figure 7. Like $\widehat{P}_4$, this geometric graph is bipartite and thickness-2, but unlike $\widehat{P}_4$, all its edges are crossing edges, i.e., $(\widehat{P}_5)_\times = \widehat{P}_5$. Also note that under the canoncial thickness edge 2-coloring, which is the only possible one, the two monochromatic subgraphs are not vertex-disjoint, as $3 \in V(\widehat{P}_5^1) \cap V(\widehat{P}_5^2)$.

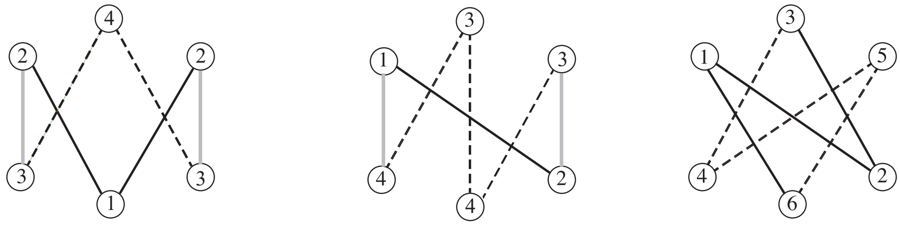

Figure 6. Three realizations of C_6.

Definition. If ϵ is a thickness edge coloring of $\overline{G}$, then any vertex belonging to two monochromatic subgraphs is called *bicolored under* ϵ. More generally, any vertex belonging to two or more monochromatic subgraphs is called *multicolored under* ϵ.

Bicolored vertices play an important role in the characterization of $\mathcal{P}_n$-colorability for all $n \geq 5$.

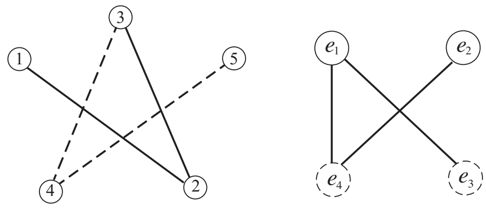

Figure 7. $\widehat{P}_5$ and $EX(\widehat{P}_5)$.

Theorem 9. *A geometric graph* $\overline{G}$ *is* $\mathcal{P}_5$*-colorable if and only if there exist a bipartition of* $V(\overline{G}) = X \cup Y$ *and a thickness edge* 2*-coloring of* $\overline{G}$ *such that*

1. *all bicolored vertices are in* X *(i.e.,* $V(\overline{G}^1) \cap V(\overline{G}^2) \subseteq X$*), and*
2. *no two bicolored vertices are co-crossing.*

The set-theoretic relationship between the bipartition and vertices of the monochromatic subgraphs is shown in the Venn diagram in Figure 8. (Again, only shaded regions contain vertices.)

Proof. First note that $\widehat{P}_5$ satisfies both conditions 1 and 2, with bipartition $\{1,3,5\} \cup \{2,4\}$ and the canonical thickness edge 2-coloring. Assume $f : \overline{G} \to \widehat{P}_5$. Pull back this bipartition and thickness edge 2-coloring to obtain a bipartition on $V(\overline{G})$ and a thickness edge 2-coloring on $\overline{G}$. Since inverse images commute with all set operations, condition 1 is satisfied. Next, all bicolored vertices in $\overline{G}$

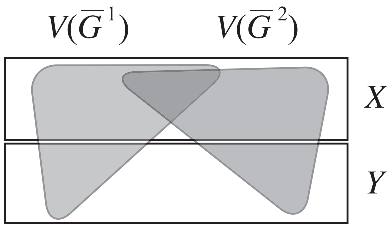

Figure 8. $V(\overline{G})$ for $\mathcal{P}_5$-colorability.

must be mapped to vertex 3 in $\widehat{P}_5$. Since co-crossing vertices cannot have the same homomorphic image, no two bicolored vertices in $\overline{G}$ are co-crossing. Thus condition 2 is satisfied.

