ARC-DISJOINT HAMILTONIAN CYCLES IN ROUND DECOMPOSABLE LOCALLY SEMICOMPLETE DIGRAPHS

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Abstract

Let \( D = (V, A) \) be a digraph; if there is at least one arc between every pair of distinct vertices of \( D \), then \( D \) is a semicomplete digraph. A digraph \( D \) is locally semicomplete if for every vertex \( x \), the out-neighbours of \( x \) induce a semicomplete digraph and the in-neighbours of \( x \) induce a semicomplete digraph. A locally semicomplete digraph without 2-cycle is a local tournament. In 2012, Bang-Jensen and Huang [J. Combin Theory Ser. B 102 (2012) 701–714] concluded that every 2-arc-strong locally semicomplete digraph which is not the second power of an even cycle has two arc-disjoint strong spanning subdigraphs, and proposed the conjecture that every 3-arc-strong local tournament has two arc-disjoint Hamiltonian cycles. According to Bang-Jensen, Guo, Gutin and Volkmann, locally semicomplete digraphs have three subclasses: the round decomposable; the non-round decomposable which are not semicomplete; the non-round decomposable which are semicomplete. In this paper, we prove that every 3-strong round decomposable locally semicomplete digraph has two arc-disjoint Hamiltonian cycles, which implies that the conjecture holds for the round decomposable local tournaments. Also, we characterize the 2-strong round decomposable local tournaments each of which contains a Hamiltonian path \( P \) and a Hamiltonian cycle arc-disjoint from \( P \).

Keywords: locally semicomplete digraph, local tournament, round decomposable, arc-disjoint, Hamiltonian cycle, Hamiltonian path.

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1. Terminology and Introduction

In this paper, we consider finite digraph without loops and multiple arcs. The main source for terminology and notation is [1].

For an integer \( n \), \([n]\) will denote the set \( \{1, 2, 3, \ldots, n\} \).

Let \( D = (V, A) \) be a digraph; if there is an arc from a vertex \( x \) to \( y \), we say that \( x \) dominates \( y \) and denote it by \( x \to y \). If \( V_1 \) and \( V_2 \) are arc-disjoint subsets of vertices of \( D \) such that there is no arc from \( V_2 \) to \( V_1 \) and \( a \to b \) for all \( a \in V_1 \) and \( b \in V_2 \), then we say that \( V_1 \) completely dominates \( V_2 \) and denote this by \( V_1 \Rightarrow V_2 \). We shall use the same notation when \( A \) and \( B \) are subdigraphs of \( D \). Let \( N^-(x) \) (respectively, \( N^+(x) \)) denote the set of vertices dominating (respectively, dominated by) \( x \) in \( D \) and say that \( N^-(x) \) (respectively, \( N^+(x) \)) is the in-neighbours of \( x \) (respectively, the out-neighbours of \( x \)).

Let \( H \) be a subdigraph of \( D \); if \( V(D) = V(H) \), we say that \( H \) is a spanning subdigraph of \( D \). If every arc of \( A(D) \) with both end-vertices in \( V(H) \) is in \( A(H) \), we say that \( H \) is induced by \( X = V(H) \) and denote this by \( D(X) \). We also use the notation \( D - X \), where \( X \subseteq V \), for digraph \( D(V(D) \setminus V(X)) \).

Let \( D_1, D_2 \) be two subdigraphs of a digraph \( D \). The union \( D_1 \cup D_2 \) is the digraph \( D \) with vertex set \( V(D_1) \cup V(D_2) \) and arc set \( A(D_1) \cup A(D_2) \).

Paths and cycles in a digraph are always directed. Let \( P \) be a directed path of digraph \( D \). If \( V(P) = V(D) \), then \( P \) is a Hamiltonian path of \( D \). Similarly, let \( C \) be a directed cycle of digraph \( D \). If \( V(C) = V(D) \), then \( C \) is a Hamiltonian cycle of \( D \).

Let \( P_1, P_2, \ldots, P_q \) be paths which are pairwise vertex-disjoint. If \( F = P_1 \cup P_2 \cup \cdots \cup P_q \) is a spanning subdigraph of \( D \), then \( F \) is a \( q \)-path factor of \( D \). Let \( C_1, C_2, \ldots, C_q \) be cycles which are pairwise vertex-disjoint. If \( F = C_1 \cup C_2 \cup \cdots \cup C_q \) is a spanning subdigraph of \( D \), then \( F \) is a \( q \)-cycle factor of \( D \).

