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# MORE ABOUT THE HEIGHT OF FACES IN 3-POLYTOPES

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**Abstract**

The height of a face in a 3-polytope is the maximum degree of its incident vertices, and the height of a 3-polytope, $h$, is the minimum height of its faces. A face is pyramidal if it is either a 4-face incident with three 3-vertices, or a 3-face incident with two vertices of degree at most 4. If pyramidal faces are allowed, then $h$ can be arbitrarily large, so we assume the absence of pyramidal faces in what follows.

In 1940, Lebesgue proved that every quadrangulated 3-polytope has $h \leq 11$. In 1995, this bound was lowered by Avgustinovich and Borodin to 10. Recently, Borodin and Ivanova improved it to the sharp bound 8.

For plane triangulation without 4-vertices, Borodin (1992), confirming the Kotzig conjecture of 1979, proved that $h \leq 20$, which bound is sharp. Later, Borodin (1998) proved that $h \leq 20$ for all triangulated 3-polytopes. In 1996, Horňák and Jendrol' proved for arbitrarily polytopes that $h \leq 23$. Recently, Borodin and Ivanova obtained the sharp bounds 10 for triangle-free polytopes and 20 for arbitrary polytopes.

In this paper we prove that any polytope has a face of degree at most 10 with height at most 20, where 10 and 20 are sharp.

**Keywords:** plane map, planar graph, 3-polytope, structural properties, height of face.

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## 1. Introduction

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [30], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs.

A plane map is *normal* (NPM) if each its vertex and face is incident with at least three edges. Clearly, every 3-polytope is an NPM.

The *degree* $d(x)$ of a vertex or face $x$ in an NPM $M$ is the number of incident edges. A *$k$-vertex* or *$k$-face* is one of degree $k$, a *$k^+$-vertex* has degree at least $k$, a *$k^-$-face* has degree at most $k$, and so on.

The *height* $h(f)$ of a face $f$ in $M$ is the maximum degree of its incident vertices. The *height* $h(M)$ (or simply $h$) of a map $M$ is the minimum height of faces in $M$.

A 3-face is *pyramidal* if it is incident with at least two $4^-$-vertices, and a 4-face is *pyramidal* if it is incident with at least three 3-vertices.

If $M$ has pyramidal faces, then $h$ can be arbitrarily large. Indeed, every face $f$ of the Archimedean $(3,3,3,n)$- and $(4,4,n)$-solids satisfies $h(f)=n$. We consider NPMs without pyramidal faces in what follows.

We now recall some results about the structure of $5^-$-faces in $M$ without pyramidal faces. By $\delta$ denote the minimum degree of vertices in $M$. We say that $f$ is a *face of type* $(k_1,k_2,\ldots)$ or simply $(k_1,k_2,\ldots)$*-face* if the set of its incident vertices is majorized by the vector $(k_1,k_2,\ldots)$.

In 1940, Lebesgue [26] gave an approximated description of $5^-$-faces in NPMs.

**Theorem 1** (Lebesgue [26]). *Every normal plane map has a $5^-$-face of one of the following types:*

$$(3,6,\infty),\ (3,7,41),\ (3,8,23),\ (3,9,17),\ (3,10,14),\ (3,11,13),$$
$$(4,4,\infty),\ (4,5,19),\ (4,6,11),\ (4,7,9),\ (5,5,9),\ (5,6,7),$$
$$(3,3,3,\infty),\ (3,3,4,11),\ (3,3,5,7),\ (3,4,4,5),\ (3,3,3,3,5).$$

The classical Theorem 1, along with other ideas in [26], has numerous applications to coloring problems on plane graphs (first examples of such applications and a recent survey can be found in [4, 28]). In 2002, Borodin [7] strengthened Theorem 1 in six parameters without worsening the others. However, the question in [7] of the best possible version(s) of Theorem 1 remains open, even for the special case of quadrangulations. Precise descriptions are obtained for NPMs with $\delta=5$ (Borodin [3]) and $\delta\geq 4$ (Borodin, Ivanova [9]), and also for triangulations (Borodin, Ivanova, Kostochka [15]).

