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ON THE NUMBER OF DISJOINT 4-CYCLES IN REGULAR TOURNAMENTS [1]

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Abstract

In this paper, we prove that for an integer $r \geq 1$, every regular tournament T of degree $3r-1$ contains at least $\frac{21}{16}r - \frac{10}{3}$ disjoint directed 4-cycles. Our result is an improvement of Lichiardopol's theorem when taking $q = 4$ [Discrete Math. **310** (2010) 2567–2570]: for given integers $q \geq 3$ and $r \geq 1$, a tournament T with minimum out-degree and in-degree both at least $(q-1)r-1$ contains at least r disjoint directed cycles of length q.

Keywords: regular tournament, C_4-free, disjoint cycles.

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1. Introduction

This paper considers only digraphs. For a digraph D, we write $V(D)$ for the vertex set of D, and the order of D is the cardinality of $V(D)$. We write $A(D)$ for the set of the arcs of D. Two or several subgraphs are *independent* or *disjoint* if they are pairwise vertex-disjoint.

We say that a vertex y is an *out-neighbor* (*in-neighbor*) of a vertex x if (x, y) (respectively (y, x)) is an arc of D. The number of out-neighbors of x is the *out-degree* $d^+(x)$ of x, and the number of in-neighbors of x is the *in-degree* $d^-(x)$ of x. The *minimum out-degree* $\delta^+(D)$ of D is the smallest of the out-degrees of the vertices of D, and the *minimum in-degree* $\delta^-(D)$ of D is the smallest of the in-degrees of the vertices of D.

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A *path* of length m of a digraph D is a sequence P with $P = (x_1, \ldots, x_{m+1})$ of distinct vertices of D such that $(x_i, x_{i+1}) \in A(D)$ for $1 \leq i \leq m$. If $\{x_1, \ldots, x_{m+1}\} = V(D)$, then P is a Hamiltonian path. A *cycle* of length m in D is a sequence C with $C = (x_1, \ldots, x_m, x_1)$ such that the vertices $x_1, \ldots, x_m$ are distinct, $(x_i, x_{i+1}) \in A(D)$ for $1 \leq i \leq m-1$, and $(x_m, x_1) \in A(D)$. If $\{x_1, \ldots, x_m\} = V(D)$, then C is a Hamiltonian cycle. A cycle of length 3 is a *triangle*. A triangle (x, y, z, x) will often be denoted by (x, u, x), where u is the arc (y, z).

A *tournament* is a digraph T such that for any two distinct vertices x and y, exactly one of the ordered pairs (x, y) and (y, x) is an arc of T. A *regular tournament* of degree d is a tournament T such that $d^+(x) = d^-(x) = d$ for every vertex x. Necessarily the order of T is $2d + 1$. For a subset S of $V(T)$, $T[S]$ denotes the *subtournament* induced by the vertices of S.

It is well-known (Redei's Theorem) that any tournament contains a Hamiltonian path, and (Camion's Theorem) a tournament is strong if and only if it contains a Hamiltonian cycle. It is also known (Moon's Theorem) that a strong tournament T of order $|T|$ is pancyclic, i.e., it has cycles of all lengths $3, \ldots, |T|$. In particular this means that if C is a q-cycle of T, then the tournament $T[V(C)]$ has cycles of all lengths $3, \ldots, q$. A *C_q-free tournament* is a tournament T without a q-cycle.

In 1981, Bermond and Thomassen [3] conjectured that for any positive integer r, any digraph of minimum out-degree at least $2r - 1$ contains at least r disjoint directed cycles. It is trivially true when $r = 1$. It was proved by Thomassen [8] when $r = 2$ in 1983. The case $r = 3$ was proved by Lichiardopol *et al.* in [5]. It is still open for large values of r. In 2014, Bang-Jensen *et al.* proved the conjecture for tournaments in [2]. Lichiardopol proposed a conjecture for tournaments [7]: for given integers $q \geq 3$ and $r \geq 1$, a tournament T with minimum out-degree at least $(q-1)r - 1$ contains at least r disjoint q-cycles. In 2012, Lichiardopol [6] proved that for an integer $r \geq 1$, every regular tournament T of degree $2r - 1$ contains at least $\frac{7}{6}r - \frac{7}{3}$ disjoint directed cycles. By pancyclic property of tournaments, the following is easy to see.