A digraph \( D = (V, A) \) is called strongly connected (or just strong) if there exists a path from \( x \) to \( y \) and a path from \( y \) to \( x \) in \( D \) for every choice of distinct vertices \( x, y \) of \( D \), and \( D \) is \( k \)-arc-strong (respectively, \( k \)-strong) if \( D - X \) is strong for every subset \( X \subseteq A \) (respectively, \( X \subseteq V \)) of size at most \( k - 1 \). Note that a digraph with only one vertex is strong.

A digraph \( D \) is semicomplete if, for every pair \( x, y \) of vertices of \( D \), either \( x \) dominates \( y \) or \( y \) dominates \( x \) (or both). A digraph \( D \) is locally semicomplete if for every vertex \( x \), the out-neighbours of \( x \) induce a semicomplete digraph and the in-neighbours of \( x \) induce a semicomplete digraph. A semicomplete digraph without 2-cycle is a tournament and a locally semicomplete digraph without 2-cycle is a local tournament.

A digraph \( R \) on \( r \) vertices is round if we can label its vertices \( x_1, x_2, \ldots, x_r \) so that for each \( i \), we have \( N^+_R(x_i) = \{x_{i+1}, x_{i+2}, \ldots, x_{i+d_R(x_i)}\} \) and \( N^-_R(x_i) = \{x_{i-d_R(x_i)}, \ldots, x_{i-1}\} \) (all subscripts are taken modulo \( r \)). Note that every round
digraph is locally semicomplete, a round digraph without 2-cycle is a local tournament. If a local tournament $R$ is round then there exists a unique (up to cyclic permutations) labeling of vertices of $R$ which satisfies the properties in the definition. We refer to this as the round labeling of $R$. See Figure 1(a) for an example of a round digraph $R$. Observe that the ordering $x_1, x_2, \ldots, x_6$ is a round labeling of $R$. The second power of a cycle $C_n$, denoted by $C_n^2$, is the digraph obtained from $C_n$ by adding the arcs \( \{x_ix_{i+2} : i \in [n]\} \), where $C_n = x_1x_2 \cdots x_nx_1$ and subscripts are modulo $n$. See Figure 1(b) for the second power of an 8-cycle.

Figure 1. A round digraph and the second power of an 8-cycle.

Let $R$ be a digraph with vertex set \( \{x_i : i \in [r]\} \), and let $D_1, D_2, \ldots, D_r$ be digraphs which are pairwise vertex-disjoint. Let $D = R[D_1, D_2, \ldots, D_r]$, $r \geq 2$, be the new digraph obtained from $R$ by replacing $x_i$ with $D_i$ and adding arc from every vertex of $D_i$ to every vertex of $D_j$ if and only if $x_i \rightarrow x_j$ in $R$. If $R$ is a round digraph and each $D_i$ is a strong semicomplete digraph, it is easy to see that $D = R[D_1, D_2, \ldots, D_r]$ is a locally semicomplete digraph. We call $D$ a round decomposable locally semicomplete digraph. If a round decomposable locally semicomplete digraph $D = R[D_1, D_2, \ldots, D_r]$ has no 2-cycle (i.e., the round digraph $R$ has no 2-cycle and each $D_i, i \in [r]$, is a strong tournament or a single vertex), then we say that $D$ is a round decomposable local tournament.

Locally semicomplete digraphs were introduced in 1990 by Bang-Jensen [2]. The following theorem, due to Bang-Jensen, Guo, Gutin and Volkmann, states a full classification of locally semicomplete digraphs.

**Theorem 1.1** [3]. Let $D$ be a locally semicomplete digraph. Then exactly one of the following possibilities holds.

(a) $D$ is round decomposable with a unique round decomposition $R[D_1, D_2, \ldots, D_r]$, where $R$ is a round locally semicomplete on $r \geq 2$ vertices and $D_i$ is a strong semicomplete digraph for $i = 1, 2, \ldots, r$;

(b) $D$ is a round decomposable locally semicomplete digraph.
(b) $D$ is not round decomposable and not semicomplete;
(c) $D$ is a semicomplete digraph which is not round decomposable.

In [3], Bang-Jensen et al. also characterized the structure of locally semicomplete digraph which is not round decomposable and not semicomplete. If $D$ is restricted to a local tournament, we have the following result.

**Corollary 1.2.** Let $D$ be a local tournament. Then exactly one of the following possibilities holds.

(a) $D$ is round decomposable with a unique round decomposition $R[D_1, D_2, \ldots, D_r]$, where $R$ is a round local tournament on $r \geq 2$ vertices and $D_i$ is a strong tournament for $i = 1, 2, \ldots, r$;
(b) $D$ is not round decomposable and not a tournament;
(c) $D$ is a tournament which is not round decomposable.