Conversely, assume there is a vertex bipartition $V(\overline{G}) = X \cup Y$ and thickness edge 2-coloring satisfying conditions 1 and 2. Suppose these conditions are satisfied vacuously because there are no bicolored vertices, i.e., $V(\overline{G}^1) \cap V(\overline{G}^2) = \emptyset$. Since all edges of $\overline{G}$, both crossing and non-crossing, have been colored, no vertex in $V(\overline{G}^1)$ has a neighbor in $V(\overline{G}^2)$, and so no vertex in $V({\overline{G}_\times}^1)$ has a neighbor in $V({\overline{G}_\times}^2)$. Thus both conditions of Theorem 7 are satisfied, meaning $\overline{G}$ is $\mathcal{P}_4$-colorable and hence $\mathcal{P}_5$-colorable.

Now assume that there are bicolored vertices in $\overline{G}$. Define $f : \overline{G} \to \widehat{P}_5$ by

$$f(v) = \begin{cases} 1, & v \in X \cap \left[V(\overline{G}^1) \backslash V(\overline{G}^2)\right], \\ 3, & v \in V(\overline{G}^1) \cap V(\overline{G}^2), \\ 5, & v \in X \cap \left[V(\overline{G}^2) \backslash V(\overline{G}^1)\right], \\ 2, & v \in Y \cap V(\overline{G}^1), \\ 4, & v \in Y \cap V(\overline{G}^2). \end{cases}$$

Using the fact each edge in $\overline{G}$ is either in $E(\overline{G}^1)$ or $E(\overline{G}^2)$, it is easy to verify that f preserves adjacency. Next, suppose that uv crosses xy in $\overline{G}$. Without loss of generality assume $uv \in E(\overline{G}^1)$ and $xy \in E(\overline{G}^2)$ and that $u, x \in X$. By construction, $f(u) = 1$ or 3, $f(v) = 2$, $f(x) = 3$ or 5 and $f(y) = 4$. Since u and x are co-crossing, by condition 2 they cannot both be bicolored, so at most one of $f(u)$ and $f(x)$ is 3. Since $e_1 = 12$ crosses both $e_3 = 34$ and $e_4 = 45$, and $e_2 = 23$ crosses $e_4 = 45$ in $\widehat{P}_5$, f preserves crossings and is therefore a geometric homomorphism. ■

Example 10. The geometric graph $\overline{C}_{12}$ in Figure 9 is obtained by identifying endpoints of three copies of $\widehat{P}_5$. Two different thickness edge 2-colorings are shown. Under the one on the left, the bicolored vertices are $\{1, 3, 5, 7, 9, 11\}$, while under the one on the right, the bicolored vertices are $\{3, 5, 7, 11\}$. It is easy to verify that 3, 7 and 11 are bicolored under any thickness edge 2-coloring of $\overline{C}_{12}$; in addition, either one or all of 1, 5 or 9 must be also be bicolored. All bicolored vertices will be in the same 'odd' partite, but there will always be a pair of co-crossing bicolored vertices. Hence $\overline{C}_{12}$ is not $\mathcal{P}_5$-colorable.

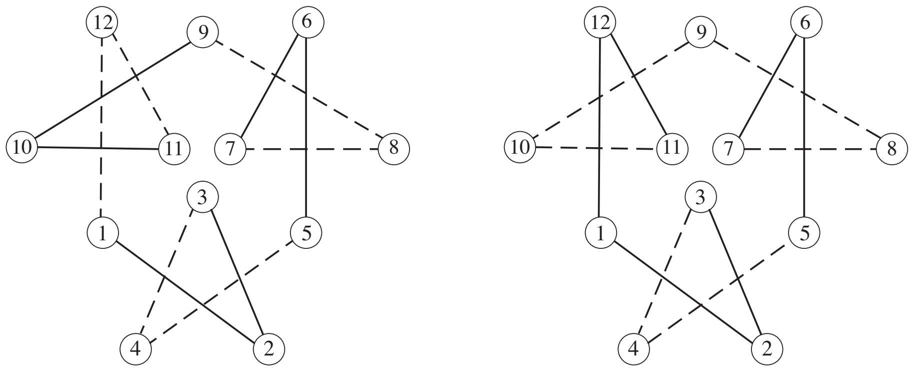

Figure 9. Two thickness edge 2-colorings of $\overline{C}_{12}$.

