SOME RESULTS ON THE INDEPENDENCE POLYNOMIAL OF UNICYCLIC GRAPHS

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Abstract

Let $G$ be a simple graph on $n$ vertices. An independent set in a graph is a set of pairwise non-adjacent vertices. The independence polynomial of $G$ is the polynomial $I(G, x) = \sum_{k=0}^{n} s(G, k)x^k$, where $s(G, k)$ is the number of independent sets of $G$ with size $k$ and $s(G, 0) = 1$. A unicyclic graph is a graph containing exactly one cycle. Let $C_n$ be the cycle on $n$ vertices. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs $G$ on $n$ vertices (except two of them), $I(G, t) > I(C_n, t)$ for sufficiently large $t$. Finally for every $n \geq 3$ we find all connected graphs $H$ such that $I(H, x) = I(C_n, x)$.

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1. Introduction

Throughout this paper we will consider only simple graphs, the graphs with no loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The order of $G$ denotes the number of vertices of $G$. Let $e$ be an edge of $G$. By $e = uv$ we mean that $e$ is an edge between vertices $u$ and $v$. For every vertex $v \in V(G)$, the closed neighborhood of $v$ denoted by $N[v]$ is defined as $\{u \in V(G) | uv \in E(G)\} \cup \{v\}$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the disjoint union of $G_1$ and $G_2$, denoted by $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The
graph $rG$ denotes the disjoint union of $r$ copies of $G$. For every vertex $v \in V(G)$, the degree of $v$ is the number of edges incident with $v$. A pendant vertex is a vertex of degree one. For a vertex $v \in V(G)$, $G \setminus v$ denotes the graph obtained from $G$ by removing $v$. A unicyclic graph is a graph containing exactly one cycle. We denote the complete graph of order $n$, the complete bipartite graph with part sizes $m,n$, the cycle of order $n$, and the path of order $n$, by $K_n$, $K_{m,n}$, $C_n$, and $P_n$, respectively. Also $K_{1,n}$ is called a star.

A set $S \subseteq V(G)$ is an independent set if there is no edge between the vertices of $S$. If $S$ is an independent set with $|S| = k$, then $S$ is called a $k$-independent set. By $s(G,k)$ we mean the number of $k$-independent sets of $G$. The independence number of $G$, $\alpha(G)$, is the maximum cardinality of an independent set of $G$. The independence polynomial of $G$, $I(G,x)$, is defined as $I(G,x) = \sum_{k=0}^{\alpha(G)} s(G,k)x^k$, where $s(G,k)$ is the number of independent sets of $G$ of size $k$ and $s(G,0) = 1$.

This polynomial was introduced by Gutman and Harary in [10]. For example for every $n \geq 1$, $\alpha(K_n) = 1$ and $s(K_n,1) = n$. Thus $I(K_n,x) = 1 + nx$. The independence polynomial has very nice properties, see [5, 6, 13] for more details.

There are many polynomials associated with graphs. For example chromatic polynomial, clique polynomial, domination polynomial, edge cover polynomial and matching polynomial, see [1]–[16]. One of the most important problems related to graph polynomials is the following:

**Problem.** Which graphs are uniquely determined by their graph polynomials?

In many papers, researchers study the problem defined above for graph polynomials. For example in [3] the authors show that the complete graphs, the cycles and some complete bipartite graphs are determined by their edge cover polynomials. In [2] it is proved that the cycles are determined by their domination polynomials. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs $G$ on $n$ vertices except the cycle $C_n$ and the graph $D_n$ (see Figure 3), $I(G,t) > I(C_n,t)$ for sufficiently large $t$. We show that for every $n \geq 4$ there is only one connected graph $H$ such that $H \not\cong C_n$ and $I(H,x) = I(C_n,x)$.

2. **The Independence Polynomials of Unicyclic Graphs**

In this section we study the independence polynomials of unicyclic graphs. We need the following basic properties of independence polynomials.