According to the classification of locally semicomplete digraphs, many nice properties of semicomplete digraphs (tournaments) are generalized to locally semicomplete digraphs (local tournaments), see [5–8]. Recently, some new problems on locally semicomplete digraphs, such as the out-arc pancyclicity, the number of Hamiltonian cycles, the kings, the H-force set and so on, were studied in [9–13]. In particular, Bang-Jensen and Huang investigated the decomposition of locally semicomplete digraphs and proved the theorem below.

**Theorem 1.3** [4]. A 2-arc-strong locally semicomplete digraph $D$ has two arc-disjoint strong spanning subdigraphs if and only if $D$ is not the second power of an even cycle.

Meanwhile, they proposed the following conjecture.

**Conjecture 1.4** [4]. Every 3-strong local tournament has two arc-disjoint Hamiltonian cycles.

In this paper, we prove the following theorem in Section 3 which implies that the conjecture holds for the subclass of local tournaments—the round decomposable local tournaments.

**Theorem 1.5.** Every 3-strong round decomposable locally semicomplete digraph has two arc-disjoint Hamiltonian cycles.

Also, in the following theorem, we give a characterization of the 2-strong round decomposable local tournaments each of which contains a Hamiltonian path $P$ and a Hamiltonian cycle arc-disjoint from $P$. This theorem will be proved in Section 4.
Theorem 1.6. Every 2-strong round decomposable local tournament has a Hamiltonian path and a Hamiltonian cycle which are arc-disjoint if and only if it is not the second power of an even cycle.

To show the main results, we introduce the following definition and theorem due to Thomassen. A tournament is called transitive if it contains no cycle. It is easy to see that, for a transitive tournament $T$, there is a unique vertex ordering $v_1, v_2, \ldots, v_n$ of $T$, such that $v_i \to v_j$ for all $1 \leq i < j \leq n$. A tournament is almost transitive if it is obtained from the transitive tournament $T$ by reversing the arc $v_1v_n$.

Theorem 1.7 [11]. Every tournament which is strong and which is not an almost transitive tournament of odd order has two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.

We also use the following facts several times.

Theorem 1.8 [1]. Every strong semicomplete digraph is vertex-pancyclic.

Theorem 1.9 [7]. A tournament is strong if and only if it has a Hamiltonian cycle.

2. Preliminaries

In this section we start with the following three lemmas which imply that every strong semicomplete digraph with at least 3 vertices contains a Hamiltonian path $Q$ and a 2-path-factor $P' \cup P''$ arc-disjoint from $Q$ such that the paths $Q, P'$ and $P''$ have distinct initial vertices and distinct terminal vertices.

Lemma 2.1. Let $D$ be a strong semicomplete digraph with at least 3 vertices. Then $D$ contains a spanning subdigraph which is a strong tournament.

Proof. By Theorem 1.8, assume that $C$ is a Hamiltonian cycle of $D$. Notice that at most one arc of each 2-cycle of $D$ is in $C$. For each 2-cycle of $D$, by deleting exactly one arc which is not in $C$, we obtain a tournament $T$ which is a spanning subdigraph of $D$ and contains a Hamiltonian cycle. Note that a tournament is strong if and only if it has a Hamiltonian cycle. The proof is complete. $\blacksquare$

Lemma 2.2. Let $T$ be a strong tournament which is not an almost transitive tournament of odd order. Then $T$ contains a Hamiltonian path $Q$ and a 2-path-factor $P' \cup P''$ arc-disjoint from $Q$ such that the paths $Q, P'$ and $P''$ have distinct initial vertices and distinct terminal vertices.
Proof. Let $|V(T)| = n$. Since $T$ is a strong tournament which is not an almost transitive tournament of odd order, if $n = 3$, then $T$ is an almost transitive tournament of odd order. So $n \geq 4$. By Theorem 1.7, $T$ contains a pair of arc-disjoint Hamiltonian paths $P$ and $Q$ such that $P$ and $Q$ have distinct initial vertices and distinct terminal vertices. Denote $P = v_1v_2 \cdots v_n$, $Q = u_1u_2 \cdots u_n$. Then $v_1 \neq u_1$, $v_n \neq u_n$. Let the vertex $u_1$ of $Q$ correspond to the vertex $v_i$ of $P$, and the vertex $u_n$ of $Q$ correspond to the vertex $v_j$ of $P$. Note that $i > 1$, $j < n$ and $i \neq j$. Now we will construct a 2-path-factor $P' \cup P''$ arc-disjoint from $Q$ such that $Q$, $P'$ and $P''$ have distinct initial vertices and distinct terminal vertices.

Case 1. $i < j$. Let $P' = v_1v_2 \cdots v_i$ and $P'' = v_{i+1}v_{i+2} \cdots v_n$.