In Section 3, we noted the close relationship between our characterizations of $\mathcal{P}_4$-colorability and $\mathcal{C}_4$-colorability. There is no analogous relationship between characterizations $\mathcal{P}_5$-colorability and $\mathcal{C}_5$-colorability. From [2], the homomorphism poset $\mathcal{C}_5$ is a chain with greatest element $\widehat{C}_5$ shown in Figure 10, which has the property that $EX(\widehat{C}_5) = C_5$. Hence if $\overline{G}$ is $\mathcal{C}_5$-colorable, then by Lemma 2, both G and $EX(\overline{G})$ are C_5-colorable. A homomorphism $EX(\overline{G}) \to C_5$ is called a thickness edge C_5-coloring of $\overline{G}$. Intuitively, it assigns colors 1 through 5 to the edges of $\overline{G}$ so that colors assigned to edges that cross each other must be consecutive mod 5. For a full discussion of $\mathcal{C}_5$-colorability, see [4]; the main theorem is given below.

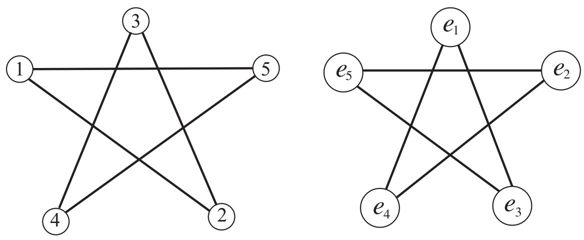

Figure 10. $\widehat{C}_5$ and $EX(\widehat{C}_5)$.

Theorem 11 [4]**.** *A geometric graph $\overline{G}$ is $\mathcal{C}_5$-colorable if and only if there exists thickness edge C_5-coloring of $\overline{G}$ such that*

1. *if* $|i - j| = 1 \mod 5$, *then* $V(\overline{G}^i) \cap V(\overline{G}^j) = \emptyset$*;*

2. *for each $1 \leq i \leq 5$, there exists a bipartition $V(\overline{G}^i) = X^i \cup Y^i$ such that $V(\overline{G}^i) \cap V(\overline{G}^{i+2}) \subseteq X^i$ and $V(\overline{G}^i) \cap V(\overline{G}^{i+3}) \subseteq Y^i$.*

However, viewing C_4 as $K_{2,2}$, the relationship between $\mathcal{P}_5$-colorability and $\mathcal{K}_{2,3}$-colorability is as nice as that between $\mathcal{P}_4$-colorability and $\mathcal{K}_{2,2}$-colorability. From [1, 5], the homomorphism poset of $K_{2,3}$ is a chain with greatest element $\widehat{K}_{2,3}$ (see Figure 11). Thus $\overline{G}$ is $\mathcal{K}_{2,3}$-colorable if and only if $\overline{G} \to \widehat{K}_{2,3}$. Our theorem characterizing $\mathcal{K}_{2,3}$-colorability differs from Theorem 9 only in that the thickness edge coloring condition is on $\overline{G}_\times$, rather than on $\overline{G}$.

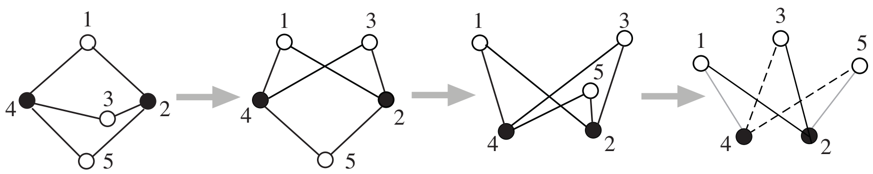

Figure 11. The homomorphism poset $\mathcal{K}_{2,3}$.