**Theorem 1** [10, 11]. Let $G$ be a graph with connected components $G_1, \ldots, G_t$. Then $I(G,x) = \prod_{i=1}^{t} I(G_i,x)$.

**Theorem 2** [10, 11]. Let $G$ be a graph and $v$ be a vertex of $G$. Then

$I(G,x) = I(G \setminus v, x) + xI(G \setminus N[v], x)$. 

Remark 3. We remark that by independence polynomials one can find the number of vertices and the number of edges of graphs. More precisely, if \( G \) is a graph with \( n \) vertices and \( m \) edges, then \( n = s(G, 1) \) and \( m = \binom{n}{2} - s(G, 2) \).

Lemma 4. Let \( T \) be a tree of order \( n \). Then there exists a positive real number \( r_n \) such that for all \( x \geq r_n \) we have

\[
I(T, x) > \begin{cases} 
  x^{\frac{n}{2}}, & \text{if } n \text{ is odd;} \\
  2x^{\frac{n}{2}}, & \text{if } n \text{ is even.}
\end{cases}
\]

Proof. Since \( T \) is a tree, \( T \) is bipartite. Assume that \( X \) and \( Y \) are partite sets of \( V(T) \). Hence \( \alpha(T) \geq |X|/|Y| \). This shows that \( \alpha(T) \geq \lfloor \frac{n}{2} \rfloor \). First assume that \( n \) is odd. Thus for all \( x \geq 1 \), \( x^{\alpha(T)} \geq x^{\lfloor \frac{n}{2} \rfloor} \). This shows that for all \( x \geq 1 \), \( I(T, x) > x^{\frac{n}{2}} \). Now assume that \( n \) is even. If \( \alpha(T) = \frac{n}{2} \), then \( |X| = |Y| = \frac{n}{2} \). Thus \( s(T, \frac{n}{2}) \geq 2 \). Hence for all \( x \geq 1 \), \( s(T, \alpha(T))x^{\alpha(T)} \geq 2x^{\frac{n}{2}} \) and so \( I(T, x) > 2x^{\frac{n}{2}} \). Otherwise suppose that \( \alpha(T) > \frac{n}{2} \). Thus \( I(T, x) - 2x^{\frac{n}{2}} \) is a polynomial with positive leading coefficient. Therefore for sufficiently large \( x \), \( I(T, x) - 2x^{\frac{n}{2}} > 0 \). This completes the proof.

Let \( G \) be a graph of order \( n \) with vertex set \( \{v_1, \ldots, v_n\} \). Let \( H_1, \ldots, H_n \) be some disjoint graphs. Assume that \( u_1 \in V(H_1), \ldots, u_n \in V(H_n) \). By \( G(H_1, \ldots, H_n; u_1, \ldots, u_n) \) we mean the graph that is obtained by identifying the vertices \( u_i \) and \( v_i \) for \( i = 1, \ldots, n \). Note that the order of \( G(H_1, \ldots, H_n; u_1, \ldots, u_n) \) is \( |V(H_1)| + \cdots + |V(H_n)| \), see Figure 1. In particular, suppose that \( H_1, \ldots, H_n \) are some stars, say \( H_1 = K_{1,m_1}, \ldots, H_n = K_{1,m_n} \) where \( m_1, \ldots, m_n \) are some non-negative integers (by \( K_{1,0} \) we mean the single vertex \( K_1 \)). In addition let \( u_i \) be the vertex of \( K_{1,m_i} \) with degree \( m_i \). Then we use \( G(m_1, \ldots, m_n) \) instead of \( G(K_{1,m_1}, \ldots, K_{1,m_n}; u_1, \ldots, u_n) \). Note that the order of \( G(m_1, \ldots, m_n) \) is \( m_1 + \cdots + m_n + n \) and \( G(0, \ldots, 0) \cong G \). See Figure 2.

