SOME RESULTS ON THE INDEPENDENCE POLYNOMIAL OF UNICYCLIC GRAPHS

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Abstract

Let $G$ be a simple graph on $n$ vertices. An independent set in a graph is a set of pairwise non-adjacent vertices. The independence polynomial of $G$ is the polynomial $I(G, x) = \sum_{k=0}^{n} s(G, k)x^k$, where $s(G, k)$ is the number of independent sets of $G$ with size $k$ and $s(G, 0) = 1$. A unicyclic graph is a graph containing exactly one cycle. Let $C_n$ be the cycle on $n$ vertices. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs $G$ on $n$ vertices (except two of them), $I(G, t) > I(C_n, t)$ for sufficiently large $t$. Finally for every $n \geq 3$ we find all connected graphs $H$ such that $I(H, x) = I(C_n, x)$.

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1. Introduction

Throughout this paper we will consider only simple graphs, the graphs with no loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The order of $G$ denotes the number of vertices of $G$. Let $e$ be an edge of $G$. By $e = uv$ we mean that $e$ is an edge between vertices $u$ and $v$. For every vertex $v \in V(G)$, the closed neighborhood of $v$ denoted by $N[v]$ is defined as $\{ u \in V(G) \mid uv \in E(G) \} \cup \{ v \}$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the disjoint union of $G_1$ and $G_2$ denoted by $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The
graph $rG$ denotes the disjoint union of $r$ copies of $G$. For every vertex $v \in V(G)$, the degree of $v$ is the number of edges incident with $v$. A pendant vertex is a vertex of degree one. For a vertex $v \in V(G)$, $G \setminus v$ denotes the graph obtained from $G$ by removing $v$. A unicyclic graph is a graph containing exactly one cycle. We denote the complete graph of order $n$, the complete bipartite graph with part sizes $m, n$, the cycle of order $n$, and the path of order $n$, by $K_n$, $K_{m,n}$, $C_n$, and $P_n$, respectively. Also $K_{1,n}$ is called a star.

A set $S \subseteq V(G)$ is an independent set if there is no edge between the vertices of $S$. If $S$ is an independent set with $|S| = k$, then $S$ is called a $k$-independent set. By $s(G,k)$ we mean the number of $k$-independent sets of $G$. The independence number of $G$, $\alpha(G)$, is the maximum cardinality of an independent set of $G$. The independence polynomial of $G$, $I(G, x)$, is defined as $I(G, x) = \sum_{k=0}^{\alpha(G)} s(G,k)x^k$, where $s(G,k)$ is the number of independent sets of $G$ of size $k$ and $s(G,0) = 1$.

This polynomial was introduced by Gutman and Harary in [10]. For example for every $n \geq 1$, $\alpha(K_n) = 1$ and $s(K_n,1) = n$. Thus $I(K_n, x) = 1 + nx$. The independence polynomial has very nice properties, see [5, 6, 13] for more details. There are many polynomials associated with graphs. For example chromatic polynomial, clique polynomial, domination polynomial, edge cover polynomial and matching polynomial, see [1]–[16]. One of the most important problems related to graph polynomials is the following:

**Problem.** Which graphs are uniquely determined by their graph polynomials?

In many papers, researchers study the problem defined above for graph polynomials. For example in [3] the authors show that the complete graphs, the cycles and some complete bipartite graphs are determined by their edge cover polynomials. In [2] it is proved that the cycles are determined by their domination polynomials. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs $G$ on $n$ vertices except the cycle $C_n$ and the graph $D_n$ (see Figure 3), $I(G, t) > I(C_n, t)$ for sufficiently large $t$. We show that for every $n \geq 4$ there is only one connected graph $H$ such that $H \not= C_n$ and $I(H, x) = I(C_n, x)$.

## 2. The Independence Polynomials of Unicyclic Graphs

In this section we study the independence polynomials of unicyclic graphs. We need the following basic properties of independence polynomials.

**Theorem 1** [10, 11]. Let $G$ be a graph with connected components $G_1, \ldots, G_t$. Then $I(G, x) = \prod_{i=1}^{t} I(G_i, x)$.

