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ARC-DISJOINT HAMILTONIAN CYCLES IN ROUND DECOMPOSABLE LOCALLY SEMICOMPLETE DIGRAPHS

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Abstract

Let $D = (V, A)$ be a digraph; if there is at least one arc between every pair of distinct vertices of D, then D is a semicomplete digraph. A digraph D is locally semicomplete if for every vertex x, the out-neighbours of x induce a semicomplete digraph and the in-neighbours of x induce a semicomplete digraph. A locally semicomplete digraph without 2-cycle is a local tournament. In 2012, Bang-Jensen and Huang [J. Combin Theory Ser. B 102 (2012) 701–714] concluded that every 2-arc-strong locally semicomplete digraph which is not the second power of an even cycle has two arc-disjoint strong spanning subdigraphs, and proposed the conjecture that every 3-strong local tournament has two arc-disjoint Hamiltonian cycles. According to Bang-Jensen, Guo, Gutin and Volkmann, locally semicomplete digraphs have three subclasses: the round decomposable; the non-round decomposable which are not semicomplete; the non-round decomposable which are semicomplete. In this paper, we prove that every 3-strong round decomposable locally semicomplete digraph has two arc-disjoint Hamiltonian cycles, which implies that the conjecture holds for the round decomposable local tournaments. Also, we characterize the 2-strong round decomposable local tournaments each of which contains a Hamiltonian path P and a Hamiltonian cycle arc-disjoint from P.

Keywords: locally semicomplete digraph, local tournament, round decomposable, arc-disjoint, Hamiltonian cycle, Hamiltonian path.

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1. Terminology and Introduction

In this paper, we consider finite digraph without loops and multiple arcs. The main source for terminology and notation is [1].

For an integer n, $[n]$ will denote the set $\{1, 2, 3, \ldots, n\}$.

Let $D = (V, A)$ be a digraph; if there is an arc from a vertex x to y, we say that x dominates y and denote it by $x \to y$. If V_1 and V_2 are arc-disjoint subsets of vertices of D such that there is no arc from V_2 to V_1 and $a \to b$ for all $a \in V_1$ and $b \in V_2$, then we say that V_1 completely dominates V_2 and denote this by $V_1 \Rightarrow V_2$. We shall use the same notation when A and B are subdigraphs of D. Let $N^-(x)$ (respectively, $N^+(x)$) denote the set of vertices dominating (respectively, dominated by) x in D and say that $N^-(x)$ (respectively, $N^+(x)$) is the in-neighbours of x (respectively, the out-neighbours of x).

Let H be a subdigraph of D; if $V(D) = V(H)$, we say that H is a spanning subdigraph of D. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $X = V(H)$ and denote this by $D\langle X\rangle$. We also use the notation $D - X$, where $X \subseteq V$, for digraph $D\langle V(D)\backslash V(X)\rangle$.

Let D_1, D_2 be two subdigraphs of a digraph D. The union $D_1 \cup D_2$ is the digraph D with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

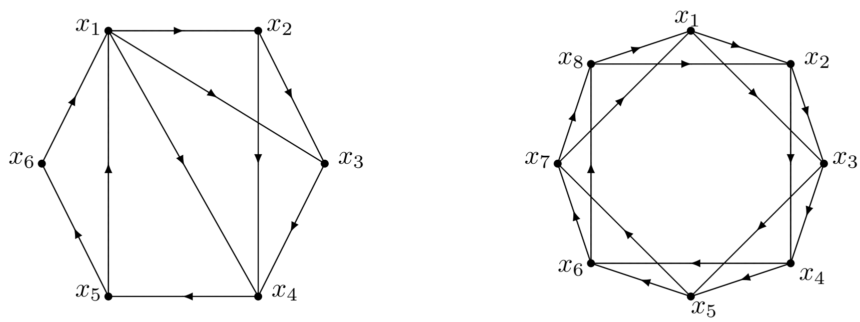
Paths and cycles in a digraph are always directed. Let P be a directed path of digraph D. If $V(P) = V(D)$, then P is a Hamiltonian path of D. Similarly, let C be a directed cycle of digraph D. If $V(C) = V(D)$, then C is a Hamiltonian cycle of D.

Let $P_1, P_2, \ldots, P_q$ be paths which are pairwise vertex-disjoint. If $\mathcal{F}= P_1 \cup P_2 \cup \cdots \cup P_q$ is a spanning subdigraph of D, then $\mathcal{F}$ is a q-path factor of D. Let $C_1, C_2, \ldots, C_q$ be cycles which are pairwise vertex-disjoint. If $\mathcal{F}= C_1 \cup C_2 \cup \cdots \cup C_q$ is a spanning subdigraph of D, then $\mathcal{F}$ is a q-cycle factor of D.