Lemma 5. Let \( k \geq 3 \) be an integer. Let \( V(C_k) = \{v_1, \ldots, v_k\} \) and \( E(C_k) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\} \). Let \( G = C_k(n_1, \ldots, n_k) \) and \( n = n_1 + \cdots + n_k + k \), where \( n_1, \ldots, n_k \) are some non-negative integers. If \( G \ncong C_n \) and \( n \geq 5 \), then for sufficiently large \( x \) we have \( I(G, x) > I(C_n, x) \).

Proof. First we note that if \( n = 3 \), then \( G \cong C_3 \). Also if \( n = 4 \), then \( G \cong C_4 \) or \( G \cong C_3(1, 0, 0) \). Since \( I(C_3(1, 0, 0), x) = I(C_4, x) = 1 + 4x + 2x^2 \), \( I(G, x) \geq I(C_4, x) \). We note that \( C_k(n_1, \ldots, n_k) \cong C_n \) if and only if \( n = k \). Now assume that \( n \geq 5 \) and \( G \cong C_n \) (\( G \not\cong C_k \)). We have one of the following cases.

(i) For some \( i \in \{1, \ldots, k\} \), \( n_i \geq 2 \). Without losing the generality assume that \( n_1 \geq 2 \). Note that \( \alpha(G) \geq n_1 + \alpha(P_{k-1}(n_2, \ldots, n_k)) \), where \( V(P_{k-1}) = \{v_2, \ldots, v_k\} \) and \( E(P_{k-1}) = \{v_2v_3, \ldots, v_{k-1}v_k\} \). Since \( P_{k-1}(n_2, \ldots, n_k) \) is a tree...
of order $n - n_1 - 1$ (by the proof of Lemma 4), $\alpha(P_{k-1}(n_2, \ldots, n_k)) \geq \lceil \frac{n - n_1 - 1}{2} \rceil$. Hence

$$\alpha(G) \geq n_1 + \left\lceil \frac{n - n_1 - 1}{2} \right\rceil = \left\lceil \frac{n + n_1 - 1}{2} \right\rceil \geq \left\lceil \frac{n + 1}{2} \right\rceil > \lfloor \frac{n}{2} \rfloor = \alpha(C_n).$$

Thus $\alpha(G) > \alpha(C_n)$. Since the coefficients of independence polynomials are positive, for sufficiently large $x$ we have $I(G, x) > I(C_n, x)$.

(ii) For $i = 1, \ldots, k$, $n_i \in \{0, 1\}$ and $n$ is odd. Since $G \not\cong C_n$, for some $i$, $n_i = 1$. Without losing the generality let $n_1 = 1$. Since $n$ is odd, similar to part (i), $\alpha(G) \geq 1 + \left\lceil \frac{n - 2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil = \alpha(C_n)$. Thus the result follows.

(iii) For $i = 1, \ldots, k$, $n_i \in \{0, 1\}$ and $n$ is even. Since $G \not\cong C_n$, for some $t$, $n_t = 1$. First suppose that there is only one $i$ such that $n_i = 1$. Without losing the generality assume that $n_1 = \cdots = n_{k-1} = 0$ and $n_k = 1$. Hence $k = n - 1$. In other words, $G \cong C_{n-1}(0, \ldots, 0, 1)$. Hence $\alpha(G) = \frac{n}{2}$. Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_1, v_{n-1}v_n\}$. Since $n \geq 6$, $\{v_1, v_3, \ldots, v_{n-3}, v_n\}$, $\{v_1, v_3, \ldots, v_{n-5}, v_{n-2}, v_n\}$ and $\{v_2, v_4, \ldots, v_{n-2}, v_n\}$ are three independent sets of $G$ with cardinality $\frac{n}{2}$. Hence $s(G, \frac{n}{2}) \geq 3$. On the
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Figure 2. The graph $C_4(1, 1, 2, 3) = G(H_1, H_2, H_3, H_4; u_1, u_2, u_3, u_4)$, where $G = C_4$ and $u_1$ is the vertex of $H_1 = K_{1,1}$ of degree one, $u_2$ is the vertex of $H_2 = K_{1,1}$ of degree one, $u_3$ is the vertex of $H_3 = K_{1,2}$ of degree two and $u_4$ is the vertex of $H_4 = K_{1,3}$ of degree three.