**Theorem 2** [10, 11]. Let $G$ be a graph and $v$ be a vertex of $G$. Then $I(G, x) = I(G \setminus v, x) + xI(G \setminus N[v], x)$. 
Remark 3. We remark that by independence polynomials one can find the number of vertices and the number of edges of graphs. More precisely, if $G$ is a graph with $n$ vertices and $m$ edges, then $n = s(G, 1)$ and $m = \binom{n}{2} - s(G, 2)$.

Lemma 4. Let $T$ be a tree of order $n$. Then there exists a positive real number $r_n$ such that for all $x \geq r_n$ we have

$$I(T, x) > \begin{cases} 2x^\frac{n}{2}, & \text{if } n \text{ is odd;} \\ x^\frac{n+1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Since $T$ is a tree, $T$ is bipartite. Assume that $X$ and $Y$ are partite sets of $V(T)$. Hence $\alpha(T) \geq |X|, |Y|$. This shows that $\alpha(T) \geq \lceil \frac{n}{2} \rceil$. First assume that $n$ is odd. Thus for all $x \geq 1$, $x^{\alpha(T)} \geq x^{\left\lceil \frac{n}{2} \right\rceil}$. This shows that for all $x \geq 1$, $I(T, x) > x^{\frac{n}{2}}$. Now assume that $n$ is even. If $\alpha(T) = \frac{n}{2}$, then $|X| = |Y| = \frac{n}{2}$. Thus $s(T, \frac{n}{2}) \geq 2$. Hence for all $x \geq 1$, $s(T, \alpha(T))x^{\alpha(T)} \geq 2x^\frac{n}{2}$ and so $I(T, x) > 2x^\frac{n}{2}$. Otherwise suppose that $\alpha(T) > \frac{n}{2}$. Thus $I(T, x) - 2x^\frac{n}{2}$ is a polynomial with positive leading coefficient. Therefore for sufficiently large $x$, $I(T, x) - 2x^\frac{n}{2} > 0$. This completes the proof.

Let $G$ be a graph of order $n$ with vertex set $\{v_1, \ldots, v_n\}$. Let $H_1, \ldots, H_n$ be some disjoint graphs. Assume that $u_1 \in V(H_1), \ldots, u_n \in V(H_n)$. By $G(H_1, \ldots, H_n; u_1, \ldots, u_n)$ we mean the graph that is obtained by identifying the vertices $u_i$ and $v_i$ for $i = 1, \ldots, n$. Note that the order of $G(H_1, \ldots, H_n; u_1, \ldots, u_n)$ is $|V(H_1)| + \cdots + |V(H_n)|$, see Figure 1. In particular, suppose that $H_1, \ldots, H_n$ are some stars, say $H_1 = K_{1, m_1}, \ldots, H_n = K_{1, m_n}$, where $m_1, \ldots, m_n$ are some non-negative integers (by $K_{1, 0}$ we mean the single vertex $K_1$). In addition let $u_i$ be the vertex of $K_{1, m_i}$ with degree $m_i$. Then we use $G(m_1, \ldots, m_n)$ instead of $G(K_{1, m_1}, \ldots, K_{1, m_n}; u_1, \ldots, u_n)$. Note that the order of $G(m_1, \ldots, m_n)$ is $m_1 + \cdots + m_n + n$ and $G(0, \ldots, 0) \cong G$. See Figure 2.

Lemma 5. Let $k \geq 3$ be an integer. Let $V(C_k) = \{v_1, \ldots, v_k\}$ and $E(C_k) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$. Let $G = C_k(n_1, \ldots, n_k)$ and $n = n_1 + \cdots + n_k + k$, where $n_1, \ldots, n_k$ are some non-negative integers. If $G \cong C_n$ and $n \geq 5$, then for sufficiently large $x$ we have $I(G, x) > I(C_n, x)$.

Proof. First we note that if $n = 3$, then $G \cong C_3$. Also if $n = 4$, then $G \cong C_4$ or $G \cong C_3(1, 0, 0)$. Since $I(C_4(1, 0, 0), x) = I(C_4, x) = 1 + 4x + 2x^2$, $I(G, x) = I(C_4, x)$. We note that $C_k(n_1, \ldots, n_k) \cong C_n$ if and only if $n = k$. Now assume that $n \geq 5$ and $G \not\cong C_n$ ( $G \not\cong C_k$). We have one of the following cases.