A digraph $D = (V, A)$ is called strongly connected (or just strong) if there exists a path from x to y and a path from y to x in D for every choice of distinct vertices x, y of D, and D is k-arc-strong (respectively, k-strong) if $D-X$ is strong for every subset $X \subseteq A$ (respectively, $X \subseteq V$) of size at most $k-1$. Note that a digraph with only one vertex is strong.

A digraph D is semicomplete if, for every pair x, y of vertices of D, either x dominates y or y dominates x (or both). A digraph D is locally semicomplete if for every vertex x, the out-neighbours of x induce a semicomplete digraph and the in-neighbours of x induce a semicomplete digraph. A semicomplete digraph without 2-cycle is a tournament and a locally semicomplete digraph without 2-cycle is a local tournament.

A digraph R on r vertices is round if we can label its vertices $x_1, x_2, \ldots, x_r$ so that for each i, we have $N_R^+(x_i) = \left\{x_{i+1}, x_{i+2}, \ldots, x_{i+d_R^+(x_i)}\right\}$ and $N_R^-(x_i) = \left\{x_{i-d_R^-(x_i)}, \ldots, x_{i-1}\right\}$ (all subscripts are taken modulo r). Note that every round digraph is locally semicomplete, a round digraph without 2-cycle is a local tournament. If a local tournament R is round then there exists a unique (up to cyclic permutations) labeling of vertices of R which satisfies the properties in the definition. We refer to this as the round labeling of R. See Figure 1(a) for an example of a round digraph R. Observe that the ordering $x_1, x_2, \ldots, x_6$ is a round labeling of R. The second power of a cycle C_n, denoted by C_n^2, is the digraph obtained from C_n by adding the arcs $\{x_i x_{i+2} : i \in [n]\}$, where $C_n = x_1 x_2 \cdots x_n x_1$ and subscripts are modulo n. See Figure 1(b) for the second power of an 8-cycle.



(a) R (b) C_8^2

Figure 1. A round digraph and the second power of an 8-cycle.

Let R be a digraph with vertex set $\{x_i : i \in [r]\}$, and let $D_1, D_2, \ldots, D_r$ be digraphs which are pairwise vertex-disjoint. Let $D = R[D_1, D_2, \ldots, D_r], r \geq 2$, be the new digraph obtained from R by replacing x_i with D_i and adding arc from every vertex of D_i to every vertex of D_j if and only if $x_i \to x_j$ in R. If R is a round digraph and each D_i is a strong semicomplete digraph, it is easy to see that $D = R[D_1, D_2, \ldots, D_r]$ is a locally semicomplete digraph. We call D a round decomposable locally semicomplete digraph. If a round decomposable locally semicomplete digraph $D = R[D_1, D_2, \ldots, D_r]$ has no 2-cycle (i.e., the round digraph R has no 2-cycle and each $D_i, i \in [r]$, is a strong tournament or a single vertex), then we say that D is a round decomposable local tournament.

Locally semicomplete digraphs were introduced in 1990 by Bang-Jensen [2]. The following theorem, due to Bang-Jensen, Guo, Gutin and Volkmann, states a full classification of locally semicomplete digraphs.

Theorem 1.1 [3]. *Let D be a locally semicomplete digraph. Then exactly one of the following possibilities holds.*

(a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, \ldots, D_r]$, where R is a round locally semicomplete on $r \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, \ldots, r$;*

(b) *D is not round decomposable and not semicomplete;*
(c) *D is a semicomplete digraph which is not round decomposable.*

In [3], Bang-Jensen *et al.* also characterized the structure of locally semicomplete digraph which is not round decomposable and not semicomplete. If D is restricted to a local tournament, we have the following result.

Corollary 1.2. *Let D be a local tournament. Then exactly one of the following possibilities holds.*

(a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, \ldots, D_r]$, where R is a round local tournament on $r \geq 2$ vertices and D_i is a strong tournament for $i = 1, 2, \ldots, r$;*
(b) *D is not round decomposable and not a tournament;*
(c) *D is a tournament which is not round decomposable.*

According to the classification of locally semicomplete digraphs, many nice properties of semicomplete digraphs (tournaments) are generalized to locally semicomplete digraphs (local tournaments), see [5–8]. Recently, some new problems on locally semicomplete digraphs, such as the out-arc pancyclicity, the number of Hamiltonian cycles, the kings, the H-force set and so on, were studied in [9–13]. In particular, Bang-Jensen and Huang investigated the decomposition of locally semicomplete digraphs and proved the theorem below.

Theorem 1.3 [4]**.** *A 2-arc-strong locally semicomplete digraph D has two arc-disjoint strong spanning subdigraphs if and only if D is not the second power of an even cycle.*

Meanwhile, they proposed the following conjecture.

Conjecture 1.4 [4]**.** *Every 3-strong local tournament has two arc-disjoint Hamiltonian cycles.*

In this paper, we prove the following theorem in Section 3 which implies that the conjecture holds for the subclass of local tournaments—the round decomposable local tournaments.