On the other hand, since $n$ is even, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. By the fact that $\alpha(G) = \alpha(C_n) = \frac{n}{2}$ and $s(G, \frac{n}{2}) > s(C_n, \frac{n}{2})$, for sufficiently large $x$ we obtain $I(G, x) > I(C_n, x)$. Now assume that there are some $i \neq j$ such that $n_i = 1$ and $n_j = 1$. This shows that $G$ has at least two vertices of degree one ($G$ has two pendant vertices). Let $u$ and $v$ be two pendant vertices of $G$. Applying Theorem 2 for vertex $u$ we obtain $I(G, x) = I(G \setminus u, x) + xI(T_1, x)$, where $T_1$ is a tree of order $n - 2$. Using Theorem 2 for $v$ and $G \setminus u$ we have

$$I(G, x) = I(G \setminus \{u, v\}, x) + xI(T_2, x) + xI(T_1, x),$$

where $T_2$ is a tree of order $n - 3$. Hence for $x \geq 0, I(G, x) > xI(T_2, x) + xI(T_1, x)$. Using Lemma 4 for trees $T_1$ and $T_2$ we obtain that for sufficiently large $x$,

$$I(G, x) > xx^{\lfloor \frac{n-3}{2} \rfloor} + 2xx^{\frac{n-2}{2}} = 3x^{\frac{n}{2}}.$$  

On the other hand, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. Hence for sufficiently large $x$, $3x^{\frac{n}{2}} > I(C_n, x)$. Thus for sufficiently large $x$, $I(G, x) > 3x^{\frac{n}{2}} > I(C_n, x)$. The proof is complete.

3. Graphs Whose Independence Polynomials Coincide with Independence Polynomials of Cycles

In this section we study the graphs $G$ such that $I(G, x) = I(C_n, x)$, where $n \geq 3$. We show that there is only one connected graph $G \not= C_n$ satisfying $I(G, x) = I(C_n, x)$. Let $n \geq 4$ be an integer. By $D_n$ we mean the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{v_{n-2}v_n\}$, see Figure 3. In addition by $D_3$ we mean the cycle $C_3$. The next result shows that the independence polynomials of $C_n$ and $D_n$ are the same.
Lemma 6. Let $n \geq 4$ be an integer. Then

(i) $I(C_n, x) = I(C_{n-1}, x) + xI(C_{n-2}, x)$, where $C_2$ is the path $P_2$.

(ii) $I(C_n, x) = I(D_n, x)$.

Proof. It is easy to check the result for $n = 4$. Thus let $n \geq 5$. Using Theorem 2 for one of the vertices of $C_n$ we obtain that

\begin{equation}
I(C_n, x) = I(P_{n-1}, x) + xI(P_{n-3}, x).
\end{equation}

On the other hand, by Theorem 2 for one of the pendant vertices of $P_t$ we have

\begin{equation}
I(P_t, x) = I(P_{t-1}, x) + xI(P_{t-2}, x), \text{ for } t \geq 2, \text{ where } I(P_0, x) = 1.
\end{equation}

Using equations (1) and (2) one can see that

\[ I(C_n, x) = I(P_{n-2}, x) + xI(P_{n-4}, x) + x(I(P_{n-3}, x) + xI(P_{n-5}, x)). \]

So by equation (1) the first part is proved. Now we prove the second part. Using Theorem 2 for the vertex $v_n$ of $D_n$ (see Figure 3) we obtain that $I(D_n, x) = I(P_{n-1}, x) + xI(P_{n-3}, x)$. Hence by equation (1), $I(D_n, x) = I(C_n, x)$. The proof is complete.

We recall that a unicyclic graph is a graph with exactly one cycle. The next result shows that among all connected unicyclic graphs the cycles have the smallest independence polynomials.