(i) For some $i \in \{1, \ldots, k\}$, $n_i \geq 2$. Without losing the generality assume that $n_1 \geq 2$. Note that $\alpha(G) \geq n_1 + \alpha(P_{k-1}(n_2, \ldots, n_k))$, where $V(P_{k-1}) = \{v_2, \ldots, v_k\}$ and $E(P_{k-1}) = \{v_2v_3, \ldots, v_{k-1}v_k\}$. Since $P_{k-1}(n_2, \ldots, n_k)$ is a tree
of order $n - n_1 - 1$ (by the proof of Lemma 4), $\alpha(P_{k-1}(n_2, \ldots, n_k)) \geq \lceil \frac{n-n_1-1}{2} \rceil$. Hence

$$\alpha(G) \geq n_1 + \left\lceil \frac{n-n_1-1}{2} \right\rceil = \left\lceil \frac{n+n_1-1}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil > \left\lceil \frac{n}{2} \right\rceil = \alpha(C_n).$$

Thus $\alpha(G) > \alpha(C_n)$. Since the coefficients of independence polynomials are positive, for sufficiently large $x$ we have $I(G, x) > I(C_n, x)$.

(ii) For $i = 1, \ldots, k$, $n_i \in \{0, 1\}$ and $n$ is odd. Since $G \not\cong C_n$, for some $i$, $n_i = 1$. Without losing the generality let $n_1 = 1$. Since $n$ is odd, similar to part (i), $\alpha(G) \geq 1 + \left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil = \alpha(C_n)$. Thus the result follows.

(iii) For $i = 1, \ldots, k$, $n_i \in \{0, 1\}$ and $n$ is even. Since $G \not\cong C_n$, for some $r$, $n_r = 1$. First suppose that there is only one $i$ such that $n_i = 1$. Without losing the generality assume that $n_1 = \cdots = n_{k-1} = 0$ and $n_k = 1$. Hence $k = n - 1$. In other words, $G \cong C_{n-1}(0, \ldots, 0, 1)$. Hence $\alpha(G) = \frac{n}{2}$. Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_1, v_{n-1}v_n\}$. Since $n \geq 6$, $\{v_1, v_3, \ldots, v_{n-3}, v_n\}$, $\{v_1, v_3, \ldots, v_{n-5}, v_{n-2}, v_n\}$ and $\{v_2, v_4, \ldots, v_{n-2}, v_n\}$ are three independent sets of $G$ with cardinality $\frac{n}{2}$. Hence $s(G, \frac{n}{2}) \geq 3$. On the
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Figure 2. The graph $C_4(1, 1, 2, 3) = G(H_1, H_2, H_3, H_4; u_1, u_2, u_3, u_4)$, where $G = C_4$ and $u_1$ is the vertex of $H_1 = K_{1,1}$ of degree one, $u_2$ is the vertex of $H_2 = K_{1,1}$ of degree one, $u_3$ is the vertex of $H_3 = K_{1,2}$ of degree two and $u_4$ is the vertex of $H_4 = K_{1,3}$ of degree three.

On the other hand, since $n$ is even, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. By the fact that $\alpha(G) = \alpha(C_n) = \frac{n}{2}$ and $s(G, \frac{n}{2}) > s(C_n, \frac{n}{2})$, for sufficiently large $x$ we obtain $I(G, x) > I(C_n, x)$. Now assume that there are some $i \neq j$ such that $n_i = 1$ and $n_j = 1$. This shows that $G$ has at least two vertices of degree one ($G$ has two pendant vertices). Let $u$ and $v$ be two pendant vertices of $G$. Applying Theorem 2 for vertex $u$ we obtain $I(G, x) = I(G \setminus u, x) + xI(T_1, x)$, where $T_1$ is a tree of order $n - 2$. Using Theorem 2 for $v$ and $G \setminus u$ we have

$$I(G, x) = I(G \setminus \{u, v\}, x) + xI(T_2, x) + xI(T_1, x),$$

where $T_2$ is a tree of order $n - 3$. Hence for $x \geq 0$, $I(G, x) > xI(T_2, x) + xI(T_1, x)$.