Theorem 1.5. *Every 3-strong round decomposable locally semicomplete digraph has two arc-disjoint Hamiltonian cycles.*

Also, in the following theorem, we give a characterization of the 2-strong round decomposable local tournaments each of which contains a Hamiltonian path P and a Hamiltonian cycle arc-disjoint from P. This theorem will be proved in Section 4.

Theorem 1.6. *Every* 2*-strong round decomposable local tournament has a Hamiltonian path and a Hamiltonian cycle which are arc-disjoint if and only if it is not the second power of an even cycle.*

To show the main results, we introduce the following definition and theorem due to Thomassen. A tournament is called transitive if it contains no cycle. It is easy to see that, for a transitive tournament T, there is a unique vertex ordering $v_1, v_2, \ldots, v_n$ of T, such that $v_i \to v_j$ for all $1 \leq i < j \leq n$. A tournament is almost transitive if it is obtained from the transitive tournament T by reversing the arc $v_1 v_n$.

Theorem 1.7 [11]**.** *Every tournament which is strong and which is not an almost transitive tournament of odd order has two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.*

We also use the following facts several times.

Theorem 1.8 [1]**.** *Every strong semicomplete digraph is vertex-pancyclic.*

Theorem 1.9 [7]**.** *A tournament is strong if and only if it has a Hamiltonian cycle.*

2. Preliminaries

In this section we start with the following three lemmas which imply that every strong semicomplete digraph with at least 3 vertices contains a Hamiltonian path Q and a 2-path-factor $P' \cup P''$ arc-disjoint from Q such that the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.

Lemma 2.1. *Let D be a strong semicomplete digraph with at least* 3 *vertices. Then D contains a spanning subdigraph which is a strong tournament.*

Proof. By Theorem 1.8, assume that C is a Hamiltonian cycle of D. Notice that at most one arc of each 2-cycle of D is in C. For each 2-cycle of D, by deleting exactly one arc which is not in C, we obtain a tournament T which is a spanning subdigraph of D and contains a Hamiltonian cycle. Note that a tournament is strong if and only if it has a Hamiltonian cycle. The proof is complete. ■

Lemma 2.2. *Let T be a strong tournament which is not an almost transitive tournament of odd order. Then T contains a Hamiltonian path Q and a* 2*-path-factor $P' \cup P''$ arc-disjoint from Q such that the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.*

Proof. Let $|V(T)| = n$. Since T is a strong tournament which is not an almost transitive tournament of odd order, if $n = 3$, then T is an almost transitive tournament of odd order. So $n \geq 4$. By Theorem 1.7, T contains a pair of arc-disjoint Hamiltonian paths P and Q such that P and Q have distinct initial vertices and distinct terminal vertices. Denote $P = v_1v_2\cdots v_n$, $Q = u_1u_2\cdots u_n$. Then $v_1 \neq u_1$, $v_n \neq u_n$. Let the vertex u_1 of Q correspond to the vertex v_i of P, and the vertex u_n of Q correspond to the vertex v_j of P. Note that $i > 1$, $j < n$ and $i \neq j$. Now we will construct a 2-path-factor $P' \cup P''$ arc-disjoint from Q such that Q, P' and P'' have distinct initial vertices and distinct terminal vertices.

Case 1. $i < j$. Let $P' = v_1v_2\cdots v_i$ and $P'' = v_{i+1}v_{i+2}\cdots v_n$.

Case 2. $i > j, j \neq 1$. Let $P' = v_1v_2\cdots v_{j-1}$ and $P'' = v_jv_{j+1}\cdots v_n$.

Case 3. $i > j, j = 1$ and $i \neq n$. Let $P' = v_1v_2\cdots v_i$ and $P'' = v_{i+1}v_{i+2}\cdots v_n$.

Case 4. $i > j, j = 1$ and $i = n$. Let $P' = v_1v_2$ and $P'' = v_3v_4\cdots v_n$.

Figure 2 shows the construction of the Hamiltonian path Q and the 2-path factor $P' \cup P''$ in the four cases. Notice that in all cases the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices, respectively, i.e., T contains a Hamiltonian path Q and a 2-path factor $P' \cup P''$ arc-disjoint from Q such that Q, P' and P'' have distinct initial vertices and distinct terminal vertices. We are done. ■