Some parameters of Lebesgue's Theorem were improved for special classes of plane graphs. In 1989, Borodin [3] proved, confirming Kotzig's conjecture [24] of

1963, that every normal plane map with $\delta = 5$ has a $(5,5,7)$-face or $(5,6,6)$-face, where all parameters are the best possible. This result also confirmed Grünbaum's conjecture [19] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph so as to obtain two components each of which has a cycle) of every 5-connected plane graph is at most 11, which bound is sharp (earlier, Plummer [29] obtained the bound 13).

For plane triangulations without 4-vertices Kotzig [25] proved that $h \le 30$, and Borodin [5] proved, confirming Kotzig's conjecture [25], that $h \le 20$; this bound is the best possible, as follows from the construction obtained from the icosahedron by twice inserting a 3-vertex into each face. Borodin [6] further showed that $h \le 20$ for every triangulated 3-polytopes.

In 1940, Lebesgue [26] proved that every quadrangulated 3-polytope satisfies $h \le 11$. In 1995, this bound was improved by Avgustinovich and Borodin [1] to 10. Recently, Borodin and Ivanova [10] improved this bound to the sharp bound 8, and obtained the best possible bound 10 for triangle-free polytopes in [11].

Borodin and Loparev [8], with the additional assumption of the absence of $(3,5,\infty)$-faces, proved that there is either a 3-face with height at most 20, or 4-face with height at most 11, or 5-face of height at most 5, where bounds 20 and 5 are best possible. We note that the height of $5^-$-faces can reach 30 in the presence $(3,5,\infty)$-face due to the construction by Horňák and Jendrol' [20]. Furthermore, Horňák and Jendrol' [20] proved that $h \le 39$, which was recently improved by Borodin and Ivanova [14] to $h \le 30$.

Other results related to Lebesgue's Theorem can be found in the above mentioned papers and also in [2, 16–18, 21–23, 27, 31].

For arbitrary polytopes, Horňák and Jendrol' [20] (1996) proved that $h \le 23$. Recently, Borodin and Ivanova [13] improved this bound to the best possible bound 20.

The purpose of this paper is to refine the general bound 20 as follows.

**Theorem 2.** *Every normal plane map without pyramidal faces has a* $10^-$*-face of height at most* 20*, where both bounds* 10 *and* 20 *are sharp.*

## 2. Proof of Theorem 2

The bound 20 is attained at the triangulation described in Introduction, obtained from the icosahedron by two-fold putting 3-vertices in all faces.

Figure 1 shows how to transform the (3,3,3,3,5) Archimedean solid into a 3-polytope with no $9^-$-faces of height at most 20, which means that 10 is sharp. In particular, Figure 1 shows a fragment of the 3-polytope obtained.

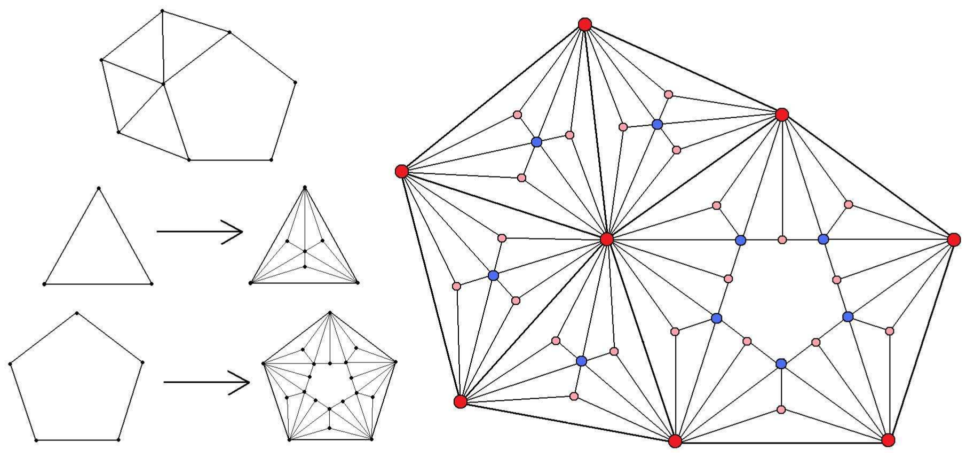

Figure 1. Each $9^-$-face is incident with a 22-vertex [12].

