Discussiones Mathematicae
Graph Theory 38 (2018) 463–476
doi:10.7151/dmgt.2015

ON THE PALETTE INDEX OF COMPLETE BIPARTITE GRAPHS [1]

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Abstract

The palette of a vertex x of a graph G determined by a proper edge colouring φ of G is the set $\{\varphi(xy) : xy \in E(G)\}$ and the diversity of φ is the number of different palettes determined by φ. The palette index of G is the minimum of diversities of φ taken over all proper edge colourings φ of G. In the article we determine the palette index of $K_{m,n}$ for $m \le 5$ and pose two conjectures concerning the palette index of complete bipartite graphs.

Keywords: edge colouring, palette index, bipartite graph.

2010 Mathematics Subject Classification: 05C15.

Let G be a finite simple graph and C a finite set of colours. A proper edge colouring is a mapping from $E(G)$ to C that assigns different colours to adjacent edges. It is well known that the minimum possible cardinality of C, the chromatic index of the graph G, is either $\Delta(G)$ (then G is a class 1 graph) or $\Delta(G)+1$ (and G is a class 2 graph).

Consider a proper edge colouring $\varphi : E(G) \to C$. The *palette* of a vertex $x \in V(G)$ (determined by the colouring φ) is the set $S_\varphi(x) := \{\varphi(xy) : xy \in E(G)\}$ and the *diversity* of φ is the number $\mathrm{dvs}(\varphi) := \left|\bigcup_{x \in V(G)}\{S_\varphi(x)\}\right|$. The *palette*

[1]The research of the first author was supported by the Slovak grant VEGA 1/0368/16, by the grant APVV-15-0116 and by the Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic for the Structural Funds of EU under the project ITMS 26220120007.

index of the graph G, denoted by $\check{s}(G)$, is the minimum of $\mathrm{dvs}(\varphi)$ over all proper edge colourings φ of G.

The palette index has been introduced by Horňák *et al.* in [2], where the main attention has been devoted to regular graphs and especially to complete graphs. It is a straightforward observation that $\check{s}(G) = 1$ if and only if G is a regular class 1 graph. If G is a d-regular class 2 graph, then $3 \le \check{s}(G) \le d + 1$: in [2] it has been proved that $\check{s}(G) \ne 2$, and if $\varphi : E(G) \to C$ is a proper colouring with $|C| = d + 1$, then $\check{s}(G) \le \mathrm{dvs}(\varphi) \le \binom{d+1}{d} = d + 1$. By the results in [2], the palette index of a cubic graph is 3 or 4, according to whether G has a perfect matching or not, respectively.

The case of 4-regular graphs has been studied by Bonvicini and Mazzuoccolo in [1]. The analysis of the palette index of 4-regular graphs is much more complicated, no characterisation similar to that for cubic graphs seems to be in sight. Nevertheless, there are 4-regular class 2 graphs having the palette index p for any $p \in \{3, 4, 5\}$.

The concept of the palette index has been generalised to designs by Lindner, Meszka and Rosa in [3], where they obtained several results for the palette index of Steiner triple systems and Steiner quadruple systems.

In this paper we are interested in the palette index of complete bipartite graphs. Since $K_{n,m} \cong K_{m,n}$ and $\check{s}(K_{m,m}) = 1$ ($K_{m,m}$ is an m-regular class 1 graph), our study will be restricted to looking for $\check{s}(K_{m,n})$ with $m < n$. It will be useful to have a special notation for intervals of integers. Namely, if $p, q \in \mathbb{Z}$, then $[p, q] := \{z \in \mathbb{Z} : p \le z \le q\}$ and $[p, \infty) := \{z \in \mathbb{Z} : z \ge p\}$. An edge colouring $\varphi : E(K_{m,n}) \to C$ of the graph $K_{m,n}$ with the (ordered) bipartition (X, Y), $X = \{x_i : i \in [1, m]\}$ and $Y = \{y_j : j \in [1, n]\}$ can be comfortably described using the $m \times n$ matrix M_φ whose element $(M_\varphi)_{i,j}$ in the ith row and the jth column is $\varphi(x_i y_j)$. Palettes (determined by φ) are therefore of two types: n-element row palettes (called in the sequel *ralettes*) $\{\varphi(x_i y_j) : j \in [1, n]\}$, $i \in [1, m]$, and m-element column palettes (*calettes*) $\{\varphi(x_i y_j) : i \in [1, m]\}$, $j \in [1, n]$.

Let $\varphi : E(K_{m,n}) \to C$ be a fixed proper colouring. For $i \in [1, m]$, let r_i be the number of ralettes appearing i times (*i-ralettes*) and for $j \in [1, n]$ let c_j be the number of calettes appearing j times (*j-calettes*). Further, let r be the number of ralettes and c the number of calettes. Notice that, given a j-calette $\widetilde{C} \subseteq C$, each colour $\tilde{c} \in \widetilde{C}$ appears at least j times in M_φ. The assumption $j \ge m + 1$ then would mean, by the pigeonhole principle, that there is a row of M_φ containing the colour $\tilde{c}$ at least twice, which contradicts the fact that φ is proper. Therefore, $c_j = 0$ for each $j \in [m + 1, n]$. We shall frequently use this (more or less) obvious statement without explicitly mentioning it. If $m \ne n$, then ralettes and calettes are distinct, hence the diversity of φ is $r + c$. Moreover, we have $r = \sum_{i=1}^{m} r_i$, $m = \sum_{i=1}^{m} i r_i$, $c = \sum_{j=1}^{m} c_j$, and $n = \sum_{j=1}^{m} j c_j$. We denote by $\mathcal{R}(\varphi)$ the set of ralettes and by $\mathcal{C}(\varphi)$ the set of calettes (both determined by φ, here a reference to φ will be useful).