Theorem 12. *A geometric graph $\overline{G}$ is $\mathcal{K}_{2,3}$-colorable if and only if there exists a bipartition of $V(\overline{G}) = X \cup Y$ and a thickness edge 2-coloring of $\overline{G}_\times$ such that*

1. *all bicolored crossing vertices are in X (i.e., $V(\overline{G}^1_\times) \cap V(\overline{G}^2_\times) \subseteq X$), and*
2. *no two bicolored crossing vertices are co-crossing.*

Proof. Note that $\widehat{K}_{2,3}$ satisfies both conditions with bipartition $\{1, 3, 5\} \cup \{2, 4\}$ and the thickness edge 2-coloring on $(\widehat{K}_{2,3})_\times = \widehat{P}_5$ being the canonical one. If $\overline{G} \to \widehat{K}_{2,3}$, then the pulled-back bipartition on $V(\overline{G})$ and thickness edge 2-coloring on $\overline{G}_\times$ will also satisfy conditions 1 and 2.

Conversely, suppose there exists a bipartition of $V(\overline{G})$ and thickness edge 2-coloring on $\overline{G}_\times$ satisfying conditions 1 and 2. Restricting the bipartition to $V(\overline{G}_\times)$, by Theorem 9 there is a geometric homomorphism $f : \overline{G}_\times \to \widehat{P}_5$. As noted earlier, geometric homomorphisms between crossing subgraphs cannot in general be extended to the parent geometric graphs. However, in this case we have the extra information that G is bipartite and that $K_{2,3}$ is complete bipartite. We can extend f by sending every non-crossing vertex in X to 1 and every non-crossing vertex in Y to 4. ■

5. GENERAL CASE: $n \geq 5$

We focus first on the case $n = 6$. By Corollary 3, $\widehat{P}_6$ is thickness-3. Although there are now three monochromatic subgraphs, no vertices belong to more than two of them, i.e., all multicolored vertices are in fact bicolored vertices. All bicolored vertices still belong to the same partite, but now they are co-crossing. Some new terminology will prove useful.

Definition. The *co-crossing closure* of a geometric graph $\overline{G}$ is the abstract graph obtained from $\overline{G}$ by adding an edge between any pair of co-crossing vertices. More generally, if $W \subseteq V(\overline{G})$, then the *co-crossing closure of* W *in* $\overline{G}$ is the abstract graph with vertex set W and edges between any two vertices of W that are either adjacent or co-crossing in $\overline{G}$.

Clearly, a geometric homomorphism $f : \overline{G} \to \overline{H}$ induces a graph homomorphism on the co-crossing closures of $\overline{G}$ and $\overline{H}$. More generally, if $W \subseteq V(\overline{G})$, then f induces a graph homomorphism from the co-crossing closure of W in $\overline{G}$ to the co-crossing closure of $f(W)$ in $\overline{H}$. In our characterization result below, we consider the co-crossing closure not just of the bicolored vertices, but of the entire partite containing the bicolored vertices.

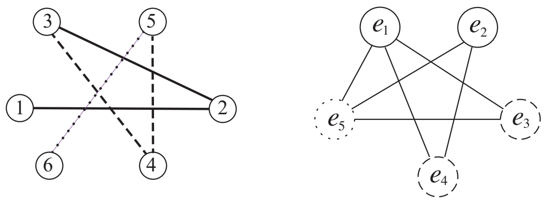

Figure 12. $\widehat{P}_6$ and $EX(\widehat{P}_6)$.