Case 2. $i > j$, $j \neq 1$. Let $P' = v_1v_2 \cdots v_{j-1}$ and $P'' = v_jv_{j+1} \cdots v_n$.

Case 3. $i > j$, $j = 1$ and $i \neq n$. Let $P' = v_1v_2 \cdots v_i$ and $P'' = v_{i+1}v_{i+2} \cdots v_n$.

Case 4. $i > j$, $j = 1$ and $i = n$. Let $P' = v_1v_2$ and $P'' = v_3v_4 \cdots v_n$.

Figure 2 shows the construction of the Hamiltonian path $Q$ and the 2-path factor $P' \cup P''$ in the four cases. Notice that in all cases the paths $Q$, $P'$ and $P''$ have distinct initial vertices and distinct terminal vertices, respectively. i.e., $T$ contains a Hamiltonian path $Q$ and a 2-path factor $P' \cup P''$ arc-disjoint from $Q$ such that $Q$, $P'$ and $P''$ have distinct initial vertices and distinct terminal vertices. We are done.

Figure 2. The 2-path factor constructed in the proof of Lemma 2.2.
Lemma 2.3. Let $T$ be a strong tournament which is an almost transitive tournament of odd order. Then $T$ contains a Hamiltonian path $Q$ and a 2-path factor $P' \cup P''$ arc-disjoint from $Q$ such that the paths $Q$, $P'$ and $P''$ have distinct initial vertices and distinct terminal vertices.

**Proof.** Let $|V(T)| = n$ and $V(T) = \{v_1, v_2, \ldots, v_n\}$. Obviously, $n \geq 3$. Since $T$ is an almost transitive tournament of odd order, assume without loss of generality that $v_i \rightarrow v_j$ for all $1 \leq i < j \leq n$ except for $v_n \rightarrow v_1$. Hence, for arbitrary $i \leq n - 2$, we must have $v_i \rightarrow v_{i+1}$ and $v_i \rightarrow v_{i+2}$.

**Case 1.** $n = 3$. Let $P' = v_2v_3$, $P'' = v_1, Q = v_3v_1v_2$. It is clear that $P' \cup P''$ is a 2-path factor of $T$ which is arc-disjoint from $Q$. The paths $Q$, $P'$ and $P''$ have distinct initial vertices and distinct terminal vertices.

**Case 2.** $n > 3$. Let $n = 2k + 1$. Suppose that $P' = v_1v_3 \cdots v_{2k-1}v_{2k+1}, P'' = v_2v_4 \cdots v_{2k-2}v_{2k}, Q = v_3v_4 \cdots v_nv_1v_2$. Obviously, $P' \cup P''$ is a 2-path-factor of $T$ which is arc-disjoint from $Q$. The paths $Q$, $P'$ and $P''$ have distinct initial vertices and distinct terminal vertices, respectively.

Figure 3 shows the construction of the Hamiltonian path $Q$ and the 2-path factor $P' \cup P''$ in the two cases of the proof.

Figure 3. The arc-disjoint 2-path factor $P' \cup P''$ and the Hamiltonian path $Q$ in an almost transitive tournament. In (a), $P' = v_2v_3, P'' = v_1, Q = v_3v_1v_2$. In (b), $P' = v_1v_3v_5, P'' = v_2v_4, Q = v_3v_4v_5v_1v_2$.

The following lemma is also useful in our proof of main results.

Lemma 2.4. Every 2-strong semicomplete digraph with at least 3 vertices contains two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.
**Proof.** Let $D$ be a 2-strong semicomplete digraph. By Lemma 2.1, $D$ contains a strong tournament $T$ as a spanning subdigraph. If $T$ is not an almost transitive tournament of odd order, by Lemma 1.7, we are done. Otherwise, $T$ is an almost transitive tournament of odd order. Assume without loss of generality that $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $v_i \rightarrow v_j$ for all $1 \leq i < j \leq n$ except $v_n \rightarrow v_1$ in $T$. In the following, we will construct two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices in $D$.

Since $D$ is 2-strong, there must exist some arc of the form $v_jv_i, i < j$ besides $v_nv_1$ in $D$. For all arcs of the form $v_jv_i, i < j$ except for $v_nv_1$ in $D$, we shall consider the following two cases.

**Case 1.** There is one arc of the form $v_jv_i, j > i + 1$ besides $v_nv_1$ in $D$. Let $v_jv_i, j > i + 1$, which is not $v_nv_1$, be an arc of $D$. Now we replace the arc $v_iv_j$ with $v_jv_i$ in $T$. Then we can get a tournament $T'$ which is a spanning subdigraph of $D$. Recall that $T$ is an almost transitive tournament of odd order. Then $T'$ is not an almost transitive tournament of odd order. Notice that $C = v_1v_2 \cdots v_n$ is still a Hamiltonian cycle of $T'$. So $T'$ is a strong tournament. By Theorem 1.7, we are done.