Now let a normal plane map $M'$ be a counterexample to Theorem 2. Starting from $M'$, we construct a counterexample $M$ to Theorem 2 with some useful properties.

The *operation* D1 consists in putting a diagonal incident with a $21^+$-vertex into a $4^+$-face $f$ that subdivides $f$ into two non-pyramidal faces. By the *operation* D2 we mean putting a 3-vertex into a face $xyz$ such that $d(x) \geq 21$, $d(y) \geq 21$, and $d(z) = 5$. Clearly, D2 does not create pyramidal faces, and each application of D1 or D2 transforms a counterexample to another counterexample with additional useful properties.

We first apply D1 to $M'$ as many times as possible, and then apply D2 as much as we can; after a finite number of steps this results in a counterexample $M$.

### 2.1. The structural properties of the counterexample $M$

**(P1)** *$M$ has no faces of degree from* 6 *to* 10*.* Since each such face $f$ is incident with a $21^+$-vertex $v$ by assumption, we apply the operation D1 to $f$ by joining $v$ with a vertex at distance at least 3 along the boundary of $f$. This results is splitting $f$ to two non-pyramidal $4^+$-faces with height at least 22, contrary to the maximality of $M$.

**(P2)** *$M$ has no $4^+$-face $f = \cdots xyz$, where $d(y) \geq 21$ and both $x$ and $z$ are $5^+$-vertices.* We can apply D1 to such a face by adding a diagonal incident with $y$, thus splitting $f$ into two non-pyramidal $3^+$-faces, a contradiction.

**(P3)** *$M$ has no 4-face $f = wxyz$, such that $d(y) \geq 21$ and $d(x) = d(z) = 3$.* Since $M$ has no pyramidal 4-faces, it would follow that $d(w) \geq 4$ and we could add the diagonal $yw$ to $f$.

**(P4)** *In $M$, a $21^+$-vertex cannot lie at distance two from a $4^+$-vertex in the boundary of an incident $4^+$-face $f$.* Otherwise, we could apply D1 by joining these vertices inside $f$.

**(P5)** *Every 5-vertex $v$ in $M$ is incident with an $11^+$-face $f$ of height at most 20.* Due to the oddness of $d(v)$, our $v$ has either two consecutive $20^-$-neighbors, or two consecutive $21^+$-neighbors.

If $v_1$ and $v_2$ are $21^+$-neighbors of $v$, then there is a 3-face $v_1vv_2$ according to D1, which means that we can apply D2, a contradiction.

Suppose $v_3$ and $v_4$ are $20^-$-neighbors of $v$. Hence there is a $4^+$-face $f = \cdots v_3vv_4$ (since $M$ has no $10^-$-face of height at most 20). If $f$ were incident with a $21^+$-vertex $z$, then we could join $v$ to $z$, contrary to the maximality of $M$. Hence $h(f) \leq 20$, which implies that $d(f) \geq 11$, as claimed.

**(P6)** *If $M$ has a 3-vertex $v$ incident with precisely two 3-faces, then $v$ has a $21^+$-neighbor and is incident with an $11^+$-face $f$ of height at most 20.* Suppose a 3-vertex $v$ is incident with a $4^+$-face $f = \cdots v_1vv_3$ and 3-faces $vv_1v_2$ and $vv_2v_3$. Note that $d(v_1) \geq 5$ and $d(v_3) \geq 5$ due to the absence of pyramidal 3-faces. On the other hand, if $d(v_1) \geq 21$, then we could apply D1 by inserting the diagonal $v_1v_3$, a contradiction. By symmetry, we have $d(v_1) \leq 20$ and $d(v_3) \leq 20$, which again implies that $h(f) \leq 20$ and $d(f) \geq 11$ by (P2). In turn, this implies that $d(v_2) \geq 21$, and we are done.