Theorem 1.1. *For an integer $r \geq 1$, every regular tournament T of degree $2r - 1$ contains at least $\frac{7}{6}r - \frac{7}{3}$ disjoint triangles.*

We consider the number of 4-cycles in a regular tournament and prove the following theorem.

Theorem 1.2. *For an integer $r \geq 1$, every regular tournament T with degree $3r - 1$ contains at least $\frac{21}{16}r - \frac{10}{3}$ disjoint 4-cycles.*

In 2012, Lichiardopol [7] proved the following theorem.

Theorem 1.3 ([7]). *For given integers $q \geq 3$ and $r \geq 1$, a tournament T with min $\{\delta^+(T), \delta^-(T)\} \geq (q-1)r - 1$ contains at least r disjoint q-cycles.*

If we take $q = 4$, it is easy to see

Theorem 1.4. *For an integer $r \geq 1$, every regular tournament T with degree $3r - 1$ contains at least r disjoint 4-cycles.*

Our result improves this lower bound to $\frac{21}{16}r - \frac{10}{3}$.

There are many analogous results on bipartite tournaments, for example, Bai *et al.* in [1] proved the following theorem.

Theorem 1.5 ([1]). *Let BT be a bipartite tournament with minimum out-degree at least $qr-1$ and let $t_1, \ldots, t_r \in [4, 2q]$ be any r even integers. Then BT contains r disjoint cycles of length $t'_1, \ldots, t'_r$ such that $t'_i = t_i$ for $t_i = 0 \pmod 4$ and $t'_i \in \{t_i, t_i + 2\}$ for $t_i = 2 \pmod 4$, where $1 \leq i \leq r$.*

2. Lemma

In this section, we list a lemma to prove Theorem 1.2.

Lemma 2.1. *Let M be a proper subset of N with $|N| = n$ and $|M| = m$. Suppose that $T[N]$ is C_4-free and $P = (x_1, x_2, \ldots, x_{n-1}, x_n)$ is a Hamiltonian path of $T[N]$. If $\{x_1, x_2, x_{n-1}, x_n\} \subseteq M$, then there is a Hamiltonian path $Q = (y_1, \ldots, y_m)$ of $T[M]$ such that $y_1 = x_1$, $y_m = x_n$.*

Proof. We construct Q from P by deleting vertices that are not contained in M by the following two steps.

Step 1. (1) If there exists x_i for $i \geq 3$ such that none of $x_i, x_{i+1}, \ldots, x_j$ belongs to M $(j \geq i+1)$, delete $x_i, x_{i+1}, \ldots, x_j$ from P.

(2) If there exists x_i for $i \geq 3$ such that $x_i \notin M, x_{i-1}, x_{i+1} \in M$ and $(x_{i-1}, x_{i+1}) \in A(T)$, delete x_i from P. Do (1) and (2) until there are no such vertices.

We claim that after Step 1 the remaining vertices can still form a path as the prior order. It is obvious for Step 1(2). For Step 1(1), we can prove that $(x_{i-1}, x_{j+1}) \in A(T)$. Suppose on the contrary that $(x_{j+1}, x_{i-1}) \in A(T)$, then $\{x_{i-1}, x_i, \ldots, x_j, x_{j+1}, x_{i-1}\}$ is a cycle of length at least 4. By property of pancyclic, it has a 4-cycle, a contradiction (since $T[N]$ is C_4-free). Denote this new path by $Q' = (z_1, \ldots, z_l)$. Clearly, Q' has the following property: if $x_i \notin M$ then $x_{i-1}, x_{i+1} \in M$ and $(x_{i+1}, x_{i-1}) \in A(T)$. Since $\{x_1, x_2, x_{n-1}, x_n\} \subseteq M$, we have $z_1 = x_1, z_2 = x_2, z_{l-1} = x_{n-1}, z_l = x_n$.