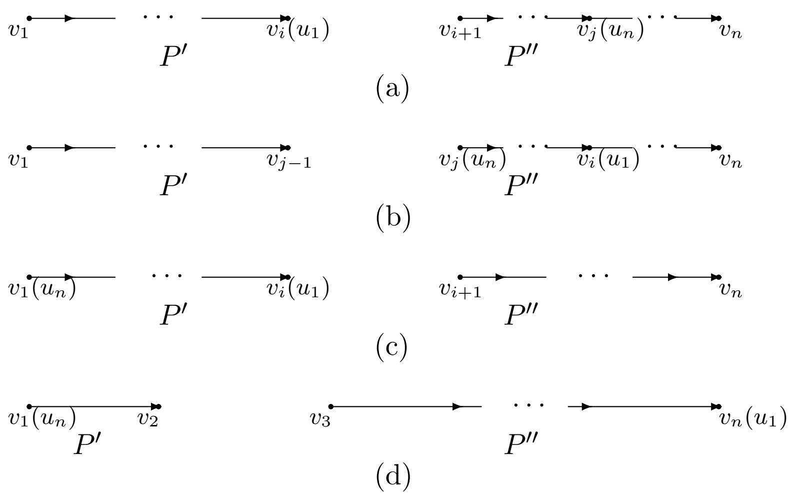


Figure 2. The 2-path factor constructed in the proof of Lemma 2.2.

Lemma 2.3. *Let T be a strong tournament which is an almost transitive tournament of odd order. Then T contains a Hamiltonian path Q and a 2-path factor $P' \cup P''$ arc-disjoint from Q such that the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.*

Proof. Let $|V(T)| = n$ and $V(T) = \{v_1, v_2, \ldots, v_n\}$. Obviously, $n \geq 3$. Since T is an almost transitive tournament of odd order, assume without loss of generality that $v_i \to v_j$ for all $1 \leq i < j \leq n$ except for $v_n \to v_1$. Hence, for arbitrary $i \leq n-2$, we must have $v_i \to v_{i+1}$ and $v_i \to v_{i+2}$.

Case 1. $n = 3$. Let $P' = v_2v_3, P'' = v_1, Q = v_3v_1v_2$. It is clear that $P' \cup P''$ is a 2-path factor of T which is arc-disjoint from Q. And the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.

Case 2. $n > 3$. Let $n = 2k+1$. Suppose that $P' = v_1v_3 \cdots v_{2k-1}v_{2k+1}, P'' = v_2v_4 \cdots v_{2k-2}v_{2k}, Q = v_3v_4 \cdots v_nv_1v_2$. Obviously, $P' \cup P''$ is a 2-path-factor of T which is arc-disjoint from Q. The paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices, respectively.

Figure 3 shows the construction of the Hamiltonian path Q and the 2-path factor $P' \cup P''$ in the two cases of the proof. ■

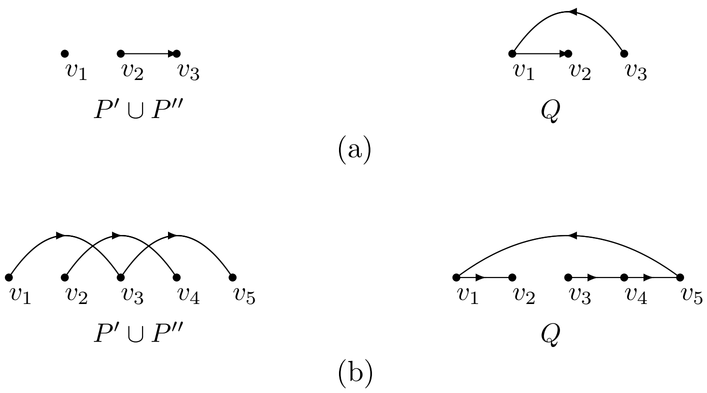


Figure 3. The arc-disjoint 2-path factor $P' \cup P''$ and the Hamiltonian path Q in an almost transitive tournament. In (a), $P' = v_2v_3, P'' = v_1, Q = v_3v_1v_2$. In (b), $P' = v_1v_3v_5, P'' = v_2v_4, Q = v_3v_4v_5v_1v_2$.

The following lemma is also useful in our proof of main results.

Lemma 2.4. *Every 2-strong semicomplete digraph with at least 3 vertices contains two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.*

Proof. Let D be a 2-strong semicomplete digraph. By Lemma 2.1, D contains a strong tournament T as a spanning subdigraph. If T is not an almost transitive tournament of odd order, by Lemma 1.7, we are done. Otherwise, T is an almost transitive tournament of odd order. Assume without loss of generality that $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $v_i \to v_j$ for all $1 \leq i < j \leq n$ except $v_n \to v_1$ in T. In the following, we will construct two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices in D.

Since D is 2-strong, there must exist some arc of the form $v_jv_i, i < j$ besides v_nv_1 in D. For all arcs of the form $v_jv_i, i < j$ except for v_nv_1 in D, we shall consider the following two cases.

Case 1. There is one arc of the form $v_jv_i, j > i+1$ besides v_nv_1 in D. Let $v_jv_i, j > i+1$, which is not v_nv_1, be an arc of D. Now we replace the arc v_iv_j with v_jv_i in T. Then we can get a tournament T' which is a spanning subdigraph of D. Recall that T is an almost transitive tournament of odd order. Then T' is not an almost transitive tournament of odd order. Notice that $C = v_1v_2\cdots v_n$ is still a Hamiltonian cycle of T'. So T' is a strong tournament. By Theorem 1.7, we are done.