### 2.2. Discharging

Euler's formula $|V| - |E| + |F| = 2$ for $M$ implies

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12, \tag{1}$$

where $V$, $E$ and $F$ are the sets of vertices, edges, and faces of $M$, respectively.

We define the *initial charge* to be $\mu(v) = d(v) - 6$ whenever $v \in V$ and $\mu(f) = 2d(f) - 6$ whenever $f \in F$. Using the properties of $M$ as a counterexample, we locally redistribute the initial charges, preserving their sum, so as the *new charge* $\mu'(x)$ becomes non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of new charges is still $-12$ according to (1).

By $v_1, v_2, \ldots, v_{d(v)}$ denote the neighbors of a vertex $v$ in a cyclic order. A 4-face $wxyz$ is *special* if $d(x) = d(w) = 3$, $4 \leq d(y) \leq 20$, and $d(z) \geq 21$. A 3-vertex $v$ is *bad* if $v$ is incident with a 3-face $v_1vv_2$, where $d(v_1) \geq 21$, $5 \leq d(v_2) \leq 20$, special face $vv_2xv_3$ and $4^+$-face $\cdots v_1vv_3$ (see Figure 2, R3). Note that $d(x) \geq 21$. A vertex incident only with 3-faces is *simplicial*.

We use the following rules of discharging (see Figure 2).

**R1.** *Every 3-vertex not incident with 3-faces receives 1 from each incident face.*

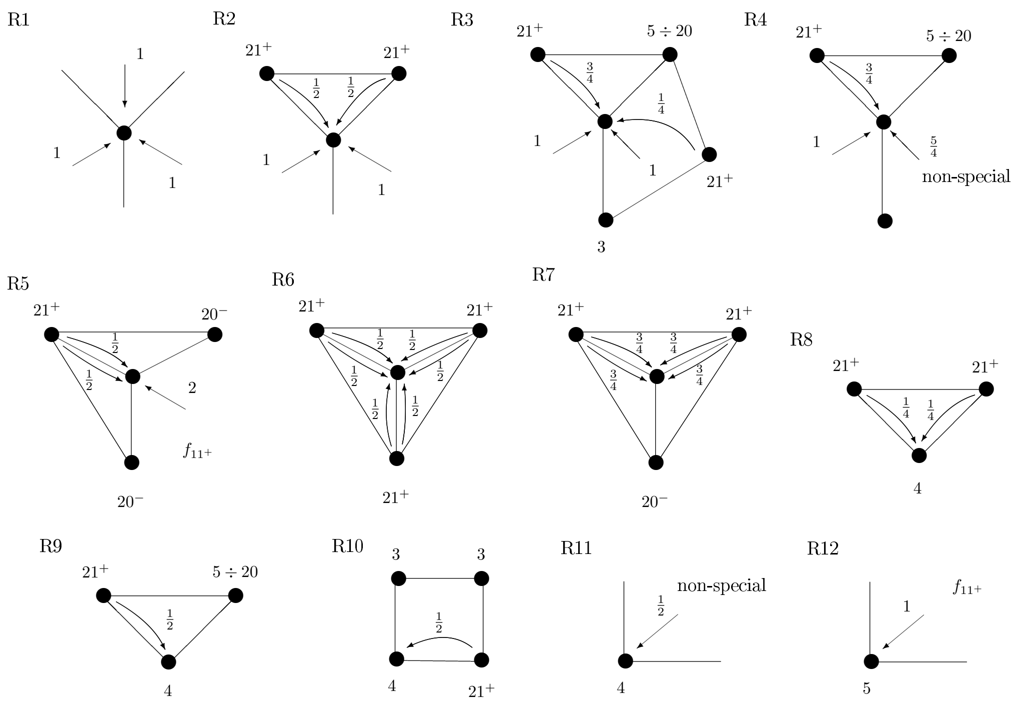


Figure 2. Rules of discharging.