Theorem 7. Let $G$ be a connected unicyclic graph of order $n$. Assume that $G \not\cong C_n$ and $G \not\cong D_n$. Then for sufficiently large $x$ we have $I(G, x) > I(C_n, x)$.

Proof. Assume that $H$ is a connected unicyclic graph of order $n$. Thus $n \geq 3$. If $n = 3$, then $H \cong C_3$. If $n = 4$, then $H \cong C_4$ or $H \cong D_4$. If $n = 5$, then $H \cong C_5$ or $H \cong D_5$ or $H \cong C_4(1,0,0,0)$ or $H \cong C_3(2,0,0)$ or $H \cong C_3(1,1,0)$. So by the fact that $G$ is unicyclic and $G \not\cong C_n$ and $G \not\cong D_n$ we obtain that $n \geq 5$. We use induction on $n$ to prove the result. If $n = 5$, then $G \cong C_4(1,0,0,0)$ or $G \cong C_3(2,0,0)$ or $G \cong C_3(1,1,0)$. One can see that $I(C_4(1,0,0,0), x) = 1 + 5x + 5x^2 + x^3$, $I(C_3(2,0,0), x) = 1 + 5x + 5x^2 + 2x^3$ and

\[ I(C_3(1,1,0), x) = 1 + 5x + 5x^2 + x^3. \]
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$I(C_3(1, 1, 0), x) = 1 + 5x + 5x^2 + x^3$. On the other hand $I(C_5, x) = 1 + 5x + 5x^2$. Thus the result holds for $n = 5$.

Now assume that $n \geq 6$. Suppose that the length of the unique cycle of $G$ is $k$. Assume that $v_1, \ldots, v_k$ are the vertices of this cycle. Since $G$ is unicyclic there are some trees $T_1, \ldots, T_k$ such that $G = C_k(T_1, \ldots, T_k; v_1, \ldots, v_k)$. If each tree $T_1, \ldots, T_k$ is a star, then by Lemma 5 the result follows. Now without losing the generality assume that $T_1$ is not a star. Let $u_1$ be a pendant vertex of $T_1$ which has the maximum distance from $v_1$ among all pendant vertices of $T_1$. We consider the three following cases for $G \setminus u_1$.

(i) Assume that $G \setminus u_1$ is the cycle $C_{n-1}$. Hence $G = C_{n-1}(1, 0, \ldots, 0)$ and $T_1 = P_2$, a contradiction (since $T_1$ is not a star). Thus this case does not happen.

(ii) Assume that $G \setminus u_1$ is the graph $D_{n-1}$. Hence $G \cong D_n$ or $G \cong H$, where $H$ is obtained by identifying the pendant vertex of $D_{n-2}$ with the non-pendant vertex of $P_3$. Thus it suffices to check the result for $H$. Let $z$ be a pendant vertex of $H$. Thus $H \setminus z \cong D_{n-1}$ and $H \setminus [z] \cong D_{n-3} + K_1$. Hence by Theorems 1 and 2, $I(H, x) = I(H \setminus z, x) + xI(H \setminus [z], x) = I(D_{n-1}, x) + (1 + x)I(D_{n-3}, x)$. So by the second part of Lemma 6 we obtain

$$I(H, x) = I(C_{n-1}, x) + x(1 + x)I(C_{n-3}, x). \quad (3)$$

On the other hand, by the first part of Lemma 6 for $n \geq 7$, $I(C_{n-3}, x) = I(C_{n-4}, x) + xI(C_{n-5}, x)$. This shows that for $x > 0$, $I(C_{n-3}, x) > I(C_{n-4}, x)$ (this inequality also holds for $n = 6$, where $C_2$ is the path $P_2$). Hence for $x > 0$, $xI(C_{n-3}, x) > xI(C_{n-4}, x)$. Thus for every $x > 0$ we have

$$(1 + x)I(C_{n-3}, x) = I(C_{n-3}, x) + xI(C_{n-3}, x) > I(C_{n-3}, x) + xI(C_{n-4}, x).$$

Therefore by the first part of Lemma 6 we obtain that

$$I(H, x) > I(C_{n-2}, x). \quad (4)$$

The equations (3) and (4) show that for $x > 0$, $I(H, x) > I(C_{n-1}, x) + xI(C_{n-2}, x)$. Hence by the first part of Lemma 6 for every $x > 0$, $I(H, x) > I(C_n, x)$.