Proposition 1. *If $m \in [1, \infty)$ and $n \in [m+1, \infty)$, then $\check{s}(K_{m,n}) \leq n+1$.*

Proof. Consider the proper colouring $\varphi : E(K_{m,n}) \to [1, n+1]$ with $M_\varphi = M_{m,n}$, where $M_{m,n}$ is the $m \times n$ matrix determined by $(M_{m,n})_{i,j} = (j-i+1) \pmod{m}$, i.e., the matrix

$$M_{m,n} = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & 1 & \dots & n-2 & n-1 \\ \dots & \dots & \dots & \dots & \dots \\ n-m+3 & n-m+4 & \dots & n-m+1 & n-m+2 \\ n-m+2 & n-m+3 & \dots & n-m & n-m+1 \end{pmatrix}.$$

Then $\mathcal{R}(\varphi) = \{[1, n]\}$, $r = |\mathcal{R}(\varphi)| = 1$, and it is easy to see that $c = |\mathcal{C}(\varphi)| = n$; so, $\check{s}(K_{m,n}) \leq \mathrm{dvs}(\varphi) = 1 + n$. ■

Note that the matrix $M_{m,n}$ from the proof of Proposition 1 can be defined for $m = n$, too, and the edge colouring φ of $K_{m,m}$ with $M_\varphi = M_{m,m}$, where

$$M_{m,m} = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & 1 & \dots & m-2 & m-1 \\ \dots & \dots & \dots & \dots & \dots \\ 3 & 4 & \dots & 1 & 2 \\ 2 & 3 & \dots & m & 1 \end{pmatrix},$$

shows that $\check{s}(K_{m,m}) = 1$.

Lemma 2. *If $m, n \in [1, \infty)$ and $n \neq m$, then $\check{s}(K_{m,m+n}) \leq \check{s}(K_{m,n}) + 1$.*

Proof. Let $\varphi : E(K_{m,n}) \to C$ be a proper colouring with $\mathrm{dvs}(\varphi) = \check{s}(K_{m,n})$. Without loss of generality we may suppose that $C \cap [1, m] = \emptyset$. The proper edge colouring $\psi : E(K_{m,m+n}) \to C \cup [1, m]$ with M_ψ equal to the block matrix $(M_\varphi \, M_{m,m})$ then satisfies $\mathcal{R}(\psi) = \{R \cup [1, m] : R \in \mathcal{R}(\varphi)\}$ and $\mathcal{C}(\psi) = \mathcal{C}(\varphi) \cup \{[1, m]\}$, which implies $|\mathcal{R}(\psi)| = |\mathcal{R}(\varphi)|$ and $|\mathcal{C}(\psi)| = |\mathcal{C}(\varphi)| + 1$. As a consequence of $n \neq m$ we have $\mathcal{R}(\varphi) \cap \mathcal{C}(\varphi) = \emptyset$, hence $\mathrm{dvs}(\varphi) = |\mathcal{R}(\varphi) \cup \mathcal{C}(\varphi)| = |\mathcal{R}(\varphi)| + |\mathcal{C}(\varphi)|$, and, having in mind that $\mathcal{R}(\psi) \cap \mathcal{C}(\psi) = \emptyset$, $\check{s}(K_{m,m+n}) \leq \mathrm{dvs}(\psi) = |\mathcal{R}(\psi)| + |\mathcal{C}(\psi)| = \mathrm{dvs}(\varphi) + 1 = \check{s}(K_{m,n}) + 1$. ■

Corollary 3. *If $m, n, p \in [1, \infty)$ and $n \neq m$, then $\check{s}(K_{m,mp+n}) \leq \check{s}(K_{m,n}) + p$.*

Lemma 4. *If $m \in [1, \infty)$, $n \in [m+1, \infty) \setminus \{2m\}$ and there is a proper colouring $\varphi : E(K_{m,n}) \to C$ such that $c_m \geq 1$, then $\check{s}(K_{m,n-m}) \leq \mathrm{dvs}(\varphi) - 1$.*

Proof. Without loss of generality we may suppose that $[1,m] \subseteq C$ is an m-calette of φ, and that there is an $m\times(n-m)$ matrix A satisfying $M_\varphi = (A\, M_{m,m})$ (permuting rows and/or columns of a matrix does not change the diversity of the corresponding edge colouring of a complete bipartite graph). Clearly, no element of A belongs to $[1,m]$. So, the colouring $\psi : E(K_{m,n-m}) \to C \setminus [1,m]$ with $M_\psi = A$ is proper, and we have

$$\begin{aligned}\mathcal{R}(\psi) &= \{R \setminus [1,m] : R \in \mathcal{R}(\varphi)\},\\ \mathcal{C}(\psi) &= \mathcal{C}(\varphi) \setminus \{[1,m]\},\end{aligned}$$

and, consequently, $\check{s}(K_{m,n-m}) \le \operatorname{dvs}(\psi) = \operatorname{dvs}(\varphi) - 1$ (notice that $\mathcal{R}(\psi) \cap \mathcal{C}(\psi) = \emptyset$ because of $n \neq 2m$). ■

Lemma 5. *If $m,n \in [1,\infty)$ and $p \in [2,\infty)$, then $\check{s}(K_{mp,np}) \le \check{s}(K_{m,n})$.*

Proof. Consider a proper colouring $\varphi : E(K_{m,n}) \to C$ with $\operatorname{dvs}(\varphi) = \check{s}(K_{m,n})$. Further, for $k \in [1,p]$ let $M(k)$ be the $m\times n$ matrix with $(M(k))_{i,j} = ((M_\varphi)_{i,j}, k)$ and let $C_p := \{(c,k) : c \in C, k \in [1,p]\}$. Denote by M_p the block matrix composed of p blocks $M(k)$ for each $k \in [1,p]$ in such a way that the block in the ith block row and the jth block column is the matrix $M((j-i+1) \pmod p)$. So,

$$M_p = \begin{pmatrix} M(1) & M(2) & \dots & M(p-1) & M(p)\\ M(p) & M(1) & \dots & M(p-2) & M(p-1)\\ \dots & \dots & \dots & \dots & \dots\\ M(3) & M(4) & \dots & M(1) & M(2)\\ M(2) & M(3) & \dots & M(p) & M(1)\end{pmatrix},$$

and the proper colouring $\psi : E(K_{mp,np}) \to C_p$ determined by $M_\psi = M_p$ satisfies

$$\begin{aligned}\mathcal{R}(\psi) &= \{\{(x,k) : x \in R, k \in [1,p]\} : R \in \mathcal{R}\},\\ \mathcal{C}(\psi) &= \{\{(y,k) : y \in C, k \in [1,p]\} : C \in \mathcal{C}\},\end{aligned}$$

which means that $\check{s}(K_{mp,np}) \le \operatorname{dvs}(\psi) = \operatorname{dvs}(\varphi) = \check{s}(K_{m,n})$. ■