Case 2. There is no arc of the form $v_jv_i, j > i+1$ besides v_nv_1 in D. This means that if $v_jv_i, j > i$ is an arc of D, then $j = i+1$. Note that there must exist two arc-disjoint paths from v_n to v_1 in D since D is 2-strong. Then we have $v_{k+1}v_k \in D$ for any $k \in [n-1]$ since otherwise there exists only one path from v_n to v_1 in D, a contradiction. Obviously, $v_1v_2\cdots v_n$ and $v_nv_{n-1}\cdots v_1$ are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. ■

For 3-strong round decomposable locally semicomplete digraphs, the following result is clear.

Lemma 2.5. *Let D be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r], r \geq 2$ is the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong semicomplete digraph or a single vertex. Then*

(a) *when $r = 2$, we have $D_1 \Rightarrow D_2 \Rightarrow D_1$;*
(b) *when $r \geq 3$, for any $i \in [r]$ with $|V(D_i)| \leq 2$, we have $D_{i-1} \Rightarrow D_{i+1}$ (subscripts are modulo r).*

For 2-strong round decomposable local tournaments, we have the similar result.

Lemma 2.6. *Let D be a 2-strong round decomposable local tournament. $D = R[D_1, D_2, \ldots, D_r], r \geq 2$ is the round decomposition of D, where R is a round* *digraph and for each $i \in [r]$, D_i is either a strong tournament or a single vertex. Then $r \geq 3$ and for any $i \in [r]$ with $|V(D_i)| = 1$, we have $D_{i-1} \Rightarrow D_{i+1}$ (subscripts are modulo r).*

3. Proof of Theorem 1.5

Let D be a 3-strong round decomposition locally semicomplete digraph, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of D. In this section, we shall prove Theorem 1.5 in three classes: there exists at least one component D_i that has more than 2 vertices; each component D_i for $i \in [r]$ is either a 2-cycle or a single vertex and there exists at least one component D_i that is a 2-cycle; each component D_i for $i \in [r]$ is a single vertex.

Theorem 3.1. *Let D be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r]$ is the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong semicomplete digraph or a single vertex. If there is a component D_i that has more than 2 vertices, then D contains two arc-disjoint Hamiltonian cycles.*

Proof. Suppose that $x_1, x_2, \ldots, x_r$ is a round labeling of R. When $|V(D_i)| \geq 3$, by Lemma 2.1, D_i contains a spanning subdigraph T_i which is a strong tournament. Combining Lemma 2.2 and Lemma 2.3, we know that D_i contains a Hamiltonian path Q_i and a 2-path-factor $P_i' \cup P_i''$ arc-disjoint from Q_i such that Q_i, P_i' and P_i'' have distinct initial vertices and distinct terminal vertices. Let u_i, u_i', u_i'' be the initial vertices of Q_i, P_i', P_i'' and v_i, v_i', v_i'' be the terminal vertices of Q_i, P_i', P_i'', respectively. When $|V(D_i)| = 2$, let $Q_i = u_iv_i, P_i' = P_i'' = v_iu_i$. When $|V(D_i)| = 1$, suppose that u_i is the only vertex in D_i. Let $Q_i = P_i' = P_i'' = u_i$. We will consider two cases below.

Case 1. $r = 2$. By Lemma 2.5, we know that $D_1 \Rightarrow D_2 \Rightarrow D_1$. Without loss of generality, assume that $|V(D_1)| \geq 3$. When $|V(D_2)| \geq 3$, let $C_1 = Q_1Q_2u_1, C_2 = P_1'P_2'P_1''P_2''u_1'$. When $|V(D_2)| = 2$, let $C_1 = Q_1Q_2u_1, C_2 = P_1'u_2P_1''v_2u_1'$. When $|V(D_2)| = 1$, notice that D_1 is a 2-strong semicomplete digraph since $D_1 = D - u_1$ and D is 3-strong. By Lemma 2.4, assume that $\widehat{P}_1$ and $\widehat{Q}_1$ are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. Let $C_1 = u_2\widehat{Q}_1u_2, C_2 = u_2\widehat{P}_1u_2$. It is easy to check that C_1 and C_2 are two arc-disjoint Hamiltonian cycles of D.

Case 2. $r \geq 3$. We can easily obtain a Hamiltonian cycle $C_1 = Q_1Q_2 \cdots Q_ru_1$. An example is shown in Figure 4(a), where $C_1 = Q_1Q_2Q_3Q_4Q_5Q_6u_1$. In the following, we shall find the other Hamiltonian cycle C_2 such that C_1 and C_2 are arc-disjoint.

Step 1. Build a 2-cycle factor $C' \cup C''$ of D.