**R2.** *Every* 3*-vertex* $v$ *incident with a unique triangle* $T = v_1vv_2$*, where* $d(v_i) \geq 21$, $1 \leq i \leq 2$*, receives* $\frac{1}{2}$ *from each* $v_i$ *through* $T$ *and* 1 *from each of the two incident* $4^+$*-faces.*

**R3.** *Every bad* 3*-vertex* $v$ *incident with a triangle* $T = v_1vv_2$ *with* $d(v_1) \geq 21$ *and* $5 \leq d(v_2) \leq 20$ *and special face* $f = vv_2xv_3$ *with* $d(x) \geq 21$ *receives* $\frac{3}{4}$ *from* $v_1$ *through* $T$*,* $\frac{1}{4}$ *from* $x$ *through* $f$*, and* 1 *from each of the two incident* $4^+$*-faces.*

**R4.** *Every* 3*-vertex* $v$ *incident with a unique triangle* $T = v_1vv_2$ *with* $d(v_1) \geq 21$ *and* $5 \leq d(v_2) \leq 20$ *and a non-special* $4^+$*-face* $f = \cdots v_2vv_3$ *receives* $\frac{3}{4}$ *from* $v_1$ *through* $T$*,* $\frac{5}{4}$ *from* $f$*, and* 1 *from the other incident* $4^+$*-face.*

**R5.** *Every* 3*-vertex* $v$ *incident with an* $11^+$*-face* $f = v_1vv_3\cdots$ *and two* 3*-faces receives* 2 *from* $f$ *and* $\frac{1}{2}$ *from the* $21^+$*-vertex* $v_2$ *through each incident* 3*-face.*

**R6.** *Every simplicial* 3*-vertex adjacent to three* $21^+$*-vertices receives* $\frac{1}{2}$ *from each of them through each incident face.*

**R7.** *Every simplicial* 3*-vertex adjacent to precisely two* $21^+$*-vertices receives* $\frac{3}{4}$ *from each of them through each incident face.*

**R8.** *Every* 4*-vertex* $v$ *incident with a triangle* $T = v_1vv_2$*, where* $d(v_i) \geq 21$, $1 \leq i \leq 2$*, receives* $\frac{1}{4}$ *from each* $v_i$ *through* $T$*.*

**R9.** *Every* 4*-vertex* $v$ *incident with a triangle* $T = v_1vv_2$*, where* $d(v_1) \geq 21$*,* $5 \leq d(v_2) \leq 20$*, receives* $\frac{1}{2}$ *from* $v_1$ *through* $T$*.*

**R10.** *Every* 4*-vertex incident with a special face* $f$ *receives* $\frac{1}{2}$ *through* $f$ *from the* $21^+$*-vertex incident with* $f$*.*

**R11.** *Every* 4*-vertex receives* $\frac{1}{2}$ *from each incident non-special* $4^+$*-face.*

**R12.** *Every* 5*-vertex* $v$ *receives* 1 *from each incident* $11^+$*-face.*

### 2.3. Proving that $\mu'(x) \geq 0$ whenever $x \in V \cup F$

*Case* 1. $f \in F$. Note that $d(f) \leq 5$ or $d(f) \geq 11$ due to (P1). We recall that every $10^-$-face is incident with a $21^+$-vertex.

Suppose $f = \cdots v_2v_1$. First suppose that $d(f) \geq 11$. If $f$ gives 2 to $v_2$ by R5, then $d(v_1) \geq 5$ and $d(v_3) \geq 5$ due to the absence of pyramidal 3-faces, so each of $v_1$ and $v_3$ receives at most 1 from $f$. If $f$ gives $\frac{5}{4}$ to $v_2$ by R4, then we can assume by symmetry that $d(v_1) \geq 5$ and again receives at most 1 from $f$.

If $v_2$ receives 2, then we move $\frac{1}{4}$ to the donations of each of $v_1$ and $v_3$, so that each of $v_1$, $v_2$, and $v_3$ now takes at most $\frac{3}{2}$ from $f$. As a result, we have $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{3}{2} = \frac{d(f)-12}{2} \geq 0$ for $d(f) \geq 12$.