Lemma 6. *If $m \in [1,\infty)$ and $n \in [m+1,\infty)$, then $\check{s}(K_{m,n}) \ge \lceil \frac{n}{m} \rceil + 1$.*

Proof. Provided that $\varphi : E(K_{m,n}) \to C$ is a proper colouring with $\operatorname{dvs}(\varphi) = \check{s}(K_{m,n})$, we have $n = \sum_{j=1}^m jc_j \le m\sum_{j=1}^m c_j$ and $\check{s}(K_{m,n}) = r+c \ge 1+\sum_{j=1}^m c_j \ge 1 + \lceil \frac{n}{m} \rceil$. ■

Corollary 7. *If $n \in [2,\infty)$ and $p \in [1,\infty)$, then $\check{s}(K_{p,np}) = n+1$.*

Proof. Any proper edge colouring of the graph $K_{1,n}$ uses n colours and has the diversity $n+1$, hence $\check{s}(K_{1,n}) = n+1$. Therefore, by Lemmas 6 and 5, $n+1 \le \check{s}(K_{p,np}) \le \check{s}(K_{1,n}) = n+1$, and we are done. ■

So, the inequality in Lemma 5 for $m = 1$ turns into the equality. We are convinced that this is true for any m.

Conjecture 1. *If $m, n \in [1, \infty)$ and $p \in [2, \infty)$, then $\check{s}(K_{mp,np}) = \check{s}(K_{m,n})$.*

We have $\check{s}(K_{m,m}) = 1$ and, by Corollary 7, $\check{s}(K_{m,mp+m}) = p + 2$ for any $m, p \in [1, \infty)$. Thus, provided that $\check{s}(K_{l,k})$ is known for each $l \in [1, m-1]$ and $k \in [1, \infty)$, to determine $\check{s}(K_{m,k})$ it is sufficient to restrict our attention to $\check{s}(K_{m,mp+n})$ with $n \in [1, m-1]$ and $p \in [1, \infty)$. We solve this problem completely step by step for $m = 2, 3, 4, 5$.

Theorem 8. *If $p \in [0, \infty)$, then $\check{s}(K_{2,2p+1}) = p + 3$.*

Proof. From Corollary 7 we know that $\check{s}(K_{2,1}) = 3$, hence, by Corollary 3, $\check{s}(K_{2,2p+1}) \leq \check{s}(K_{2,1}) + p = p + 3$ for any $p \in [0, \infty)$, and the last inequality turns into equality for $p = 0$.

Suppose there is $q \in [1, \infty)$ such that $\check{s}(K_{2,2q+1}) \leq q + 2$. Without loss of generality q can be taken to be minimum; in such a case $\check{s}(K_{2,2p+1}) = p + 3$ for every $p \in [0, q-1]$, especially $\check{s}(K_{2,2(q-1)+1}) = \check{s}(K_{2,2q-1}) = q + 2$. Let $\varphi : E(K_{2,2q+1}) \to C$ be a colouring with $\mathrm{dvs}(\varphi) = \check{s}(K_{2,2q+1})$.

Suppose first that $c_2 \geq 1$. By Lemma 4 then $q+2 = \check{s}(K_{2,2q-1}) \leq \check{s}(K_{2,2q+1}) - 1 \leq q + 1$, a contradiction.

If $c_2 = 0$, then $c = c_1 = 2q + 1$ and $q + 2 \geq \check{s}(K_{2,2q+1}) = \mathrm{dvs}(\varphi) = r + c \geq 2q + 2$, a contradiction again. ■

Theorem 9. *If $p \in [0, \infty)$ and $n \in [1, 2]$, then $\check{s}(K_{3,3p+n}) = p + 4$.*

Proof. Observe that $\check{s}(K_{3,n}) = 4$ (by Corollary 7 for $n = 1$, and by Theorem 8 for $n = 2$). Therefore, by Corollary 3, we have $\check{s}(K_{3,3p+n}) \leq p + 4$ for any $p \in [0, \infty)$ with the equality in the case $p = 0$.

Admit that there are $q \in [1, \infty)$ and $n \in [1, 2]$ such that $\check{s}(K_{3,3q+n}) \leq q+3$ and $\check{s}(K_{3,3p+n}) = p+4$ whenever $p \in [0, q-1]$. Consider a colouring $\varphi : E(K_{3,3q+n}) \to C$ with $\mathrm{dvs}(\varphi) = \check{s}(K_{3,3q+n})$.

Let $c_3 \geq 1$. In such a case, by Lemma 4, $q+3 = \check{s}(K_{3,3q-3+n}) \leq \check{s}(K_{3,3q+n}) - 1 \leq q + 2$, a contradiction.

If $c_3 = 0$, then $q+3 \geq \mathrm{dvs}(\varphi) = r+c_1+c_2$ and $3q+n = c_1+2c_2 \leq q+3-r+c_2$, hence

$$c_2 \geq 2q + n + r - 3, \tag{1}$$

and $3q + n \geq c_1 + 4q + 2n + 2r - 6$, so that

$$c_1 + q + n + 2r \leq 6, \tag{2}$$

which implies $r \leq 2$.

If $r = 1$, then $c_1 + c_2 \le q+2$ and all $3q+n$ colours of φ are of the frequency 3. Since $c_3 = 0$, each colour must be present in a 1-calette. Therefore, $c_1 \ge \left\lceil \frac{3q+n}{3} \right\rceil = q+1$, and, consequently, $q+2 \ge c_1 + c_2 \ge q+1+c_2$, which, using (1), leads to $1 \ge c_2 \ge 2q+n-2$, $q = n = c_2 = 1$ and $c_1 = 2$. Thus, each colour is present in a 1-calette and in the unique 2-calette. The two 1-calettes are then disjoint, hence $|C| = 6 \ne 3q+n$, a contradiction.