Theorem 13. *A geometric graph $\overline{G}$ is $\mathcal{P}_6$-colorable if and only if there exists a bipartition of $V(\overline{G}) = X \cup Y$ and a thickness edge 3-coloring of $\overline{G}$ such that*

1. *all multicolored vertices are in X;*
2. *the co-crossing closure of X is 3-colorable with color classes S_1, S_3, S_5 satisfying*
   (a) $X \cap V(\overline{G}^i) \subseteq S_{2i-1} \cup S_{2i+1}$ *for* $i \in \{1, 2\}$,
   (b) $X \cap V(\overline{G}^3) \subseteq S_5$.

Condition 2 is illustrated by the Venn diagram in Figure 13; the shaded ellipses correspond to the intersection of X with $V(\overline{G}^1), V(\overline{G}^2)$ and $V(\overline{G}^3)$ and the rounded rectangles correspond to the color classes S_1, S_3 and S_5 of the co-crossing closure of X. (Only shaded regions contain vertices.) Note that in particular, condition 2 implies that any multicolored vertex is bicolored.

Proof. In $\widehat{P}_6$, we have the bipartition $\{1, 3, 5\} \cup \{2, 4, 6\}$. Under the canonical thickness edge 3-coloring,

$$V(\widehat{P}_6^1) = \{1, 2, 3\},\ V(\widehat{P}_6^2) = \{3, 4, 5\} \text{ and } V(\widehat{P}_6^3) = \{5, 6\}.$$

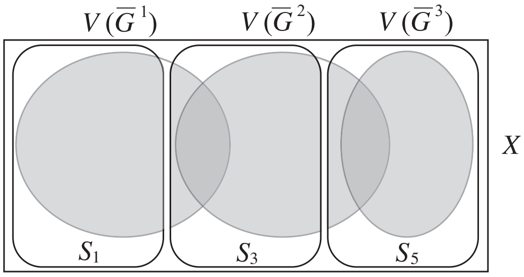

Figure 13. Condition 2 on X.

Clearly, $V(\widehat{P}_6^1) \cap V(\widehat{P}_6^3) = \emptyset$. The vertices in the partite $\{1,3,5\}$ are mutually co-crossing, so the co-crossing closure is K_3. Setting $S_1 = \{1\}, S_3 = \{3\}$ and $S_5 = \{5\}$ satisfies (a), (b) and (c) of condition 2.

Assume $f : \overline{G} \to \widehat{P}_6$. Pull back the vertex bipartition and the thickness edge 3-coloring on $\widehat{P}_6$ to $\overline{G}$. As remarked earlier, f induces a graph homomorphism on the co-crossing closure of $X \subseteq V(\overline{G})$ to that of $\{1,3,5\} \subseteq V(\widehat{P}_6)$, so we can also pull back color classes S_1, S_3 and S_5. Because inverse images commute with all set operations, this vertex bipartition, thickness edge 3-coloring and set of color classes on the co-crossing closure of X on $\overline{G}$ together satisfy conditions 1 and 2.

Conversely, suppose there is a bipartition $V(\overline{G}) = X \cup Y$, a thickness edge 3-coloring on $\overline{G}$ and 3-coloring of the co-crossing closure of X satisfying conditions 1 and 2. Define $f : \overline{G} \to \widehat{P}_6$ by

$$f(v) = \begin{cases} j, & v \in S_j, \\ 2i, & v \in Y \cap V(\overline{G}^i). \end{cases}$$

Suppose $u \in X$ is adjacent to $v \in Y$, with edge $uv \in E(\overline{G}^i)$. Then $f(v) = 2i$ and by conditions 2(b) and 2(c), $f(u) \in \{2i-1, 2i+1\}$ (if $i = 3$, then $2i+1 = 7 \notin V(\widehat{P}_6)$, so in fact $f(u) = 5$). Thus $f(u)$ and $f(v)$ are consecutive numbers in $\{1, \ldots, 6\}$ and hence must be adjacent vertices in $\widehat{P}_6$.