(iii) Suppose that $G \setminus u_1 \not\cong C_{n-1}$ and $G \setminus u_1 \not\cong D_{n-1}$. Since $G \setminus u_1$ is a connected unicyclic graph of order $n-1$, by the induction hypothesis for sufficiently large $x$, $I(G \setminus u_1, x) > I(C_{n-1}, x)$. As we defined above, $u_1$ is a pendant vertex of $T_1$ which has the maximum distance from $v_1$ among all pendant vertices of $T_1$. Assume that $w_1$ is the neighbor of $u_1$. Since $T_1$ is not a star, $d(u_1, v_1) \geq 2$. We note that $w_1 \neq v_1$. Let $deg(w_1) = t + 1$. Thus $t \geq 1$. By the definition of $u_1$, exactly $t$ neighbors of $w_1$ have degree one. Hence $G \setminus N[u_1]$ is the union of a unicyclic graph of order $n-t-1$, say $L$, with exactly $t-1$ isolated vertices. In other words, $G \setminus N[u_1] = L + (t-1)K_1$. Hence by Theorem 1, $I(G \setminus N[u_1], x) = I(L, x)(1 + x)^{t-1}$. On the other hand, by the induction hypothesis for sufficiently large $x$, $I(L, x) \geq$
Let $3 \leq n \leq 9$ and $G$ be a graph of order $n$. Assume that $I(G, x) = I(C_n, x)$. We check this question for $n \leq 9$.

**Remark 9.** Let $3 \leq n \leq 9$ and $G$ be a graph of order $n$. Assume that $I(G, x) = I(C_n, x)$. We find that if $n \in \{3, 4, 5, 7, 8\}$, then $G \cong C_n$ or $G \cong D_n$ (see Theorem 8). We obtain that $I(G, x) = I(C_6, x)$ if and only if $G \in \{C_6, D_6, K_2 + K_4 \setminus e\}$, where $e$ is an edge of $K_4$. We find that $I(G, x) = I(C_9, x)$ if and only if $G \in \{C_9, D_9, H_1, H_2, H_3\}$, where $H_1, H_2$ and $H_3$ have been shown in Figure 4. In fact $I(C_6, x) = 1 + 6x + 9x^2 + 2x^3 = (1 + 4x + x^2)(1 + 2x) = I(K_4 \setminus e, x)I(K_2, x)$ and $I(C_9, x) = 1 + 9x + 27x^2 + 30x^3 + 9x^4 = (1 + 6x + 9x^2 + 3x^3)(1 + 3x)$. These examples show that the structure of all non-connected graphs $G$ with $I(G, x) = I(C_m, x)$ is not clear, where $m \geq 10$. 

**Theorem 8.** Let $n \geq 3$ be an integer. Assume that $G$ is a connected graph such that $I(G, x) = I(C_n, x)$. Then $G \cong C_n$ or $G \cong D_n$.

**Proof.** Since $I(G, x) = I(C_n, x)$ and $C_n$ has $n$ vertices and $n$ edges, by Remark 3 we find that $G$ has exactly $n$ vertices and $n$ edges. Since the number of vertices and the number of edges of $G$ are the same and $G$ is connected, $G$ is unicyclic. If $G \not\cong C_n$ or $G \not\cong D_n$, then by Theorem 7 for large $x$ we have $I(G, x) > I(C_n, x)$, a contradiction. This completes the proof. 

Now we are in a position to prove the main result of this section.
We finish the paper by the following problem.

**Problem.** Let $n \geq 10$ be an integer. Find all non-connected graphs $G$ such that $I(G, x) = I(C_n, x)$.

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