If $r = 2$, then, by (2), $q = n = 1$, $c_1 = 0$, $c_2 = 2$ and all colours of φ are of an even frequency. However, there is one 1-ralette and one 2-ralette, which means that colours of the 1-ralette are of an odd frequency, a contradiction again. ∎

Theorem 10. *If* $p \in [1, \infty)$, *then*

$$\check{s}(K_{4,4p+n}) = \begin{cases} p+4, & \text{for } n = 1, \\ p+3, & \text{for } n = 2, \\ p+5, & \text{for } n = 3. \end{cases}$$

Proof. The polynomial $f(n) = \frac{1}{2}(3n^2 - 11n + 16)$ satisfies $f(1) = 4$, $f(2) = 3$ and $f(3) = 5$, hence we have to prove that $\check{s}(K_{4,4p+n}) = p + \frac{1}{2}(3n^2 - 11n + 16)$ whenever $p \in [1, \infty)$ and $n \in [1, 3]$. Note that in the statement of our theorem the assumption $p = 0$ should be avoided, since, by Corollary 7, $\check{s}(K_{4,1}) = 5 > f(1)$. Nevertheless, by the same corollary we see that $\check{s}(K_{4,2}) = 3 = f(2)$, while Theorem 9 yields $\check{s}(K_{4,3}) = 5 = f(3)$; so,

$$n \in [1,3] \Rightarrow \check{s}(K_{4,n}) \ge \frac{1}{2}\left(3n^2 - 11n + 16\right). \tag{3}$$

The matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \\ 2 & 3 & 4 & 1 & 6 \\ 3 & 4 & 1 & 6 & 2 \end{pmatrix}$$

shows that $\check{s}(K_{4,5}) \le 5 = 1+f(1)$, by Lemma 5 and Theorem 8 we have $\check{s}(K_{4,6}) \le \check{s}(K_{2,3}) = 4 = 1 + f(2)$, while Lemma 2 and Theorem 9 lead to $\check{s}(K_{4,7}) \le \check{s}(K_{4,3})+1 = 6 = 1+f(3)$. Therefore, Corollary 3 gives us $\check{s}(K_{4,4p+n}) \le p+f(n) = p + \frac{1}{2}(3n^2 - 11n + 16)$ for any $p \in [1, \infty)$ and $n \in [1, 3]$. Our task is to show that the last inequality is in fact the equality.

Proceeding by the way of contradiction we suppose there are $q \in [1, \infty)$ and $n \in [1, 3]$ that satisfy

$$\check{s}(K_{4,4q+n}) \le q + \frac{1}{2}\left(3n^2 - 11n + 16\right) - 1, \tag{4}$$

$$p \in [1, q-1] \Rightarrow \check{s}\left(K_{4,4p+n}\right) = p + \frac{1}{2}\left(3n^2 - 11n + 16\right). \tag{5}$$

Consider a colouring $\varphi : E(K_{4,4q+n}) \to C$ with $\mathrm{dvs}(\varphi) = \check{s}(K_{4,4q+n})$.

Admit first that $c_4 \geq 1$. If $q \geq 2$, then $p = q - 1$ is one of possible values in the assumption of the implication (5) so that $\check{s}(K_{4,4q-4+n}) = \check{s}(K_{4(q-1)+n}) = q - 1 + \frac{1}{2}(3n^2 - 11n + 16)$. On the other hand, for $q = 1$ we see, by (3), that $\check{s}(K_{4,4q-4+n}) \geq q - 1 + \frac{1}{2}(3n^2 - 11n + 16)$. Thus, the inequality

$$q - 1 + \frac{1}{2}\left(3n^2 - 11n + 16\right) \leq \check{s}(K_{4,4q-4+n}) \tag{6}$$

is true for both possibilities that are available for q. Since $c_4 \geq 1$, we can apply Lemma 4 to the colouring φ, and then, using (6) and (4), we arrive to

$$\begin{aligned} q - 1 + \frac{1}{2}\left(3n^2 - 11n + 16\right) \leq \check{s}(K_{4,4q-4+n}) &\leq \operatorname{dvs}(\varphi) - 1 = \check{s}(K_{4,4q+n}) - 1 \\ &\leq q + \frac{1}{2}\left(3n^2 - 11n + 16\right) - 2, \end{aligned}$$

a contradiction.

If $c_4 = 0$, then $r + c_1 + c_2 + c_3 = \operatorname{dvs}(\varphi) \leq q + \frac{1}{2}(3n^2 - 11n + 14)$, $8q + 2n = 2(c_1 + 2c_2 + 3c_3) \leq 2q + 3n^2 - 11n + 14 - 2r + (2c_2 + 4c_3) = 2q + 3n^2 - 11n + 14 - 2r + (4q + n - c_1 + c_3)$, hence

$$c_3 \geq 2r + c_1 + 2q - 3n^2 + 12n - 14, \tag{7}$$

which yields $q + \frac{1}{2}(3n^2 - 11n + 14) \geq r + c_1 + c_2 + 2r + c_1 + 2q - 3n^2 + 12n - 14$ and

$$3r + 2c_1 + c_2 + q \leq \frac{1}{2}\left(9n^2 - 35n + 42\right). \tag{8}$$

If $n = 2$, then (8) means that $3r + 2c_1 + c_2 + q \leq 4$, which is possible only if $r = q = 1$ (so that all colours are of the frequency 4) and $c_1 = c_2 = 0$, $c_3 = 2$, implying that all colours are of the frequency 3, a contradiction.

If $n \in \{1, 3\}$, then (8) yields $r \leq 2$. Suppose first $r = 1$ so that each colour is of the frequency 4. A colour in a 3-calette must then appear in a 1-calette, too. Therefore, at least $4c_3$ positions in the matrix M_φ are occupied by colours of 1-calettes, hence $c_1 \geq \left\lceil \frac{4c_3}{4} \right\rceil = c_3$. Then necessarily $c_1 \not\equiv c_3 \pmod 2$, for otherwise $c_1 + 2c_2 + 3c_3 \equiv 0 \pmod 2$, while $4q + n \equiv 1 \pmod 2$, a contradiction; so, $c_1 \geq c_3 + 1$. Since $-3n^2 + 12n - 14 = -5$, (7) leads to $c_3 \geq c_1 + 2q - 3 \geq c_3 + 2q - 2$, $q = 1$, $c_1 = c_3 + 1$, $4 + n = c_3 + 1 + 2c_2 + 3c_3$ and $c_3 = \frac{3+n-2c_2}{4} < 2$.

If $c_3 = 0$, then each colour in the unique 1-calette is of an odd frequency, a contradiction.

If $c_3 = 1$, then $c_2 = \frac{n-1}{2}$, $|C| = 4 + n \geq 5$, and there is a colour of a frequency smaller than 4, a contradiction: either a colour out of the 3-calette (for $n = 1$), or a colour that is in the 2-calette and out of a 1-calette (for $n = 3$).