Step 2. If none of $z_j, z_{j+2}, \ldots, z_{j+2i}$ belongs to $M (i \geq 0)$, but $z_{j-1}, z_{j+2i+1} \in M$, we delete $z_j, z_{j+2}, \ldots, z_{j+2i}$ from Q' and replace the segment $(z_{j-2}, \ldots, z_{j+2i+2})$

by $(z_{j-2},\ z_{j+2i+1},\ z_{j+2i-1}, \ldots, z_{j+1},\ z_{j-1},\ z_{j+2i+2})$. Repeat the procedure until there are no such vertices.

Since $z_1 = x_1, z_2 = x_2, z_{l-1} = x_{n-1}, z_l = x_n$, $j \geq 3$ and $j + 2i \leq l - 2$, we have $j - 2 \geq 1$ and $j + 2i + 2 \leq l$. Denote the path after Step 2 by $Q = (y_1, \ldots, y_m)$. Then it is the desired Hamiltonian path. ■

3. Proof of Theorem 1.2

The proof of this theorem is inspired mainly by the proof of the main theorem in [6]. We begin with a preliminary result. Let (x, y) be an arc of a tournament T of order n with $n \geq 3$. We define:

$$B(x, y) = \{z \in V(T) : (x, z) \in A(T), (y, z) \in A(T)\},$$

$$E(x, y) = \{z \in V(T) : (z, x) \in A(T), (y, z) \in A(T)\},$$

$$F(x, y) = \{z \in V(T) : (x, z) \in A(T), (z, y) \in A(T)\}.$$

Observe that $E(x, y)$ is the set of vertices z such that x, y and z form a triangle. We denote by $b(x, y), e(x, y)$ and $f(x, y)$ the respective cardinalities of these three sets. It is easy to see that $d^+(x) = b(x, y) + f(x, y) + 1$ and $d^+(y) = b(x, y) + e(x, y)$. It follows that $e(x, y) = f(x, y) + d^+(y) - d^+(x) + 1$. Hence if T is regular, then we have

$$e(x, y) = f(x, y) + 1. \tag{1}$$

If $u = (x, y)$, then $E(x, y), e(x, y), F(x, y)$ and $f(x, y)$ will also be denoted by $E(u)$, $e(u)$, $F(u)$ and $f(u)$, respectively.

The order of the regular tournament T of degree $3r - 1$ is $6r - 1$. By Theorem 1.4, T contains at least r disjoint 4-cycles. When $r \leq 10$, it holds that $r \geq \frac{21}{16}r - \frac{10}{3}$, and so Theorem 1.2 holds in this case. So from now on, we suppose $r \geq 11$.

Let s be the maximum number of disjoint 4-cycles of T. In particular, let $S = \{C_1, \ldots, C_s\}$ be a set of s disjoint 4-cycles with $C_i = (a_i, b_i, u_i, v_i, a_i)$ for $1 \leq i \leq s$. Let us define $V_1 = \bigcup_{1 \leq i \leq s} V(C_i)$ and $V_2 = V(T) \setminus V_1$. Let T_s be the subtournament of T induced by the vertices of V_2. Its vertices can be ordered into a Hamiltonian path $(x_1, \ldots, x_t)$ where $t = 6r - 1 - 4s$. Note that T_s is a C_4-free tournament by the maximality of s.

Suppose first that $t \leq 20$. This means $6r - 1 - 4s \leq 20$, so $s \geq \frac{3}{2}r - \frac{21}{4}$. Since $r \geq 11$ implies $\frac{3}{2}r - \frac{21}{4} \geq \frac{21}{16}r - \frac{10}{3}$, it follows that $s \geq \frac{12}{16}r - \frac{10}{3}$ and Theorem 1.2 holds in this case.

So, from now on, we suppose that $t \geq 21$ (and $r \geq 11$).

Since T_s is C_4-free, it is easy to see the following.

Claim 3.1. *For* $1 \leq i \leq t-3$, $j \geq i+3$, $(x_i, x_j) \in A(T)$.

Since $t \geq 21$, by Claim 3.1, it is easy to see that $\omega_i = (x_i, x_{t+1-i}) \in A(T)$ for each $1 \leq i \leq 7$. Denote by Ω_s the set of the independent arcs $\omega_1, \ldots, \omega_7$.