Using Lemma 4 for trees $T_1$ and $T_2$ we obtain that for sufficiently large $x$,

$$I(G, x) > xx^\left(\frac{n-3}{2}\right) + 2xx^{\frac{n-2}{2}} = 3x^\frac{n}{2}.$$

On the other hand, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. Hence for sufficiently large $x$, $3x^\frac{n}{2} > I(C_n, x)$. Thus for sufficiently large $x$, $I(G, x) > 3x^\frac{n}{2} > I(C_n, x)$. The proof is complete.

3. Graphs Whose Independence Polynomials Coincide with Independence Polynomials of Cycles

In this section we study the graphs $G$ such that $I(G, x) = I(C_n, x)$, where $n \geq 3$. We show that there is only one connected graph $G \cong C_n$ satisfying $I(G, x) = I(C_n, x)$. Let $n \geq 4$ be an integer. By $D_n$ we mean the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{v_{n-2}v_n\}$, see Figure 3. In addition by $D_3$ we mean the cycle $C_3$. The next result shows that the independence polynomials of $C_n$ and $D_n$ are the same.
Lemma 6. Let \( n \geq 4 \) be an integer. Then

(i) \( I(C_n, x) = I(C_{n-1}, x) + x I(C_{n-2}, x) \), where \( C_2 \) is the path \( P_2 \).

(ii) \( I(C_n, x) = I(D_n, x) \).

Proof. It is easy to check the result for \( n = 4 \). Thus let \( n \geq 5 \). Using Theorem 2 for one of the vertices of \( C_n \) we obtain that

\[
I(C_n, x) = I(P_{n-1}, x) + x I(P_{n-3}, x).
\]

On the other hand, by Theorem 2 for one of the pendant vertices of \( P_t \) we have

\[
I(P_t, x) = I(P_{t-1}, x) + x I(P_{t-2}, x), \text{ for } t \geq 2, \text{ where } I(P_0, x) = 1.
\]

Using equations (1) and (2) one can see that

\[
I(C_n, x) = I(P_{n-2}, x) + x I(P_{n-4}, x) + x I(P_{n-3}, x) + x I(P_{n-5}, x).
\]

So by equation (1) the first part is proved. Now we prove the second part. Using Theorem 2 for the vertex \( v_n \) of \( D_n \) (see Figure 3) we obtain that \( I(D_n, x) = I(P_{n-1}, x) + x I(P_{n-3}, x) \). Hence by equation (1), \( I(D_n, x) = I(C_n, x) \). The proof is complete.

We recall that a unicyclic graph is a graph with exactly one cycle. The next result shows that among all connected unicyclic graphs the cycles have the smallest independence polynomials.

Theorem 7. Let \( G \) be a connected unicyclic graph of order \( n \). Assume that \( G \not\cong C_n \) and \( G \not\cong D_n \). Then for sufficiently large \( x \) we have \( I(G, x) > I(C_n, x) \).

Proof. Assume that \( H \) is a connected unicyclic graph of order \( n \). Thus \( n \geq 3 \).

If \( n = 3 \), then \( H \cong C_3 \). If \( n = 4 \), then \( H \cong C_4 \) or \( H \cong D_4 \). If \( n = 5 \), then \( H \cong C_5 \) or \( H \cong D_5 \) or \( H \cong C_4(1,0,0,0) \) or \( H \cong C_3(2,0,0) \) or \( H \cong C_3(1,1,0) \). So by the fact that \( G \) is unicyclic and \( G \not\cong C_n \) and \( G \not\cong D_n \) we obtain that \( n \geq 5 \). We use induction on \( n \) to prove the result. If \( n = 5 \), then \( G \cong C_4(1,0,0,0) \) or \( G \cong C_3(2,0,0) \) or \( G \cong C_3(1,1,0) \). One can see that

\[
I(C_4(1,0,0,0), x) = 1 + 5x + 5x^2 + x^3, \quad I(C_3(2,0,0), x) = 1 + 5x + 5x^2 + 2x^3
\]
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$I(C_3(1, 1, 0), x) = 1 + 5x + 5x^2 + x^3$. On the other hand $I(C_5, x) = 1 + 5x + 5x^2$.

Thus the result holds for $n = 5$.