Let $C' = P'_1 P'_2 \cdots P'_r u'_1, C'' = P''_1 P''_2 \cdots P''_r u''_1$.

If $|V(D_i)| \geq 3$ for each $i \in [r]$, then $C' \cup C''$ is a 2-cycle factor of D. We are done.

If there exist several subscripts k's such that $|V(D_k)| \leq 2$, then $C' \cup C''$ is not a 2-cycle factor. We will obtain the desired 2-cycle factor by modifying $C' \cup C''$. For convenience, if there exist i, j satisfying $|V(D_i)| \geq 3, |V(D_j)| \geq 3$ and $|V(D_k)| \leq 2$ for each $i < k < j$ (possibly, $D_i = D_j$), we call $D_{i+1} D_{i+2} \cdots D_{j-1}$ a maximal singular segment. Here and below the subscripts are taken modulo r. For every pair of i, j such that $D_{i+1} D_{i+2} \cdots D_{j-1}$ is a maximal singular segment, we do the following:

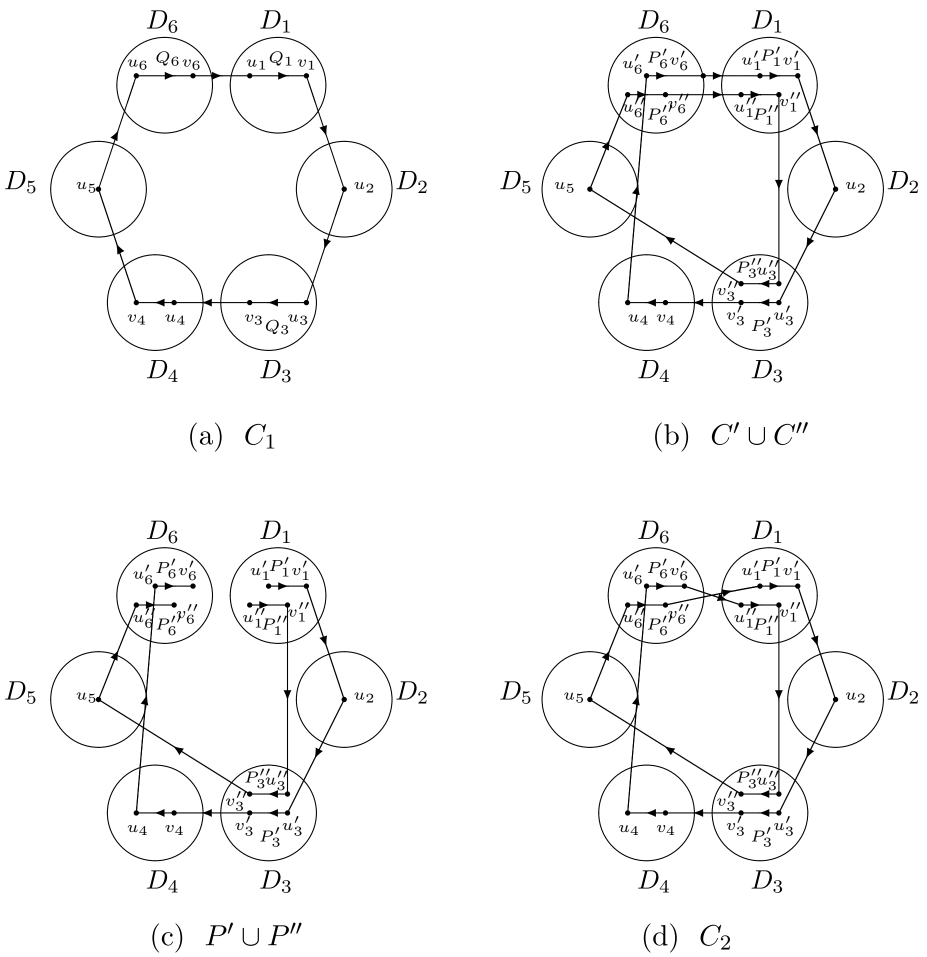
If $j - i \equiv 0 (\mathrm{mod}\ 2)$, denote $j = i + 2k$. In C', replace $v'_i P'_{i+1} P'_{i+2} \cdots P'_{i+(2k-1)} u'_{i+2k}$ with $v'_i P'_{i+1} P'_{i+3} \cdots P'_{i+(2k-1)} u'_{i+2k}$. In C'', replace $v''_i P''_{i+1} P''_{i+2} \cdots P''_{i+(2k-1)} u''_{i+2k}$ with $v''_i P''_{i+2} P''_{i+4} \cdots P''_{i+(2k-2)} u''_{i+2k}$. If $j - i \equiv 1 (\mathrm{mod}\ 2)$, denote $j = i + 2k + 1$. In C', replace $v'_i P'_{i+1} P'_{i+2} \cdots P'_{i+(2k-1)} u'_{i+2k}$ with $v'_i P'_{i+1} P'_{i+3} \cdots P'_{i+(2k-1)} u'_{i+(2k+1)}$. In C'', replace $v''_i P''_{i+1} P''_{i+2} \cdots P''_{i+(2k-1)} u''_{i+2k}$ with $v''_i P''_{i+2} P''_{i+4} \cdots P''_{i+(2k)} u''_{i+(2k+1)}$. See Figure 4(b), D_2 and $D_4 D_5$ are all maximal singular segments of D. Replace $v'_1 u_2 u'_3$ with $v'_1 u_2 u'_3$, $v''_1 u_2 u''_3$ with $v''_1 u''_3$, $v'_3 v_4 u_4 u_5 u'_6$ with $v'_3 v_4 u_4 u'_6$, and $v''_3 v_4 u_4 u_5 u''_6$ with $v''_3 u_5 u''_6$, respectively. Hence, $C' = P'_1 u_2 P'_3 u_4 P'_6 u'_1$, $C'' = P''_1 P''_3 u_5 P''_6 u''_1$. Clearly, $C' \cup C''$ is a 2-cycle factor of D.