**Case 2.** There is no arc of the form $v_jv_i, j > i + 1$ besides $v_nv_1$ in $D$. This means that if $v_jv_i, j > i$ is an arc of $D$, then $j = i + 1$. Note that there must exist two arc-disjoint paths from $v_n$ to $v_1$ in $D$ since $D$ is 2-strong. Then we have $v_kv_n \in D$ for any $k \in [n - 1]$ since otherwise there exists only one path from $v_n$ to $v_1$ in $D$, a contradiction. Obviously, $v_1v_2 \cdots v_n$ and $v_nv_{n-1} \cdots v_1$ are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.

For 3-strong round decomposable locally semicomplete digraphs, the following result is clear.

**Lemma 2.5.** Let $D$ be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r], r \geq 2$ is the round decomposition of $D$, where $R$ is a round digraph and for each $i \in [r]$, $D_i$ is either a strong semicomplete digraph or a single vertex. Then

(a) when $r = 2$, we have $D_1 \Rightarrow D_2 \Rightarrow D_1$;
(b) when $r \geq 3$, for any $i \in [r]$ with $|V(D_i)| \leq 2$, we have $D_{i-1} \Rightarrow D_{i+1}$ (subscripts are modulo $r$).

For 2-strong round decomposable local tournaments, we have the similar result.

**Lemma 2.6.** Let $D$ be a 2-strong round decomposable local tournament. $D = R[D_1, D_2, \ldots, D_r], r \geq 2$ is the round decomposition of $D$, where $R$ is a round digraph and for each $i \in [r]$ with $|V(D_i)| \leq 2$, we have $D_{i-1} \Rightarrow D_{i+1}$ (subscripts are modulo $r$).
digraph and for each \( i \in [r] \), \( D_i \) is either a strong tournament or a single vertex. Then \( r \geq 3 \) and for any \( i \in [r] \) with \( |V(D_i)| = 1 \), we have \( D_{i-1} \Rightarrow D_{i+1} \) (subscripts are modulo \( r \)).

3. Proof of Theorem 1.5

Let \( D \) be a 3-strong round decomposition locally semicomplete digraph, and let \( D = R[D_1, D_2, \ldots, D_r] \) be the round decomposition of \( D \). In this section, we shall prove Theorem 1.5 in three classes: there exists at least one component \( D_i \) that has more than 2 vertices; each component \( D_i \) for \( i \in [r] \) is either a 2-cycle or a single vertex and there exists at least one component \( D_i \) that is a 2-cycle; each component \( D_i \) for \( i \in [r] \) is a single vertex.

**Theorem 3.1.** Let \( D \) be a 3-strong round decomposable locally semicomplete digraph. \( D = R[D_1, D_2, \ldots, D_r] \) is the round decomposition of \( D \), where \( R \) is a round digraph and for each \( i \in [r] \), \( D_i \) is either a strong semicomplete digraph or a single vertex. If there is a component \( D_i \) that has more than 2 vertices, then \( D \) contains two arc-disjoint Hamiltonian cycles.

**Proof.** Suppose that \( x_1, x_2, \ldots, x_r \) is a round labeling of \( R \). When \( |V(D_i)| \geq 3 \), by Lemma 2.1, \( D_i \) contains a spanning subdigraph \( T_i \) which is a strong tournament. Combining Lemma 2.2 and Lemma 2.3, we know that \( D_i \) contains a Hamiltonian path \( Q_i \) and a 2-path-factor \( P_i' \cup P_i'' \) arc-disjoint from \( Q_i \) such that \( Q_i, P_i' \) and \( P_i'' \) have distinct initial vertices and distinct terminal vertices. Let \( u_i, u_i', u_i'' \) be the initial vertices of \( Q_i, P_i', P_i'' \) and \( v_i, v_i', v_i'' \) be the terminal vertices of \( Q_i, P_i', P_i'' \), respectively. When \( |V(D_i)| = 2 \), let \( Q_i = u_i v_i, P_i' = P_i'' = v_i u_i \). When \( |V(D_i)| = 1 \), suppose that \( u_i \) is the only vertex in \( D_i \). Let \( Q_i = P_i' = P_i'' = u_i \).

We will consider two cases below.