Now assume that $n \geq 6$. Suppose that the length of the unique cycle of $G$
is $k$. Assume that $v_1, \ldots, v_k$ are the vertices of this cycle. Since $G$ is unicyclic
there are some trees $T_1, \ldots, T_k$ such that $G = C_k(T_1, \ldots, T_k; v_1, \ldots, v_k)$. If each
tree $T_1, \ldots, T_k$ is a star, then by Lemma 5 the result follows. Now without losing
the generality assume that $T_1$ is not a star. Let $u_1$ be a pendant vertex of $T_1$
which has the maximum distance from $v_1$ among all pendant vertices of $T_1$. We
consider the three following cases for $G \setminus u_1$.

(i) Assume that $G \setminus u_1$ is the cycle $C_{n-1}$. Hence $G = C_{n-1}(1, 0, \ldots, 0)$ and
$T_1 = P_2$, a contradiction (since $T_1$ is not a star). Thus this case does not happen.

(ii) Assume that $G \setminus u_1$ is the graph $D_{n-1}$. Hence $G \cong D_n$ or $G \cong H$, where
$H$ is obtained by identifying the pendant vertex of $D_{n-2}$ with the non-neighbor
vertex of $P_3$. Thus it suffices to check the result for $H$. Let $z$ be a pendant vertex
of $H$. Thus $H \setminus z \cong D_{n-1}$ and $H \setminus N[z] \cong D_{n-3} + K_1$. Hence by Theorems 1 and
2, $I(H, x) = I(H \setminus z, x) + xI(H \setminus N[z], x) = I(D_{n-1}, x) + x(1 + x)I(D_{n-3}, x)$. So
by the second part of Lemma 6 we obtain

\[(3) \quad I(H, x) = I(C_{n-1}, x) + x(1 + x)I(C_{n-3}, x).\]

On the other hand, by the first part of Lemma 6 for $n \geq 7$, $I(C_{n-3}, x) = I(C_{n-4}, x) + xI(C_{n-5}, x)$. This shows that for $x > 0$, $I(C_{n-3}, x) > I(C_{n-4}, x)$
(this inequality also holds for $n = 6$, where $C_2$ is the path $P_2$). Hence for $x > 0$,$xI(C_{n-3}, x) > xI(C_{n-4}, x)$. Thus for every $x > 0$ we have

\[(1 + x)I(C_{n-3}, x) = I(C_{n-3}, x) + xI(C_{n-3}, x) > I(C_{n-3}, x) + xI(C_{n-4}, x).\]

Therefore by the first part of Lemma 6 we obtain that

\[(4) \quad for ~ x > 0, (1 + x)I(C_{n-3}, x) > I(C_{n-2}, x).\]

The equations (3) and (4) show that for $x > 0$, $I(H, x) > I(C_{n-1}, x) + xI(C_{n-2}, x)$.
Hence by the first part of Lemma 6 for every $x > 0$, $I(H, x) > I(C_n, x)$.