Step 2. Build a 2-path factor $P' \cup P''$ based on the 2-cycle factor $C' \cup C''$.

Since there is a component D_i that has more than 2 vertices for some $i \in [r]$, without loss of generality, assume that $|V(D_r)| \geq 3$. Let w' be the successor of v'_r in C', and w'' be successor of v''_r in C''. By the construction process of $C' \cup C''$, if $|V(D_1)| \leq 2$, we have $w' \in D_1, w'' \in D_2$, and if $|V(D_1)| \geq 3$ we have $w', w'' \in D_1$. We obtain P', P'' by deleting arc $v'_r w', v''_r w''$ of C', C'', respectively. It is easy to check that $P' \cup P''$ is a 2-path factor of D. See Figure 4(c). We obtain $P' = P'_1 u_2 P'_3 v_4 u_4 P'_6$ by deleting arc $v'_6 u'_1$ of C', and $P'' = P''_1 P''_3 u_5 P''_6$ by deleting $v''_6 u''_1$.

Step 3. Build a Hamiltonian cycle C_2 based on the 2-path factor $P' \cup P''$.

If $|V(D_1)| \leq 2$, then we have $w' \in D_1$ and $w'' \in D_2$. By Lemma 2.5, since D is 3-strong, D_r must completely dominate D_2. This implies that there exist the arcs $v'_r w''$ and $v''_r w'$. If $|V(D_1)| \geq 3$, then we have $w', w'' \in D_1$. Since D_r completely dominates D_1, there also exist the arcs $v'_r w''$ and $v''_r w'$. Now the initial vertices of P', P'' are w', w'', respectively. The terminal vertices of P', P'' are v'_r, v''_r, respectively. Hence, add the arcs $v'_r w''$ and $v''_r w'$ into the 2-path factor $P' \cup P''$, and we obtain the Hamiltonian cycle $C_2 = P' P'' w'$. It is easy to check that C_1 is arc-disjoint from C_2. See Figure 4(d). $C_2 = P'_1 u_2 P'_3 v_4 u_4 P'_6 P''_1 P''_3 u_5 P''_6 u'_1$ is a Hamiltonian cycle arc-disjoint from C_1. ■



(a) C_1 (b) $C' \cup C''$ (c) $P' \cup P''$ (d) C_2

Figure 4. (a) The Hamiltonian cycle C_1. (b) The 2-cycle factor $C' \cup C''$. (c) The 2-path factor $P' \cup P''$. (d) The Hamiltonian cycle C_2.

Theorem 3.2. *Let D be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r]$ is the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a 2-cycle or a single vertex. If there is a component D_i that is a 2-cycle, then D contains two arc-disjoint Hamiltonian cycles.*

Proof. When $|V(D_i)| = 2$, let $Q_i = u_iv_i, P_i = v_iu_i$. When $|V(D_i)| = 1$, suppose that u_i is the only vertex in D_i. Let $Q_i = P_i = u_i$. Obviously, $C_1 = Q_1Q_2 \cdots Q_ru_1$ is a Hamiltonian cycle of D. Assume without loss of generality that $|V(D_1)| = 2$. If r is even, then let $C_2 = v_1P_3P_5 \cdots P_{r-1}u_1P_2P_4 \cdots P_rv_1$. If r is odd, then let

$C_2 = P_1P_3 \cdots P_rP_2P_4 \cdots P_{r-1}v_1$. It is easy to check that C_1 and C_2 are two arc-disjoint Hamiltonian cycles. ■

Theorem 3.3. *Let R be a 3-strong round digraph. Then R contains two arc-disjoint Hamiltonian cycles.*

Proof. Let $x_1, x_2, \ldots, x_r$ be the unique (up to cyclic permutations) round labeling of R. Since R is 3-strong round digraph, the vertex x_i dominates the vertices x_{i+1}, x_{i+2} and x_{i+3} for each $i \in [r]$ (subscripts are modulo r).