Now let $r = 2$ so that, by (7), $c_3 \geq c_1 + 2q - 1 \geq 1$. Due to $3c_3 = 4q + n - c_1 - 2c_2$ then $c_1 + 2q - 1 \leq c_3 = \frac{1}{3}(4q + n - c_1 - 2c_2)$, which leads to

$$2c_1 + c_2 + q \leq \frac{n+3}{2}. \tag{9}$$

If $c_1 = 0$, then from $c_3 \geq 1$ it follows that there is a colour of the frequency 3, hence $r_1 = r_3 = 1$ (in the case $r_2 = 2$ all colours would be of an even frequency). Then, however, there is a colour belonging to the 1-ralette, but not to the 3-ralette, and such a colour is of the frequency 1, which means that $c_1 \geq 1$, a contradiction.

If $c_1 \geq 1$, then from (9) we obtain $c_1 = q = 1$, $n = 3$, $c_2 = 0$ and $c_3 = 2$. The two 3-calettes, say C_3^1 and C_3^2, are disjoint. Let M_φ^* be the submatrix of M_φ created from the columns with calettes C_3^1 and C_3^2. The set $R_i(M_\varphi^*)$ of colours of the ith row of M_φ^* consists of six colours of $C_3^1 \cup C_3^2$, hence $|R_i^*| = 2$ for the set $R_i^* = \left(C_3^1 \cup C_3^2\right) \setminus R_i(M_\varphi^*)$, $i = 1, 2, 3, 4$. Each of the eight colours of $C_3^1 \cup C_3^2$ is missing in M_φ^* (it has the frequency 3 there), therefore $\{R_i^* : i \in [1, 4]\}$ is a set of four pairwise disjoint 2-element sets. As a consequence then (having in mind that M_φ is created from M_φ^* by adding just one column) any two distinct sets in $\mathcal{R}(\varphi)$ differ in at least two colours, which means that $r = 4$, a contradiction. ■

From Theorems 8, 9 and 10 we know that $\check{s}(K_{m,m+1}) = m + 2$ for $m \in [2, 4]$. Later we shall see that $\check{s}(K_{5,6}) = 7$ (Theorem 13). Moreover, the next two propositions show that the difference between m and $\check{s}(K_{m,m+1})$ can be arbitrarily large if m is large enough.

Proposition 11. *If $k \in [3, \infty)$, then $\check{s}(K_{2k-1,2k}) \leq \left\lceil \frac{k+9}{2} \right\rceil$.*

Proof. Let $\varphi : E(K_{2k-1,2k}) \to [1, 5k-1]$ be the proper colouring with the matrix M_φ equal to

$$\begin{pmatrix}
1 & 2 & \dots & k-1 & k & k+1 & k+2 & \dots & 2k-1 & 2k \\
k & 1 & \dots & k-2 & k-1 & 2k & k+1 & \dots & 2k-2 & 2k-1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
2 & 3 & \dots & k & 1 & k+2 & k+3 & \dots & 2k & k+1 \\
2k+1 & 2k+2 & \dots & 3k-1 & a_1 & 3k+1 & 3k+2 & \dots & 4k-1 & b_1 \\
3k-1 & 2k+1 & \dots & 3k-2 & a_2 & 4k-1 & 3k+1 & \dots & 4k-2 & b_2 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
2k+2 & 2k+3 & \dots & 2k+1 & a_{k-1} & 3k+2 & 3k+3 & \dots & 3k+1 & b_{k-1}
\end{pmatrix},$$

where $a_{2i-1} = b_{2i} = 4k + 2i - 1$ and $a_{2i} = b_{2i-1} = 4k + 2i$ for $i \in \left[1, \frac{k-1}{2}\right]$ if k is odd, while $a_{2i-1} = b_{2i} = 4k + 2i - 1$, $a_{2i} = b_{2i-1} = 4k + 2i$ for $i \in \left[1, \frac{k-4}{2}\right]$ and $a_{k-3} = b_{k-1} = 5k - 3$, $a_{k-2} = b_{k-3} = 5k - 2$, $a_{k-1} = b_{k-2} = 5k - 1$ if k is even. It is easy to see that $\mathrm{dvs}(\varphi) = \left\lceil \frac{k+9}{2} \right\rceil$. ■

Proposition 12. *If* $k \in [3, \infty)$*, then* $\check{s}(K_{2k,2k+1}) \leq 2k - 3\left\lceil \frac{k}{2} \right\rceil + 6$.

Proof. Consider the proper colouring $\varphi : E(K_{2k,2k+1}) \to [1, 4k]$ with the matrix M_φ equal to

$$\begin{pmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & 2k-1 & 2k & 2k+1 \\ k & 1 & \dots & k-1 & 2k+1 & k+1 & \dots & 2k-2 & 2k-1 & 2k \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 3 & \dots & 1 & k+3 & k+4 & \dots & 2k+1 & k+1 & k+2 \\ 2k+2 & 2k+3 & \dots & 3k+1 & k+2 & k+3 & \dots & 2k & 2k+1 & k+1 \\ 3k+1 & 2k+2 & \dots & 3k & 3k+2 & 3k+3 & \dots & 4k & a_1 & b_1 \\ 3k & 3k+1 & \dots & 3k-1 & 4k & 3k+2 & \dots & 4k-1 & a_2 & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2k+3 & 2k+4 & \dots & 2k+2 & 3k+3 & 3k+4 & \dots & 3k+2 & a_{k-1} & b_{k-1} \end{pmatrix},$$

where $a_{2i-1} = b_{2i} = 2i - 1$ and $a_{2i} = b_{2i-1} = 2i$ for $i \in \left[1, \frac{k-1}{2}\right]$ if k is odd, while $a_{2i-1} = b_{2i} = 2i - 1$, $a_{2i} = b_{2i-1} = 2i$ for $i \in \left[1, \frac{k-4}{2}\right]$ and $a_{k-3} = b_{k-1} = k - 3$, $a_{k-2} = b_{k-3} = k - 2$, $a_{k-1} = b_{k-2} = k - 1$ if k is even. One can easily check that $\operatorname{dvs}(\varphi) = 2k - 3\left\lceil \frac{k}{2} \right\rceil + 6$. ■

Using Propositions 12 and 11 we see that if $k \in [1, \infty)$, then

$$\check{s}(K_{4k+l,4k+l+1}) \leq \begin{cases} k+5, & \text{for } l = 1, 2, \\ k+6, & \text{for } l = 3, \\ k+7, & \text{for } l = 4. \end{cases}$$

Thus $\check{s}(K_{4k+l,4k+l+1}) \leq k + l + 4$ for $k \in [1, \infty]$ and $l \in [1, 4]$.