**Case 1.** \( r = 2 \). By Lemma 2.5, we know that \( D_1 \Rightarrow D_2 \Rightarrow D_1 \). Without loss of generality, assume that \( |V(D_1)| \geq 3 \). When \( |V(D_2)| \geq 3 \), let \( C_1 = Q_1 u_2 u_1, C_2 = P_1' P_1'' P_1'' u_1' \). When \( |V(D_2)| = 2 \), let \( C_1 = Q_1 u_2 u_1, C_2 = P_1' u_2 P_1'' u_1' \). When \( |V(D_2)| = 1 \), notice that \( D_1 \) is a 2-strong semicomplete digraph since \( D_1 = D - u_1 \) and \( D \) is 3-strong. By Lemma 2.4, assume that \( \hat{P}_1 \) and \( \hat{Q}_1 \) are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. Let \( C_1 = u_2 Q_1 u_2, C_2 = u_2 \hat{P}_1 u_2 \). It is easy to check that \( C_1 \) and \( C_2 \) are two arc-disjoint Hamiltonian cycles of \( D \).

**Case 2.** \( r \geq 3 \). We can easily obtain a Hamiltonian cycle \( C_1 = Q_1 Q_2 \cdots Q_r u_1 \). An example is shown in Figure 4(a), where \( C_1 = Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 u_1 \). In the following, we shall find the other Hamiltonian cycle \( C_2 \) such that \( C_1 \) and \( C_2 \) are arc-disjoint.
Step 1. Build a 2-cycle factor $C' \cup C''$ of $D$.

Let $C' = P'_1P'_2 \cdots P'_iu'_1$, $C'' = P''_1P''_2 \cdots P''_iu''_1$. If $|V(D_k)| \geq 3$ for each $i \in [r]$, then $C' \cup C''$ is a 2-cycle factor of $D$. We are done.

If there exist several subscripts $k$’s such that $|V(D_k)| \leq 2$, then $C' \cup C''$ is not a 2-cycle factor. We will obtain the desired 2-cycle factor by modifying $C' \cup C''$. For convenience, if there exist $i, j$ satisfying $|V(D_i)| \geq 3, |V(D_j)| \geq 3$ and $|V(D_k)| \leq 2$ for each $i < k < j$ (possibly, $D_i = D_j$), we call $D_{i+1}D_{i+2} \cdots D_{j-1}$ a maximal singular segment. Here and below the subscripts are taken modulo $r$.

For every pair of $i, j$ such that $D_{i+1}D_{i+2} \cdots D_{j-1}$ is a maximal singular segment, we do the following:

If $j - i \equiv 0 \pmod{2}$, denote $j = i + 2k$. In $C'$, replace $v_i'P_{i+1}'P_{i+2}' \cdots P_{i+(2k-1)'}u_{i+2k}'$ with $v_i'P_{i+1}'P_{i+3}' \cdots P_{i+(2k-1)'}u_{i+2k}'$. In $C''$, replace $v''_iP_{i+1}''P_{i+2}'' \cdots P_{i+(2k-1)''}u_{i+2k}''$ with $v''_iP_{i+1}''P_{i+3}'' \cdots P_{i+(2k-1)''}u_{i+2k}''$. If $j - i \equiv 1 \pmod{2}$, denote $j = i + 2k+1$. In $C'$, replace $v_i'P_{i+1}'P_{i+2}' \cdots P_{i+(2k-1)'}u_{i+2k}'$ with $v_i'P_{i+1}'P_{i+3}' \cdots P_{i+(2k-1)'}u_{i+2k}'$. In $C''$, replace $v''_iP_{i+1}''P_{i+2}'' \cdots P_{i+(2k-1)''}u_{i+2k}''$ with $v''_iP_{i+1}'P_{i+3}' \cdots P_{i+(2k-1)'}u_{i+2k}'$.

See Figure 4(b), $D_2$ and $D_4D_5$ are all maximal singular segments of $D$. Replace $v'_1u_2u_3$ with $v'_1u_2u_3$, $v'_1u_2u_3$ with $v'_1u_2u_3$, $v'_1u_2u_3$ with $v'_1u_2u_3$, and $v''_1u_2u_3$ with $v''_1u_2u_3$, respectively. Hence, $C' = P'_1u_2P'_3u_4u_5u_6', C'' = P''_1P''_3u_5P''_6$. Clearly, $C' \cup C''$ is a 2-cycle factor of $D$.

Step 2. Build a 2-path factor $P' \cup P''$ based on the 2-cycle factor $C' \cup C''$.