If $d(f) = 11$, then there exist two consecutive vertices in the boundary of $f$, say $v_1$ and $v_2$, such that each of them takes less than 2, in fact at most $\frac{5}{4}$, from $f$. Furthermore, $f$ gives at most 1 to one of $v_1$ and $v_2$. After above movings of $\frac{1}{4}$, each of $v_1$, $v_2$ takes at most $\frac{5}{4}$ from $f$. This implies that $\mu'(f) \geq 2 \times 11 - 6 - 2 \times \frac{5}{4} - (11-2) \times \frac{3}{2} = 0$.

Now suppose $d(f) = 5$. If $f$ does not give $\frac{5}{4}$ by R4, then $\mu'(f) \geq 2 \times 5 - 6 - 4 \times 1 = 0$ since $f$ is incident with a $21^+$-vertex by assumption. Otherwise, the boundary of $f$ must have a path consisting of a 3-vertex $v_1$, a vertex $v_2$ of degree between 5 and 20, and a $21^+$-vertex $v_3$ due to (P4). However, this contradicts the maximality of $M$, since we can add the diagonal $v_1v_3$ without creating pyramidal faces.

Next suppose that $d(f) = 4$. Note that $f$ can give 1 or $\frac{5}{4}$ to 3-vertices by R1–R4 and $\frac{1}{2}$ to 4-vertices by R11. It remains to assume according to (P4) that $f$ is incident with at most two 3-vertices. If $f$ is incident with precisely two 3-vertices, then R4 is not applied to $f$, which implies $\mu'(f) = 2 \times 4 - 6 - 2 \times 1 = 0$ by R1–R3. Otherwise, we have $\mu'(f) \geq 2 - \frac{5}{4} - \frac{1}{2} > 2 - 1 - 2 \times \frac{1}{2} = 0$ due to R4 and R11.

Finally, if $d(f) = 3$ then $f$ does not participate in R1–R12, whence $\mu'(f) = \mu(f) = 0$.

*Case* 2. $v \in V$. Note that the charge is given according to R2–R10 only from $21^+$-vertices to $4^-$-vertices. Moreover, $v$ gives at most $\frac{3}{4}$ through each incident face. If $d(v) \geq 24$, then $\mu'(f) \geq d(v) - 6 - d(v) \times \frac{3}{4} = \frac{d(v)-24}{4} \geq 0$.

Suppose that $21 \le d(v) \le 23$. If $v$ gives $\frac{3}{4}$ through each face, then a 23-vertex has a *deficiency* $\frac{1}{4}$, and 22- and 21-vertices have deficiencies $\frac{1}{2}$ and $\frac{3}{4}$, respectively. In what follows, we will make sure that in fact $v$ *saves* something at certain faces with respect to the level of $\frac{3}{4}$. To estimate the total donation of $v$, we need the following observations.

(S1) $v$ gives nothing through a non-special $4^+$-face, which means that $v$ saves $\frac{3}{4}$ at such a face.

(S2) The saving of $v$ at an incident $(5^+, 5^+, 21^+)$-face is $\frac{3}{4}$.

(S3) Through a special $(3, 3, 5^+, 21^+)$-face, $v$ can transfer $\frac{1}{4}$ to a bad 3-vertex by R3, and so saves $\frac{1}{2}$ at such a face.

(S4) Through a special $(3, 3, 4, 21^+)$-face, $v$ transfers $\frac{1}{2}$ by R10, and so saves $\frac{1}{4}$.

(S5) $v$ transfers at most $\frac{1}{2}$ through a 3-face incident with a 4-vertex by R8, R9, and saves at least $\frac{1}{4}$.

(S6) As follows from (S4) and (S5), the presence of a 4-vertex $w$ adjacent to $v$ implies the total saving at least $\frac{1}{2}$ at the two faces incident with the edge $vw$.

(S7) Each participation of $v$ in R5 or R6 results in saving of $\frac{1}{4} + \frac{1}{4}$.

(S8) As follows from (S1)–(S5), the saving of $v$ at an incident face $f$ can equal zero only if $f$ is a 3-face incident with a 3-vertex, which happens only when one of R3, R4, and R7 is applied.