Theorem 13. *If* $p \in [1, \infty)$ *and* $n \in [1, 4]$*, then* $\check{s}(K_{5,5p+n}) = p + 5$.

Proof. First realise that $\check{s}(K_{5,1}) = 6$ (by Corollary 7), $\check{s}(K_{5,2}) = \check{s}(K_{5,3}) = \check{s}(K_{5,4}) = 5$ (use Theorems 8, 9, 10) and $\check{s}(K_{5,6}) \leq 6$ (Proposition 11 with $k = 3$), which means that we have the inequality

$$\check{s}(K_{5,n}) \geq 5, \tag{10}$$

as well as (with help of Lemma 2 for $n \in [2, 4]$) the inequality

$$\check{s}(K_{5,5+n}) \leq 1 + 5 = 6. \tag{11}$$

So, by Corollary 3, $\check{s}(K_{5,5p+n}) \leq p + 5$ for any $p \in [1, \infty)$.

Suppose that there are $q \in [1, \infty)$ and $n \in [1, 4]$ such that $\check{s}(K_{5,5q+n}) \leq q + 4$ and $\check{s}(K_{5,5p+n}) = p + 5$ whenever $p \in [1, q - 1]$. Let $\varphi : E(K_{5,5q+n}) \to C$ be a proper colouring with $\operatorname{dvs}(\varphi) = \check{s}(K_{5,5q+n})$.

Admit first that $c_5 \ge 1$. If $q \ge 2$, then, because of the assumptions on q, $q+4 \le q-1+5 = \check{s}(K_{5,5(q-1)+n})$. The inequality $q+4 \le \check{s}(K_{5,5q-5+n})$ is true for $q=1$, too (see (10)). Thus, by Lemma 4, we obtain $q+4 \le \check{s}(K_{5,5q+n-5}) \le \mathrm{dvs}(\varphi)-1 = \check{s}(K_{5,5q+n})-1 \le q+4-1$, a contradiction.

For the rest of the proof we have $c_5 = 0$ and

$$5q+n = \sum_{j=1}^{4} jc_j, \tag{12}$$

$$q+4 \ge \mathrm{dvs}(\varphi) = r + \sum_{j=1}^{4} c_j. \tag{13}$$

For $k \in [1,\infty)$ the inequality (13) gives us $k(q+4-r) \ge \sum_{j=1}^{4} kc_j$, which combined with (12) leads to $(k-5)q + k(4-r) - n \ge \sum_{j=1}^{4}(k-j)c_j$ and

$$kr + \sum_{j=1}^{4}(k-j)c_j + (5-k)q + n \le 4k.$$

The last inequality for $k = 4,5$ reads as

$$4r + 3c_1 + 2c_2 + c_3 + q + n \le 16, \tag{14}$$

$$5r + 4c_1 + 3c_2 + 2c_3 + c_4 + n \le 20, \tag{15}$$

and we see that $r \le 3$.

If $r=1$, then all colours are of the frequency 5. Let $p_{k,l}$ denote the number of positions in the matrix M_φ that are occupied by a colour belonging to a k-calette and to an l-calette as well, $(k,l) \in \{(1,3),(1,4),(2,3)\}$. Any two different calettes of a frequency at least 3 are disjoint, therefore $p_{1,4} = 5c_4$ and $p_{1,3} + p_{2,3} = 10c_3$. We have

$$c_1 \ge \left\lceil \frac{p_{1,3}+p_{1,4}}{5} \right\rceil \ge \left\lceil \frac{5c_4}{5} \right\rceil = c_4 \tag{16}$$

and $c_2 \ge \lceil \frac{p_{2,3}}{10} \rceil$, hence

$$c_1 + c_2 \ge \frac{2p_{1,3} + 2p_{1,4} + p_{2,3}}{10} \ge \frac{p_{1,3}}{10} + c_3 + c_4. \tag{17}$$

Now using (15), (16) and (17) we obtain $15 \ge c_1 + 3(c_1+c_2) + 2c_3 + c_4 + n \ge c_4 + 3(c_3+c_4) + 2c_3 + c_4 + n = 5(c_3+c_4) + n$, hence

$$c_3 + c_4 \le \frac{15-n}{5} < 3. \tag{18}$$

Further, (14) yields $3c_1 + 2c_2 + c_3 \le 12 - q - n \le 10$ and $c_1 \le 3$.

If $c_1 = 3$, then $2c_2 + c_3 \le 3 - q - n \le 1$, which implies $c_2 = 0$ and $c_3 \le 1$. If $c_3 = 1$, then $q = n = 1$ and the number of colours is $5q + n = 6$, but at most five of them (those of the 3-calette) can have the frequency 5, a contradiction. If $c_3 = 0$, then (15) yields $c_4 \le 3 - n \le 2$. In order to have the frequency 5 each of $5q + n \ge 6$ colours must be in a 4-calette, hence, by (18), $c_4 = 2$, $n = 1$ and $q = 2$. However, at most 10 out of 11 colours can have the frequency 5 (those belonging to a 4-calette), a contradiction.

If $c_1 \le 2$, then, by (16), $c_4 \le 2$. In the case $c_4 = 2$ we obtain, because of (16) and (18), $c_1 = 2$ and $c_3 = 0$. Besides that, (15) yields $3c_2 \le 5 - n$ and $c_2 \le 1$. If $c_2 = 0$, then $10 = c_1 + 4c_4 = 5q + n \not\equiv 0 \pmod 5$, a contradiction. On the other hand, if $c_2 = 1$, then colours of the unique 2-calette do not have the frequency 5, which is impossible.

If $c_4 = 1$, then each colour of the unique 4-calette C_4 must belong to a 1-calette, and, consequently, $c_1 > 1$ (each 1-calette is distinct from C_4), which yields $c_1 = 2$; so, by (15), $3c_2 + 2c_3 \le 6 - n \le 5$ and $c_2 \le 1$. All 5 colours of C_4 are present (exactly once) in the two 1-calettes, hence at least one colour, that is out of C_4, appears in the two 1-calettes exactly once (the number of positions in columns of M_φ corresponding to 1-calettes occupied by colours that do not belong to C_4 is an odd number $10 - 5 = 5$), which means that it cannot have the frequency 5 (here we use the inequality $c_2 \le 1$), a contradiction.