Since there is a component $D_k$ that has more than 2 vertices for some $i \in [r]$, without loss of generality, assume that $|V(D_r)| \geq 3$. Let $w'$ be the successor of $v'_i$ in $C'$, and $w''$ be successor of $v''_i$ in $C''$. By the construction process of $C' \cup C''$, if $|V(D_1)| \leq 2$, we have $w' \in D_1$, $w'' \in D_2$, and if $|V(D_1)| \geq 3$ we have $w', w'' \in D_1$. We obtain $P', P''$ by deleting arc $v'_i w', v''_i w''$ of $C', C''$, respectively. It is easy to check that $P' \cup P''$ is a 2-path factor of $D$. See Figure 4(c). We obtain $P = P'_1u_2P'_3u_4u_5P'_6$ by deleting arc $v'_1u'_1$ of $C'$, and $P'' = P''_1P''_3u_5P''_6$ by deleting $v''_1u''_1$.

Step 3. Build a Hamiltonian cycle $C_2$ based on the 2-path factor $P' \cup P''$.

If $|V(D_1)| \leq 2$, then we have $w' \in D_1$ and $w'' \in D_2$. By Lemma 2.5, since $D$ is 3-strong, $D_1$ must completely dominate $D_2$. This implies that there exist the arcs $v'_iw''$ and $v''_iw'$. If $|V(D_1)| \geq 3$, then we have $w', w'' \in D_1$. Since $D_r$ completely dominates $D_1$, there also exist the arcs $v'_iw''$ and $v''_iw'$. Now the initial vertices of $P', P''$ are $w', w''$, respectively. The terminal vertices of $P', P''$ are $v'_r, v''_r$, respectively. Hence, add the arcs $v'_iw''$ and $v''_iw'$ into the 2-path factor $P' \cup P''$, and we obtain the Hamiltonian cycle $C_2 = P'P''w'$. It is easy to check that $C_1$ is arc-disjoint from $C_2$. See Figure 4(d). $C_2 = P'_1u_2P'_3u_4u_5P'_4P''_1P''_3u_5P''_6u'_1$ is a Hamiltonian cycle arc-disjoint from $C_1$. ■
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Let \( D \) be a 3-strong round decomposable locally semicomplete digraph. \( D = R[D_1, D_2, \ldots, D_r] \) is the round decomposition of \( D \), where \( R \) is a round digraph and for each \( i \in [r] \), \( D_i \) is either a 2-cycle or a single vertex. If there is a component \( D_i \) that is a 2-cycle, then \( D \) contains two arc-disjoint Hamiltonian cycles.

**Theorem 3.2.** When \( |V(D_i)| = 2 \), let \( Q_i = u_iv_i, P_i = v_iu_i \). When \( |V(D_i)| = 1 \), suppose that \( u_i \) is the only vertex in \( D_i \). Let \( Q_i = P_i = u_i \). Obviously, \( C_1 = Q_1Q_2 \cdots Q_ru_1 \) is a Hamiltonian cycle of \( D \). Assume without loss of generality that \( |V(D_1)| = 2 \). If \( r \) is even, then let \( C_2 = v_1P_3P_5 \cdots P_{r-1}u_1P_2P_4 \cdots P_rv_1 \). If \( r \) is odd, then let

![Diagrams](https://via.placeholder.com/150)

Figure 4. (a) The Hamiltonian cycle \( C_1 \). (b) The 2-cycle factor \( C' \cup C'' \). (c) The 2-path factor \( P' \cup P'' \). (d) The Hamiltonian cycle \( C_2 \).
$C_2 = P_1P_3 \cdots P_rP_4 \cdots P_r-1v_1$. It is easy to check that $C_1$ and $C_2$ are two arc-disjoint Hamiltonian cycles.

**Theorem 3.3.** Let $R$ be a 3-strong round digraph. Then $R$ contains two arc-disjoint Hamiltonian cycles.

**Proof.** Let $x_1, x_2, \ldots, x_r$ be the unique (up to cyclic permutations) round labeling of $R$. Since $R$ is 3-strong round digraph, the vertex $x_i$ dominates the vertices $x_{i+1}$, $x_{i+2}$ and $x_{i+3}$ for each $i \in [r]$ (subscripts are modulo $r$).

If $r$ is odd, denote $r = 2k + 1$. Then $R$ contains two arc-disjoint Hamiltonian cycles $C_1 = x_1 x_2 x_3 \cdots x_{2k+1} x_1$ and $C_2 = x_1 x_3 \cdots x_{2k+1} x_2 x_1$.

If $r$ is even, we consider two cases, $r = 4m + 2$ or $r = 4m$.

**Case 1.** $r = 4m + 2$. $R$ contains two arc-disjoint Hamiltonian cycles $C_1 = x_1 x_2 x_4 x_6 \cdots x_{4m+2} x_3 x_5 x_7 \cdots x_{4m+1} x_1$ and $C_2 = x_1 x_4 x_5 x_8 x_9 \cdots x_{4m-4} x_{4m-3} x_{4m}$ $x_{4m+1} x_{2} x_3 x_6 x_7 \cdots x_{4m-6} x_{4m-5} x_{4m-2} x_{4m-1} x_{4m+2} x_1$.