*Subcase* 2.1. $d(v) = 23$. To cover the deficiency of $\frac{1}{4}$, it suffices to have a face with a positive saving at $v$. Otherwise, according to (S8), the vertex $v$ is simplicial and the degrees of neighbors of $v$ alternate from 3 to $5^+$. The latter is impossible due to the oddness of $d(v)$.

*Subcase* 2.2. $d(v) = 22$. According to (S6), we can assume that $v$ has no 4-neighbors, which implies that we are done unless $v$ is simplicial due to (S1) and (S3). If so, then the degrees of neighbors of $v$ must alternate from 3 to $5^+$ in view of (S2). We now look at the eleven $5^+$-neighbors of $v$. By parity, there should exist a 3-neighbor, say $v_2$, such that either $d(v_1) \ge 21$ and $d(v_3) \ge 21$, or $d(v_1) \le 20$ and $d(v_3) \le 20$. This results in saving $2 \times \frac{1}{4}$ by $v$ at the two 3-faces incident with the edge $vv_2$ by (S7) due to R5 or R6 in view of (P4), as desired.

*Subcase* 2.3. $d(v) = 21$. We recall that now we need to find a total saving of $\frac{3}{4}$. We can assume that $v$ has no two consecutive $5^+$-neighbors, for otherwise this yields a 3-face by (P2), which takes nothing from $v$ by (S2), and we are done.

Since $d(v)$ is odd, there are two consecutive $4^-$-vertices $v_1$ and $v_{21}$, which form a $4^+$-face $f_{21} = \cdots v_1 v v_{21}$ due to the absence of any pyramidal face. If $d(v_1) = d(v_{21}) = 4$, then $v$ saves $\frac{3}{4}$ at the non-special face $f_{21}$ by our rules, and the same is true if $d(v_1) = d(v_{21}) = 3$. Therefore, we can assume that $d(v_1) = 3$ and $d(v_{21}) = 4$. This means that we are done unless $d(f_{21}) = 4$ and, moreover, $f_{21}$ is special and participates in R10. Hence $v$ saves $\frac{1}{4}$ at $f_{21}$.

Since at least $\frac{1}{4}$ is also saved at the face $f_{20} = \cdots v_{20}vv_{21}$ as mentioned in (S6), we can assume that $v$ has no saving at the other 19 faces.

According to (S8), all these 19 faces are triangles incident with 3-vertices. Due to the absence of pyramidal faces, we have $d(v_1) = d(v_3) = \cdots = d(v_{19}) = 3$, and each of these 3-vertices, except $v_1$, is simplicial and participates in R7. Hence, the degrees of $v_2, v_4, \ldots, v_{20}$ alternate from $21^+$ to $20^-$.

If $d(v_2) \geq 21$, then our $v$ saves another $\frac{1}{4}$ at the face $v_1vv_2$ according to R2, hence it remains to assume that $d(v_2) \leq 20$. This implies that $d(v_{20}) \geq 21$, which means that $d(f_{20}) = 3$ due to (P4), and $v$ actually saves as much as $\frac{1}{2}$ at $f_{20}$ according to R8. Due to $\frac{1}{4}$ saved at the face $f_{21}$, we have $\mu'(v) \geq 0$, as desired.

*Subcase* 2.5. $6 \leq d(v) \leq 20$. Since $v$ does not participate in R1–R12, it follows that $\mu'(v) = \mu(v) = d(v) - 6 \geq 0$.

*Subcase* 2.6. $d(v) = 5$. Note that $v$ is incident with an $11^+$-face due to (P5), so $\mu'(v) \geq 5 - 6 + 1 = 0$ by R12.

*Subcase* 2.7. $d(v) = 4$. Note that $v$ receives $\frac{1}{2}$ by R8–R11 from or through each incident face, whence $\mu'(v) \geq -2 + 4 \times \frac{1}{2} = 0$.

*Subcase* 2.8. $d(v) = 3$. A small case analysis based on the number of incident 3-faces shows in view of (P6) that we always have $\mu'(v) = -3 + 3 = 0$ by R1–R7.

Thus we have proved that $\mu'(x) \geq 0$ for all $x \in V \cup F$, this contradicts (1) and completes the proof of Theorem 2.

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