Suppose finally that $c_4 = 0$, which, because of (18), yields $c_3 \le 2$. Due to (15) then

$$(19) \qquad 3(c_1 + c_2) \le 4c_1 + 3c_2 \le 15 - n - 2c_3.$$

If $c_3 = 2$, (19) leads to $c_1 + c_2 \le 3$. The two 3-calettes are disjoint, $|C| = 5q + n \ge 2 \cdot 5$, hence $q \ge 2$, $5q + n \ge 11$ and there is a colour out of the 3-calettes. In order to have the frequency 5 such a colour must be in a 1-calette and in two 2-calettes so that $c_1 = 1$, $c_2 = 2$, $q = 2$, $n = 1$ and $|C| = 11$. However, the unique 1-calette is disjoint with the 3-calettes, which implies $|C| \ge 15$, a contradiction.

If $c_3 = 1$, because of (19) we have $3(c_1 + c_2) \le 13 - n$ and $c_1 + c_2 \le 4$. The assumption $q \ge 2$ then leads to $11 \le 10 + n \le 5q + n = c_1 + 2c_2 + 3c_3 \le c_1 + 2(4 - c_1) + 3 = 11 - c_1 \le 11$, $n = 1$, $q = 2$, $c_1 = 0$ and $c_2 = 4$. In such a case any of 6 colours not belonging to the unique 3-calette has an even frequency, a contradiction. If $q = 1$, then, by (13), $4 = q + 3 \ge c_1 + c_2 + 1$ and $c_1 + c_2 \le 3$. Since $|C| = 5q + n \ge 6$, there is a colour out of the unique 3-calette C_3, hence necessarily (its frequency is 5) $c_1 = 1$, $c_2 = 2$ and $|C| = 8$. However, the unique 1-calette is disjoint with C_3 so that $|C| \ge 10$, a contradiction.

If $c_3 = 0$, from (19) we know that $c_1 + c_2 \le 4$, hence $c_1 \ge 1$ and $c_2 \ge 3$ (observe that each colour must be present in a 1-calette and in two 2-calettes, which is impossible if $c_2 = 2$ only: there is a colour belonging to exactly one

2-calette). Then, however, $c_1 = 1$ and $c_2 = 3$; since each colour is present in exactly two 2-calettes, we have $4|C| = 10c_2 = 30$, a contradiction.

If $r = 2$, then, by (13), $q + 2 \geq \sum_{j=1}^{4} c_j$, and from (14) it follows that $3c_1 + 2c_2 + c_3 \leq 8 - q - n$, hence $c_1 \leq 2$.

The assumption $c_1 = 2$ implies $2c_2 + c_3 \leq 2 - q - n$, $q = n = 1$, $c_2 = c_3 = 0$ and $c_4 = 1$. Let C_4 be the unique 4-calette. Each row of M_φ contains one or two colours from $C \setminus C_4$. Further, any colour from $C \setminus C_4$ belongs to at most two rows of M_φ (each calette, in which it appears, is one of the two 1-calettes), all ralettes appear at most twice, hence $r > 2$, a contradiction.

Suppose now that $c_1 \leq 1$. The assumption $r = 2$ means that either $r_1 = r_4 = 1$ or $r_2 = r_3 = 1$.

If $r_1 = r_4 = 1$, let R_i be the unique i-ralette, $i = 1, 4$. All possible colour frequencies are 1 (colours of $R_1 \setminus R_4$), 4 (colours of $R_4 \setminus R_1$) and 5 (colours of $R_1 \cap R_4$). Since $R_1 \neq R_4$, we have $1 \geq c_1 = |R_1 \setminus R_4| \geq 1$, $c_1 = 1$ and $|R_1 \cap R_4| = |R_1| - |R_1 \setminus R_4| = 5q - n - 1 \geq 5$. At most 4 colours of the unique 1-calette C_1 are of the frequency 5 (one of them is of the frequency 1), hence there is a colour of the frequency 5 not belonging to C_1; it must be in a 2-calette C_2 and in a 3-calette C_3 so that $\min(c_2, c_3) \geq 1$. From (14) we obtain $2c_2 + c_3 \leq 5 - q - n \leq 3$ and $c_2 = c_3 = q = n = 1$. A colour of $C_2 \setminus C_3 \neq \emptyset$ then has the frequency either 2 or 3, a contradiction.

If $r_2 = r_3 = 1$, let R_i be the i-ralette, $i = 2, 3$. There are 3 possible colour frequencies, namely 2 (colours of $R_2 \setminus R_3 \neq \emptyset$), 3 (colours of $R_3 \setminus R_2 \neq \emptyset$) and 5 (colours of $R_2 \cap R_3$). Since $c_1 \leq 1$, from the fact that the frequency 2 really applies we obtain $c_2 \geq 1$. Besides that $c_4 = 0$; indeed, if C_4 is a 4-calette, it contains a (contradictory) colour of the frequency 4: either an arbitrary one (provided that $c_1 = 0$) or a colour of $C_4 \setminus C_1 \neq \emptyset$ (if C_1 is the unique 1-calette).

If $c_1 = 1$, then (14) yields $2c_2 + c_3 \leq 5 - q - n \leq 3$, hence $c_2 = 1$. If C_i is the i-calette, $i = 1, 2$, then a colour of $C_1 \setminus C_2 \neq \emptyset$ has its frequency either 1 or 4, a contradiction.

If $c_1 = 0$, then the existence of a colour of the frequency 3 yields $c_3 \geq 1$. Further, by (15), $2(c_2 + c_3) \leq 3c_2 + 2c_3 \leq 10 - n \leq 9$, $c_2 + c_3 \leq 4$, and, by (14), $2c_2 \leq 8 - q - n - c_3 \leq 5$, whence $c_2 \leq 2$. Note that the assumption $c_2 = c_3$ leads to $5c_2 = 2c_2 + 3c_3 = 5q + n \not\equiv 0 \pmod 5$, a contradiction. As a consequence then $(c_2, c_3) \in \{(1, 2), (1, 3), (2, 1)\}$.