Claim 3.2. *For* $1 \leq i \leq 7$, $f(\omega_i) \geq t-2i-2$, $e(\omega_i) \geq t-2i-1$.

Proof. Since T_s is C_4-free, by Claim 3.1, there are at most two vertices (they are x_{i+2}, x_{t-i-1}) between x_i and x_{t+1-i} that do not belong to $F(\omega_i)$. So we get $f(\omega_i) \geq t-2i-2$. By equation (1), we get $e(\omega_i) \geq t-2i-1$. □

Put $e(\Omega_s) = \sum_{1 \leq i \leq 7} e(\omega_i)$. Then we have

Claim 3.3. $e(\Omega_s) \geq 7t - 63$.

Proof. By Claim 3.2, we get $e(\omega_i) \geq t-2i-1$. It follows that $e(\Omega_s) = \sum_{1 \leq i \leq 7} e(\omega_i) \geq \sum_{1 \leq i \leq 7}(t-2i-1)$, so $e(\Omega_s) \geq 7t-63$. □

Let $W = \{x_8, \ldots, x_{t-7}\}$ be the set of vertices between x_7 and x_{t-6}, $F_W(\omega_i)$ denote the vertices in W that belong to $F(\omega_i)$, and $f_W(\omega_i) = |F_W(\omega_i)|$. Since $t \geq 21$, there are at least seven vertices between x_7 and x_{t-6}. Similarly to the proof of Claim 3.2, there are at least five of these vertices in M belonging to $F(\omega_i)$, for each $1 \leq i \leq 7$, i.e., $f_W(\omega_i) \geq 5$.

Claim 3.4. *For each* $1 \leq i \leq 7$, $E(\omega_i) \cap V_2 = \emptyset$.

Proof. If $E(\omega_i) \cap V_2 \neq \emptyset$, then there exists a vertex x_j such that $x_j \in E(\omega_i) \cap V_2$. Since $f_W(\omega_i) \geq 5$, there is a vertex x_k with $k \neq j$ such that $x_k \in F_W(\omega_i)$. Thus $(x_i, x_k, x_{t+1-i}, x_j, x_i)$ is a 4-cycle of T_s, a contradiction. □

By Claim 3.4, the set $E(\omega_i)$ does not contain any vertex of T_s.

For a vertex $x \in V_1$, let $E_{\Omega_s}(x)$ denote the set of the arcs $\omega_i \in \Omega_s$ such that $x \in E(\omega_i)$, and put $e_{\Omega_s}(x) = |E_{\Omega_s}(x)|$. For a 4-cycle C_i of S, let $e_{\Omega_s}(C_i) = \sum_{x \in V(C_i)} e_{\Omega_s}(x)$.

We then get $e(\Omega_s) = \sum_{x \in V_1} e_{\Omega_s}(x) = \sum_{1 \leq i \leq s} e_{\Omega_s}(C_i)$, by double-counting, and interchanging the order of summation. Then we get

Claim 3.5. *If a vertex* v *of a* 4*-cycle* C *of* S *satisfies* $e_{\Omega_s}(v) \geq 2$*, then* $e_{\Omega_s}(w) = 0$ *for every vertex* w *of* C *distinct from* v.

Proof. If $e_{\Omega_s}(w) > 0$, then there exists an arc ω_j of Ω_s such that $w \in E(\omega_j)$. Since $e_{\Omega_s}(v) \geq 2$, there exists an arc ω_k of Ω_s with $k \neq j$ such that $v \in E(\omega_k)$. Since $f_W(\omega_j) \geq 5$ and $f_W(\omega_k) \geq 5$, there exist two distinct vertices $x, y \in W$ such that $x \in F_W(\omega_j), y \in F_W(\omega_k)$. Clearly, $C' = (w, x_j, x, x_{t+1-j}, w)$ and $C'' = (v, x_k, y, x_{t+1-k}, v)$ are two disjoint 4-cycles. Now $(S \setminus \{C\}) \cup \{C', C''\}$ is a collection of $s+1$ disjoint 4-cycles, which is impossible by the maximality of s. So the result is proved. □

Let $U_s = \{x \in V_1 : e_{\Omega_s}(x) \geq 4\}$, and let $u_s = |U_s|$. Clearly, this claim implies that every 4-cycle C of S which is disjoint from U_s, satisfies $e_{\Omega_s}(C) \leq 4$. It implies also that every 4-cycle of S contains at most one vertex of U_s.