(iii) Suppose that $G \setminus u_1 \not\cong C_{n-1}$ and $G \setminus u_1 \not\cong D_{n-1}$. Since $G \setminus u_1$ is a connected
unicyclic graph of order $n - 1$, by the induction hypothesis for sufficiently large $x$,$I(G \setminus u_1, x) > I(C_{n-1}, x)$. As we defined above, $u_1$ is a pendant vertex of $T_1$ which
has the maximum distance from $v_1$ among all pendant vertices of $T_1$. Assume
that $w_1$ is the neighbor of $u_1$. Since $T_1$ is not a star, $d(u_1, w_1) \geq 2$. We note
that $w_1 \neq v_1$. Let $deg(w_1) = t + 1$. Thus $t \geq 1$. By the definition of $u_1$, exactly $t$
neighbors of $w_1$ have degree one. Hence $G \setminus N[u_1]$ is the union of a unicyclic
graph of order $n - t - 1$, say $L$, with exactly $t - 1$ isolated vertices. In other words,$G \setminus N[u_1] = L + (t-1)K_1$. Hence by Theorem 1, $I(G \setminus N[u_1], x) = I(L, x)(1+x)^{t-1}$.
On the other hand, by the induction hypothesis for sufficiently large $x$, $I(L, x) \geq
$I(C_{n-t-1}, x)$ (if $L \neq C_{n-t-1}$ and $L \neq D_{n-t-1}$, $I(L, x) > I(C_{n-t-1}, x)$ for large $x$). Since $n \geq t + 4$, similar to the previous part one can see that for $x > 0$,
$(1 + x)I(C_{n-t-1}, x) > I(C_{n-t}, x)$. Hence for $x > 0$, $(1 + x)^2I(C_{n-t-1}, x) > (1 + x)I(C_{n-t}, x)$. Similarly for $x > 0$, $(1 + x)I(C_{n-t}, x) > I(C_{n-t+1})$. By applying this method $t - 1$ times, we obtain that if $t \geq 2$,
\[
(5) \quad \text{for } x > 0, \ (1 + x)^{t-1}I(C_{n-t-1}, x) > I(C_{n-2}, x).
\]
Hence for $t \geq 1$ we conclude that
\[
(6) \quad \text{for } x > 0, \ (1 + x)^{t-1}I(C_{n-t-1}, x) \geq I(C_{n-2}, x).
\]
The equation (6) shows that for sufficiently large $x$,
\[
I(G \setminus N[u_1], x) = (1 + x)^{t-1} \geq (1 + x)^{t-1}I(C_{n-t-1}, x) \geq I(C_{n-2}, x).
\]
Since for large $x$, $I(G \setminus u_1, x) > I(C_{n-1}, x)$, by Theorem 2, the equation (5) and
the first part of Lemma 6, we find that for large $x$,
\[
I(G, x) = I(G \setminus u_1, x) + xI(G \setminus N[u_1], x) > I(C_{n-1}, x) + xI(C_{n-2}, x) = I(C_n, x).
\]
The proof is complete. 

Now we are in a position to prove the main result of this section.

**Theorem 8.** Let $n \geq 3$ be an integer. Assume that $G$ is a connected graph such
that $I(G, x) = I(C_n, x)$. Then $G \cong C_n$ or $G \cong D_n$.

**Proof.** Since $I(G, x) = I(C_n, x)$ and $C_n$ has $n$ vertices and $n$ edges, by Remark 3
we find that $G$ has exactly $n$ vertices and $n$ edges. Since the number of vertices and
the number of edges of $G$ are the same and $G$ is connected, $G$ is unicyclic. If $G \cong C_n$ or $G \cong D_n$, then by Theorem 7 for large $x$ we have $I(G, x) > I(C_n, x)$,
a contradiction. This completes the proof.

Let $n \geq 3$ be an integer. One might ask whether there is a disconnected graph $G$
satisfying $I(G, x) = I(C_n, x)$. We check this question for $n \leq 9$.

**Remark 9.** Let $3 \leq n \leq 9$ and $G$ be a graph of order $n$. Assume that $I(G, x) = I(C_n, x)$. We find that if $n \in \{3, 4, 5, 7, 8\}$, then $G \cong C_n$ or $G \cong D_n$ (see
Theorem 8). We obtain that $I(G, x) = I(C_n, x)$ if and only if $G \in \{C_6, D_6, K_2 + K_4 \setminus e\}$, where $e$ is an edge of $K_4$. We find that $I(G, x) = I(C_9, x)$ if and only if
$G \in \{C_9, D_9, H_1, H_2, H_3\}$, where $H_1, H_2$ and $H_3$ have been shown in Figure 4. In
fact $I(C_9, x) = 1 + 6x + 9x^2 + 2x^3 = (1 + 4x + x^2)(1 + 2x) = I(K_4 \setminus e, x)I(K_2, x)$ and
$I(C_9, x) = 1 + 9x + 27x^2 + 30x^3 + 9x^4 = (1 + 6x + 9x^2 + 3x^3)(1 + 3x)$. These examples
show that the structure of all non-connected graphs $G$ with $I(G, x) = I(C_m, x)$
is not clear, where $m \geq 10$. 

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Figure 4. All non-connected graphs $G$ such that $I(G, x) = I(C_n, x)$.

We finish the paper by the following problem.

**Problem.** Let $n \geq 10$ be an integer. Find all non-connected graphs $G$ such that $I(G, x) = I(C_n, x)$.

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**References**


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