Suppose first that $c_2 = 1$, $c_3 \in [2, 3]$, and let C_2 be the unique 2-calette. Since $c_1 = 0$ and $c_2 = 1$, each colour of $R_2 \setminus R_3$ (of the frequency 2) belongs to C_2, and so $R_2 \setminus R_3 \subseteq C_2 \setminus R_3$. On the other hand, $C_2 \setminus R_3 = C_2 \setminus (C_2 \cap R_3) = C_2 \cap R_2 \subseteq R_2$, hence $C_2 \setminus R_3 \subseteq R_2 \setminus R_3$, $R_2 \setminus R_3 = C_2 \setminus R_3 = C_2 \cap R_2$, $|R_2 \setminus R_3| = |C_2 \cap R_2| = 2$ and $|R_2 \cap R_3| = |R_2| - |R_2 \setminus R_3| = 5q + n - 2 = (2 + 3c_3) - 2 = 3c_3$. In order to have the frequency 5, each colour of $R_2 \cap R_3 \neq \emptyset$ must belong to C_2. However, $6 \leq 3c_3 = |R_2 \cap R_3| \leq |C_2| = 5$, a contradiction.

If $(c_2, c_3) = (2, 1)$, let C_2^1, C_2^2 be the two 2-calettes and C_3 the unique 3-calette. Clearly, $C_2^1 \cap C_2^2 = \emptyset$ (a colour in $C_2^1 \cap C_2^2$ would not have an appropriate frequency). Consider the 3×4 submatrix M'_φ of M_φ corresponding to rows with ralette R_3 and to columns with calette C_2^1 or C_2^2. The frequency of a colour $c \in R_3 \cap (C_2^1 \cup C_2^2)$ (which occupies a position in M'_φ) is at least 3 (because of $c_3 \in R_3$); as c appears in exactly one of the calettes C_2^1, C_2^2, and its frequency is more than 2, it must appear in C_3, too, and have the frequency 5. Since $C_2^1 \cap C_2^2 = \emptyset$, the colour c occupies at most two positions in M'_φ. The total number of positions in M'_φ is 12, hence $|R_3 \cap (C_2^1 \cup C_2^2)| \geq \frac{12}{2} = 6$, and at least 6 colours of C have the frequency 5. However, a colour having the frequency 5 must necessarily belong to C_3, so the number of colours of the frequency 5 is at most $|C_3| = 5$, a contradiction.

Finally, if $r = 3$, then (14) yields $3c_1 + 2c_2 + c_3 \leq 4 - q - n \leq 2$ and $c_1 = 0$. Suppose there exists a 4-calette C_4 and consider a colour $\tilde{c} \in C_4$. Because of $c_1 = 0$ the frequency of $\tilde{c}$ is 4, hence $\tilde{c}$ is missing in exactly one row of M_φ. On the other hand, in each row of M_φ a colour of C_4 is missing (otherwise there would be a colour of C_4 of the frequency 5). Therefore, in each row of M_φ a "private colour of C_4" is missing, so that $r = 5$, a contradiction. Thus, $c_4 = 0$, $c_2 + c_3 \leq 2c_2 + c_3 \leq 2$, $6 \leq 5q + n = 2c_2 + 3c_3 \leq 4 + c_3$, $c_3 = 2$, $c_2 = 0$, $q = n = 1$, and all colours are of the frequency 3. The assumption $r = 3$ means that either $r_1 = 1$ and $r_2 = 2$ or $r_1 = 2$ and $r_3 = 1$. However, the latter possibility cannot apply (each colour of a 1-ralette has its frequency in the set $\{1, 2, 4, 5\}$)). If $r_1 = 1$, $r_2 = 2$ and the two 2-ralettes are R_2^1, R_2^2, then $R_2^1 \cap R_2^2 = \emptyset$ (a colour in $R_2^1 \cap R_2^2$ would have its frequency either 4 or 5), $|C| \geq 2(5q + n) = 12$, and there is a colour having its frequency at most $\frac{5(5q+n)}{12} = \frac{30}{12} < 3$, a contradiction. ■

Because of Theorems 8, 9, 10 and 13 we believe that the following is true.

Conjecture 2. *If $m, p \in [1, \infty)$ and $n \in [m + 1, 2m - 1]$, then $\check{s}(K_{m,mp+n}) = \check{s}(K_{m,n}) + p$.*

In our final theorem we prove a weaker version of Conjecture 2.

Theorem 14. *If $m \in [6, \infty)$, $n \in [m + 1, 2m - 1]$ and $p \in [(m - 2)n, \infty)$, then $\check{s}(K_{m,mp+n}) = \check{s}(K_{m,n}) + p$.*

Proof. By Corollary 3 we have $\check{s}(K_{m,mp+n}) \leq \check{s}(K_{m,n}) + p$. If our theorem is not true, there is $q \in [(m - 2)n, \infty]$ such that $\check{s}(K_{m,mq+n}) < \check{s}(K_{m,n}) + q$. Without loss of generality we may suppose that $\check{s}(K_{m,mp+n}) = \check{s}(K_{m,n}) + p$ for any $p \in [0, q - 1]$. Let $\varphi : E(K_{m,mq+n}) \to C$ be a proper colouring with $\mathrm{dvs}(\varphi) = \check{s}(K_{m,mq+n})$. By the pigeonhole principle then $c_j = 0$ for $j \in [m+1, n]$. Moreover, $c_m = 0$, for otherwise, by Lemma 4, $\check{s}(K_{m,n})+q-1 = \check{s}(K_{m,mq-m+n}) \leq$

$\mathrm{dvs}(\varphi) - 1 = \check{s}(K_{m,mq+n}) - 1 < \check{s}(K_{m,n}) + q - 1$, a contradiction. Therefore, $mq + n = \sum_{j=1}^{m-1} jc_j \leq (m-1)\sum_{j=1}^{m-1} c_j$ and

$$c = \sum_{j=1}^{m-1} c_j \geq \left\lceil \frac{mq+n}{m-1} \right\rceil = q + \left\lceil \frac{q+n}{m-1} \right\rceil \geq q + \left\lceil \frac{(m-2)n+n}{m-1} \right\rceil = q + n.$$

Consequently, by Proposition 1, $\check{s}(K_{m,n}) + q > \check{s}(K_{m,mq+n}) = \mathrm{dvs}(\varphi) = r + c \geq 1 + n + q \geq \check{s}(K_{m,n}) + q$, a contradiction. ■

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Received 22 March 2016
Revised 20 October 2016
Accepted 15 December 2016