Now, we choose S such that u_s is as large as possible. Suppose first that $u_s = 0$. Since $e(\Omega_s) = \sum_{1 \leq i \leq s} e_{\Omega_s}(C_i)$, from Claim 3.3 and Claim 3.5, we get $7t - 63 \leq 4s$. That is $7(6r - 1 - 4s) - 63 \leq 4s$, so $32s \geq 42r - 70$. Hence $s \geq \frac{21}{16}r - \frac{35}{16} > \frac{21}{16}r - \frac{10}{3}$. Therefore, Theorem 1.2 holds in this case.

Suppose now $u_s > 0$. By Claim 3.5, without loss of generality, we may suppose that the u_s vertices of U_s are $a_1, \ldots, a_{u_s}$. We denote $\Delta_s = \{C_1, \ldots, C_{u_s}\}$. Note that $\Delta_s \subset S$ when $u_s < s$. For each 4-cycle C_i of Δ_s we have $e_{\Omega_s}(C_i) = e_{\Omega_s}(a_i) \leq 7$.

We denote $U'_s = \bigcup_{1 \leq i \leq u_s} \{b_i, u_i, v_i\}$ (where $V(C_i) = \{a_i, b_i, u_i, v_i\}$) and $V'_s = V_2 \cup U'_s$. Clearly, $|V'_s| = 3u_s + t$.

Claim 3.6. *The subtournament induced by the set V'_s is C_4-free.*

Proof. On the contrary, let C' be a 4-cycle of $T[V'_s]$ with $C' = (w, x, y, z, w)$. Since $T[V_2]$ is C_4-free, two cases are possible.

Case 1. C' contains exactly one vertex of U'_s. Let w be this vertex; there exists i with $1 \leq i \leq u_s$ such that $w \in V(C_i)$, and $w \neq a_i$. Since $e_{\Omega_s}(a_i) \geq 4$, there exists an arc ω_j of $E_{\Omega_s}(a_i)$ disjoint from x, y, z. Since $f_W(\omega_j) \geq 5$, there exists a vertex $a \in W$ distinct from x, y, z such that $a \in F_W(\omega_j)$. Clearly, C' and C'', where $C'' = (a_i, x_j, a, x_{t+1-j}, a_i)$, are disjoint 4-cycles. Now $(S \setminus \{C_i\}) \cup \{C', C''\}$ is a collection of $s + 1$ disjoint 4-cycles, a contraction.

Case 2. C' contains at least two vertices of U'_s. Denote the set of these vertices by Γ. Then $2 \leq |\Gamma| \leq 4$. Let m be the number of the 4-cycles of Δ_s containing at least one vertex of Γ. Then $1 \leq m \leq |\Gamma| \leq 4$. Without loss of generality, we may suppose that $C_1, \ldots, C_m$ with $C_i = (a_i, b_i, u_i, v_i, a_i)$ for $1 \leq i \leq m$ are these 4-cycles. Note that $a_i \in U_s$. Since $e_{\Omega_s}(a_i) \geq 4$, there exist m independent arcs, say $\omega_1, \ldots, \omega_m$, of Ω_s which are disjoint with $V(C') \setminus \Gamma$, such that $\omega_i \in e_{\Omega_s}(a_i)$ for each $1 \leq i \leq m$. Since $f_W(\omega_i) \geq 5$ (for each $1 \leq i \leq m$), there exist m vertices $\gamma_1, \ldots, \gamma_m$ of W distinct from the vertices of $V(C') \setminus \Gamma$ such that $\gamma_i \in F_W(\omega_i)$. Clearly, $C^i = (a_i, x_i, \gamma_i, x_{t+1-i}, a_i)$, $1 \leq i \leq m$, and C' are $m + 1$ disjoint 4-cycles. Now $(S \setminus \{C_1, \ldots, C_m\}) \cup \{C', C^1, \ldots, C^m\}$ is a collection of $s+1$ disjoint 4-cycles, a contraction. □

Since the subtournament $T[V'_s]$ is C_4-free, let $(\alpha_1, \ldots, \alpha_{\gamma_s})$ be a Hamiltonian path of $T[V'_s]$, where $\gamma_s = 3u_s + t = |V'_s|$.