Next, suppose that in $\overline{G}$, edge uv crosses edge xy, with $uv \in E(\overline{G}^i)$ and $xy \in E(\overline{G}^j)$, where $i \neq j$. Without loss of generality, $u, x \in X$ and $v, y \in Y$. Hence $f(v) = 2i$ and $f(y) = 2j$, so $f(v) \neq f(y)$ in the 'even' partite of $\widehat{P}_6$. Since u and x are co-crossing, by condition 2, $f(u) \neq f(v)$ in the 'odd' partite of $\widehat{P}_6$. Any pair of nonadjacent edges crosses in $\widehat{P}_6$, so we are done. ∎

We now provide a characterization of $\mathcal{P}_n$-colorability for all $n \geq 5$. It is easy to see that it is a generalization of our characterization of $\mathcal{P}_6$-colorability, the only difference being that condition 2(c) splits into two cases. It is also a generalization of our characterization of $\mathcal{P}_5$-colorability; conditions 1 and 2 of Theorem 9 are captured by conditions 1 and 2(a) of Theorem 14.

Theorem 14. *Assume $n \geq 5$; let $k = \lceil (n-1)/2 \rceil$ and $\ell = n - k$. A geometric graph $\overline{G}$ is $\mathcal{P}_n$-colorable if and only if there exists a bipartition of $V(\overline{G}) = X \cup Y$ and a thickness edge k-coloring of $\overline{G}$ such that*

1. *all multicolored vertices are in X;*
2. *the co-crossing closure of X is ℓ-colorable, with color classes $S_1, S_3, \ldots, S_{2\ell-1}$ satisfying*
   (a) $X \cap V(\overline{G}^i) \subseteq S_{2i-1} \cup S_{2i+1}$ *for all* $1 \leq i < k$,
   (b) $X \cap V(\overline{G}^k) \subseteq \begin{cases} S_{2k-1}, & n \text{ even}, \\ S_{2k-1} \cup S_{2k+1}, & n \text{ odd}. \end{cases}$

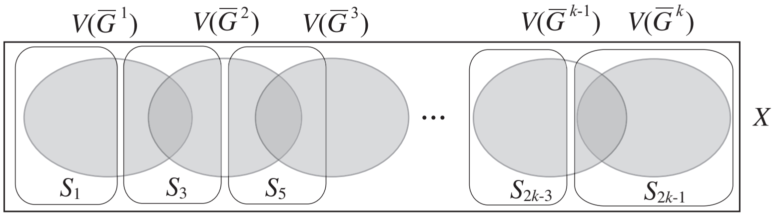

Figure 14. Condition 2 on X, n even.

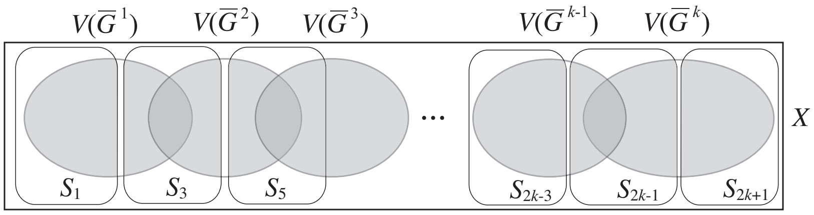

Figure 15. Condition 2 on X, n odd.

Proof. The bipartition of $V(\widehat{P}_n)$ has the set of odd-labeled vertices as one partite and the set of even-labeled vertices as the other. Under the canonical thickness edge k-coloring,

$$V(\widehat{P}_n^i) = \{2i-1, 2i, 2i+1\}, \text{ for all } 1 \leq i < k,$$
$$V(\widehat{P}_n^k) = \begin{cases} \{2k-1, 2k, 2k+1\}, & n \text{ odd}, \\ \{2k-1, 2k\}, & n \text{ even}. \end{cases}$$

Next let $S_{2i-1} = \{2i-1\}$ for $1 \leq i \leq k$; if n is odd, let $S_{2\ell-1} = S_{2k+1}$. It is easy to verify that this bipartition, thickness edge k-coloring and set of color classes on X of $\widehat{P}_n$ together satisfy conditions 1 and 2. If $f : \overline{G} \to \widehat{P}_n$, then pull everything back to obtain a bipartition and thickness edge k-coloring on $\overline{G}$ and a set of color classes on the co-crossing closure of X that satisfy conditions 1 and 2.