If r is odd, denote $r = 2k+1$. Then R contains two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_3 \cdots x_{2k+1}x_1$ and $C_2 = x_1x_3 \cdots x_{2k+1}x_2x_4 \cdots x_{2k}x_1$.

If r is even, we consider two cases, $r = 4m+2$ or $r = 4m$.

Case 1. $r = 4m+2$. R contains two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_4x_6 \cdots x_{4m+2}x_3x_5x_7 \cdots x_{4m+1}x_1$ and $C_2 = x_1x_4x_5x_8x_9 \cdots x_{4m-4}x_{4m-3}x_{4m}$ $x_{4m+1}x_2x_3x_6x_7 \cdots x_{4m-6}x_{4m-5}x_{4m-2}x_{4m-1}x_{4m+2}x_1$.

Case 2. $r = 4m$. If $r = 4m$, R contains two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_4x_6 \cdots x_{4m}x_3x_5x_7 \cdots x_{4m-1}x_1$ and $C_2 = x_1x_3x_4x_7x_8 \cdots x_{4m-3}x_{4m-1}$ $x_{4m}x_2x_5x_6x_9x_{10} \cdots x_{4m-7}x_{4m-6}x_{4m-3}x_{4m-2}x_1$.

The theorem holds. ■

Combining with Theorem 3.1, Theorem 3.2 and Theorem 3.3, the proof of Theorem 1.5 is complete.

4. Proof of Theorem 1.6

Let D be a 2-strong round decomposable local tournament, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong tournament or a single vertex. We prove Theorem 1.6 by dividing into two cases: there is a strong component D_i that is not a single vertex; each strong component D_i for $i \in [r]$ is a single vertex, i.e., $D = R$ is a round diraph.

In the proof of Theorem 3.1, the condition that D is 3-strong is necessary only when $r = 2$ or when $r \geq 3$ and $|V(D_i)| = 2$ for some $i \in [r]$. In other cases, the condition that D is 2-strong is sufficient. When $D = R[D_1, D_2, \ldots, D_r]$ is a round decomposable local tournament, we always have $r \geq 3$ and $|V(D_i)| \neq 2$ for each $i \in [r]$. Thus the proof of Theorem 3.1 can be used to prove the following theorem.

Theorem 4.1. *Let D be a 2-strong round decomposable local tournament, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong tournament or a single vertex.*

If there is a component D_i that is not a single vertex, then D contains two arc-disjoint Hamiltonian cycles.

Theorem 4.2. *Let R be a 2-strong round digraph. Then R contains a Hamiltonian cycle and a Hamiltonian path which are arc-disjoint if and only if R is not the second power of an even cycle.*

Proof. Firstly, we show the 'only if' part. Let R be a digraph with the vertex set $\{x_1, x_2, \ldots, x_r\}$ and the ordering $x_1, x_2, \ldots, x_r$ be the unique (up to cyclic permutations) round labeling of vertices of R. Suppose to the contrary that R is the second power of an even cycle. Obviously, $C = x_1x_2 \cdots x_rx_1$ is the unique Hamiltonian cycle of R. We obtain two vertex-disjoint $\frac{r}{2}$-cycles by deleting arcs of C from R. Hence, R will not contain a Hamiltonian path P arc-disjoint from the Hamiltonian cycle C, a contradiction. Thus R is not the second power of an even cycle.

To show the 'if' part, let R be a 2-strong round digraph. This means that x_i dominates x_{i+1} and x_{i+2} for each $i \in [r]$ (all subscripts are modulo r). Then R contains C_r^2 as a spanning subdigraph of R. Since R is not the second power of an even cycle, we discuss two cases below.

Case 1. $r = 2k+1$. It is obvious that C_{2k+1}^2 can be decomposed into two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_3 \cdots x_{2k}x_{2k+1}x_1$ and $C_2 = x_1x_3x_5 \cdots x_{2k+1}x_2x_4x_6 \cdots x_{2k}x_1$. It is certain that R contains a Hamiltonian cycle and a Hamiltonian path which are arc-disjoint.

Case 2. $r = 2k$. Since R is not the second power of an even cycle, there exists a vertex x_i dominating x_{i+3}. Without loss of generality, assume that x_1 dominates x_4. Thus R can be decomposed into a Hamiltonian cycle $C_1 = x_1x_2x_3 \cdots x_{2k-1}x_{2k}x_1$ and a Hamiltonian path $P_2 = x_3x_5x_7 \cdots x_{2k-1}x_1x_4x_6 \cdots x_{2k}x_2$. ■

Combining with Theorem 4.1 and Theorem 4.2, the proof of Theorem 1.6 is complete.

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