Claim 3.7. *There exists a set S' of s disjoint 4-cycles such that $\{\alpha_1, \alpha_2, \alpha_{\gamma_s - 1}, \alpha_{\gamma_s}\} \subseteq V(T_{s'})$.*

Proof. Let p be the number of the vertices of $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$ which are in U'_s. When $p = 0$, we take $S' = S$ and clearly the result is proved. Now suppose that $p \geq 1$ and let m be the number of the 4-cycles of Δ_s containing at least one vertex of $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$. Without loss of generality, we may suppose that $C_1, C_2, \ldots, C_m$ (with $C_i = (a_i, b_i, u_i, v_i, a_i)$, $1 \leq i \leq m$) are these 4-cycles. Note that $a_i \in U_s$ for each $1 \leq i \leq m$. We have $1 \leq m \leq p \leq 4$ with $m \geq 2$ when $p = 4$. Since $e_{\Omega_s}(a_i) \geq 4$ for each $1 \leq i \leq m$, there exist m independent arcs, without loss of generality, say $\omega_1, \ldots, \omega_m$, of Ω_s with $\omega_i \in E_{\Omega_s}(a_i)$ for each $1 \leq i \leq m$. Since $f_W(\omega_i) \geq 5$, there exist m distinct vertices $y_i \in W$ for each $1 \leq i \leq m$. This yields m disjoint 4-cycles $C'_i = (a_i, x_i, y_i, x_{t+1-i}, a_i)$ for each $1 \leq i \leq m$, and these 4-cycles do not contain any vertex of $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$. Then $S' = (S \setminus \{C_1, \ldots, C_m\}) \cup \{C'_1, \ldots, C'_m\}$ is a set of s disjoint 4-cycles. The vertices $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$ are in $T_{s'}$, and the vertices of $V(T_{s'})$ are vertices of $T[V'_s]$. □

Recall that T_s is the C_4-free subtournament induced by the vertices of T not contained in a 4-cycle of S, and that the vertices of T_s can be ordered into a Hamiltonian path which we denote here by $(x_1^S, \ldots, x_t^S)$. Clearly, this notation (and the other using S as subscript or superscript) is valid for every set of s disjoint 4-cycles.

Let $N = V'_s, M = V(T_{s'}), P = (\alpha_1, \ldots, \alpha_{\gamma_s})$, by Claim 3.7 and Lemma 2.1, it is easy to see that

Claim 3.8. *There exists a set S' of s disjoint 4-cycles such that $x_1^{S'} = \alpha_1$, $x_t^{S'} = \alpha_{\gamma_s}$.*

Now we can achieve the proof of Theorem 1.2. We work on the set S' of s disjoint 4-cycles constructed in Claim 3.7. Here $\Omega_{s'}$ is the set of the independent arcs $\omega_i^{S'}$ with $\omega_i^{S'} = (x_i^{S'}, x_{t+1-i}^{S'})$ for each $1 \leq i \leq 7$.

First, since $e(\omega_1^{S'}) \geq t + 3u_s - 3$, we have $e(\Omega_{s'}) \geq 7t - 63 + 3u_s$.

On the other hand, since $e_{\Omega_{s'}}(C) \leq 7$ when C is a 4-cycle of $\Delta_{s'}$, and $e_{\Omega_{s'}} \leq 4$ when C is not a 4-cycle of $\Delta_{s'}$ (by Claim 3.5), we deduce $e(\Omega_{s'}) \leq 7u_{s'} + 4(s - u_{s'})$. It follows that $7t - 63 + 3u_s \leq 3u_{s'} + 4s$.

As $u_{s'} \leq u_s$ (by the maximality of u_s), it follows that $7t - 63 + 3u_s \leq 3u_s + 4s$. Hence $7t - 63 \leq 4s$, which gives $s \geq \frac{21}{16}r - \frac{35}{16} > \frac{21}{16}r - \frac{10}{3}$. So Theorem 1.2 is proved. ■

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