Conversely, suppose there is a bipartition of $V(\overline{G}) = X \cup Y$, a thickness edge k-coloring on $E(\overline{G})$ and a k-coloring on the co-crossing closure of X that together satisfy conditions 1 and 2. Again, we define $f : \overline{G} \to \widehat{P}_n$ by

$$f(v) = \begin{cases} j, & v \in S_j, \\ 2i, & v \in Y \cap V(\overline{G}^i). \end{cases}$$

From here, the proof is identical to that of Theorem 13. ■

Note that determining whether $\overline{G}$ has a thickness edge k-coloring is equivalent to determining whether $EX(\overline{G})$ is k-colorable, which is an NP-complete problem for $k \geq 3$. For this reason, this characterization result does not lead to a polynomial algorithm for deciding whether a geometric graph is $\mathcal{P}_n$-colorable for $n \geq 6$.

Condition 2 can be phrased in terms of list colorings. Assign any bicolored vertex in $V(\overline{G}^i) \cap V(\overline{G}^j)$ with the singleton list $\{i + j\}$; in effect, the bicolored vertices are pre-colored. If n is even, then also pre-color any vertex in $X \cap V(\overline{G}^k)$ with $2k-1$. Any 'unicolored' vertex in $X \cap V(\overline{G}^i)$ is assigned the list $\{2i-1, 2i+1\}$. Condition 2 says that there will be an ℓ-coloring of the co-crossing closure of X such that any vertex will be assigned a color from its list.

Intuitively, this theorem states that a geometric graph is $\mathcal{P}_n$-colorable if and only if it is bipartite and we can decompose its edges into a stack of plane graph 'layers' such that consecutive layers are 'attached' (via shared vertices) in a special way: all attaching vertices lie within one partite, and no two vertices attaching the same two layers can be co-crossing.

6. Open Questions

1. We showed that $\mathcal{P}_4$- and $\mathcal{C}_4$-colorability (equivalently, $\mathcal{K}_{2,2}$-colorability) are closely related. Although the characterization of $\mathcal{P}_5$-colorability is not very similar to that of $\mathcal{C}_5$-colorability, it is closely related to the characterizations of $\mathcal{K}_{2,3}$-colorability. Can we characterize preimages of general geometric cycles and complete bipartite graphs? Determining necessary and sufficient conditions for $\mathcal{C}_6$-colorability is complicated by the fact that $\mathcal{C}_6$ has two maximal elements; see [2]. The structure of $\mathcal{C}_n$ for $n \geq 7$ has not yet been determined. As for complete bipartite graphs, the posets $\mathcal{K}_{2,n}$ for all n and $\mathcal{K}_{3,3}$ have been determined; neither are chains, but both have a greatest element (see [5] and [3]). However, the structure of $\mathcal{K}_{m,n}$ for $m \geq 3$ in general has not been determined.

2. An *obstruction* result characterizing H-colorability is a biconditional of the form $G \not\to H$ if and only if $J \to G$ for some graph J. A classic example is König's theorem, which can be phrased as: $G \not\to K_2$ if and only if $C_\ell \to G$ for some odd integer $\ell \geq 3$. In [12], Hell and Zhu characterize preimages of oriented paths with the following obstruction result: if P is an oriented path, then a digraph D satisfies $D \not\to P$ if and only if there exists an oriented path W such that $W \to D$ and $W \not\to P$. Can preimages of geometric paths be characterized with an obstruction result?

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