**Case 2.** $r = 4m$. If $r = 4m$, $R$ contains two arc-disjoint Hamiltonian cycles $C_1 = x_1 x_2 x_4 x_6 \cdots x_{4m} x_3 x_5 x_7 \cdots x_{4m-1} x_1$ and $C_2 = x_1 x_3 x_4 x_7 x_8 \cdots x_{4m-3} x_{4m-1}$ $x_{4m} x_2 x_5 x_6 x_9 x_{10} \cdots x_{4m-7} x_{4m-6} x_{4m-3} x_{4m-2} x_1$.

The theorem holds.

Combining with Theorem 3.1, Theorem 3.2 and Theorem 3.3, the proof of Theorem 1.6 is complete.

4. Proof of Theorem 1.6

Let $D$ be a 2-strong round decomposable local tournament, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of $D$, where $R$ is a round digraph and for each $i \in [r]$, $D_i$ is either a strong tournament or a single vertex. We prove Theorem 1.6 by dividing into two cases: there is a strong component $D_i$ that is not a single vertex; each strong component $D_i$ for $i \in [r]$ is a single vertex, i.e., $D = R$ is a round digraph.

In the proof of Theorem 3.1, the condition that $D$ is 3-strong is necessary only when $r = 2$ or when $r \geq 3$ and $|V(D_i)| = 2$ for some $i \in [r]$. In other cases, the condition that $D$ is 2-strong is sufficient. When $D = R[D_1, D_2, \ldots, D_r]$ is a round decomposable local tournament, we always have $r \geq 3$ and $|V(D_i)| \neq 2$ for each $i \in [r]$. Thus the proof of Theorem 3.1 can be used to prove the following theorem.

**Theorem 4.1.** Let $D$ be a 2-strong round decomposable local tournament, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of $D$, where $R$ is a round digraph and for each $i \in [r]$, $D_i$ is either a strong tournament or a single vertex.
If there is a component $D_i$ that is not a single vertex, then $D$ contains two arc-disjoint Hamiltonian cycles.

**Theorem 4.2.** Let $R$ be a 2-strong round digraph. Then $R$ contains a Hamiltonian cycle and a Hamiltonian path which are arc-disjoint if and only if $R$ is not the second power of an even cycle.

**Proof.** Firstly, we show the ‘only if’ part. Let $R$ be a digraph with the vertex set $\{x_1, x_2, \ldots, x_r\}$ and the ordering $x_1, x_2, \ldots, x_r$ be the unique (up to cyclic permutations) round labeling of vertices of $R$. Suppose to the contrary that $R$ is the second power of an even cycle. Obviously, $C = x_1x_2 \cdots x_rx_1$ is the unique Hamiltonian cycle of $R$. We obtain two vertex-disjoint $r/2$-cycles by deleting arcs of $C$ from $R$. Hence, $R$ will not contain a Hamiltonian path $P$ arc-disjoint from the Hamiltonian cycle $C$, a contradiction. Thus $R$ is not the second power of an even cycle.

To show the ‘if’ part, let $R$ be a 2-strong round digraph. This means that $x_i$ dominates $x_{i+1}$ and $x_{i+2}$ for each $i \in [r]$ (all subscripts are modulo $r$). Then $R$ contains $C_r^2$ as a spanning subdigraph of $R$. Since $R$ is not the second power of an even cycle, we discuss two cases below.

**Case 1.** $r = 2k + 1$. It is obvious that $C_{2k+1}^2$ can be decomposed into two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_3 \cdots x_{2k}x_{2k+1}x_1$ and $C_2 = x_1x_3x_5 \cdots x_{2k+1}x_2x_4x_6 \cdots x_{2k}x_1$. It is certain that $R$ contains a Hamiltonian cycle and a Hamiltonian path which are arc-disjoint.

**Case 2.** $r = 2k$. Since $R$ is not the second power of an even cycle, there exists a vertex $x_i$ dominating $x_{i+3}$. Without loss of generality, assume that $x_1$ dominates $x_4$. Thus $R$ can be decomposed into a Hamiltonian cycle $C_1 = x_1x_2x_3 \cdots x_{2k-1}x_{2k}x_1$ and a Hamiltonian path $P_2 = x_3x_5x_7 \cdots x_{2k-1}x_1x_4x_6 \cdots x_{2k}x_2$. ■

Combining with Theorem 4.1 and Theorem 4.2, the proof of Theorem 1.6 is complete.

**References**


