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THE COMPLEXITY OF SECURE DOMINATION PROBLEM IN GRAPHS

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Abstract

A dominating set of a graph G is a subset $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex in D. A dominating set S of G is called a secure dominating set if each vertex $u \in V(G) \setminus S$ has one neighbor v in S such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The secure domination problem is to determine a minimum secure dominating set of G. In this paper, we first show that the decision version of the secure domination problem is NP-complete for star convex bipartite graphs and doubly chordal graphs. We also prove that the secure domination problem cannot be approximated within a factor of $(1-\varepsilon)\ln|V|$ for any $\varepsilon > 0$, unless NP$\subseteq$DTIME$\big(|V|^{O(\log\log|V|)}\big)$. Finally, we show that the secure domination problem is APX-complete for bounded degree graphs.

Keywords: secure domination, star convex bipartite graph, doubly chordal graph, NP-complete, APX-complete.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. In a graph G with vertex set $V(G)$ and edge set $E(G)$, the *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* is $N_G[v] = \{v\} \cup N_G(v)$. For a subset $S \subseteq V(G)$, the *open neighborhood* of S is $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V(G) \setminus S$ such that $N_G(u) \cap S = \{v\}$. The *S-external private neighborhood* of $v \in S$, denoted by $epn_G(v, S)$, is the set of all S-external private neighbors of v. The *degree* $d_G(v)$ of v is defined as the cardinality of $N_G(v)$. If $d_G(v) = 0$, then v is said to be an *isolated vertex*. Let $\Delta(G)$ represent the *maximum degree* of G. In all cases above, we omit the subscript G when the graph G is clear from the context.

For a subset $S \subseteq V(G)$, the *subgraph induced* by S is the graph $G[S]$ with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$. If $G[S]$ is a complete subgraph of G, then S is called a *clique* of G. An *independent set* of G is a subset S of $V(G)$ such that $G[S]$ has no edge. We say that a graph G is a *split graph* if $V(G)$ can be partitioned into an independent set and a clique. A graph G is said to be *bipartite* if the vertex set $V(G)$ can be partitioned into two disjoint sets X and Y such that two endpoints of every edge of G lie in X and Y, respectively. We call a partition (X, Y) of $V(G)$ a *bipartition*. We write $G = (X, Y, E)$ for a bipartite graph with bipartition (X, Y) of $V(G)$. A *chord* of a cycle is an edge joining two vertices on the cycle that are not adjacent on the cycle. A *chordal graph* is a graph in which every cycle of length at least four has a chord.

Domination and its variations in graphs have been widely investigated as they have many applications in the real world and other disciplines such as computer networks, social networks, location theory, etc. For a detailed survey on this subject, we refer to the books [15,16] by Haynes, Hedetniemi and Slater. A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) \setminus D$ has at least one neighbor in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The *domination problem* is to find a minimum dominating set of a graph.

Another variation of domination, *secure domination* was introduced by Cockayne *et al.* [10]. A dominating set S of G is called a *secure dominating set* if each vertex $u \in V(G) \setminus S$ is adjacent to a vertex v in S such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The *secure domination number* $\gamma_s(G)$ of G is the minimum cardinality of a secure dominating set of G. The *secure domination problem* is to find a minimum secure dominating set of a graph. An application of secure domination for graph protection is presented in [6, 10]. Many of secure domination results in the literature concentrated on establishing tight bounds on $\gamma_s(G)$ for various graph classes in terms of different graph invariants, for example, in [5, 9, 10, 19–21]. Recently, the secure critical graphs are investigated in [7, 13].

On the complexity side of the secure domination problem, Merouane and Chellali [20] showed that the decision version of the secure domination problem is NP-complete for bipartite graphs and split graphs (a subclass of chordal graphs). Burger *et al.* [6] presented a linear time algorithm for the secure domination problem in trees. However, to the best of our knowledge, no result has been obtained on the approximability of the secure domination problem.

In this paper we continue to study the complexity of the secure domination problem. Firstly, we show that the decision version of the secure domination problem is NP-complete for star convex bipartite graphs and doubly chordal graphs. We also prove that the secure domination problem cannot be approximated within a factor of $(1-\varepsilon)\ln|V|$ for any $\varepsilon > 0$, unless NP$\subseteq$DTIME$\big(|V|^{O(\log\log|V|)}\big)$. Finally, we show that the secure domination problem is APX-complete for graphs with maximum degree 4.

2. PRELIMINARIES

A *tree* is a connected graph without cycles. A *star* is a special tree $T = (X, F)$, where $X = \{x_0, x_1, x_2, \ldots, x_n\}$ and $F = \{x_0x_i \mid 1 \leq i \leq n\}$. The vertex x_0 is said to be the *central* vertex, and $\{x_1, x_2, \ldots, x_n\}$ are called *leaves*. A bipartite graph $G = (X, Y, E)$ is called *star convex bipartite* [17] if there exists an associated star $T = (X, F)$ such that $N_G(y)$ induces a subtree of T for each vertex $y \in Y$.

Given a graph $G = (V, E)$, a vertex $v \in V$ is a *simplicial vertex* of G if $N_G[v]$ forms a clique of G. An ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of G is a *perfect elimination ordering* (PEO) of G if v_i is a simplicial vertex of the induced subgraph $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for all i, $1 \leq i \leq n$. A vertex $u \in N_G[v]$ is a *maximum neighbor* of v in G if $N_G[w] \subseteq N_G[u]$ holds for each $w \in N_G[v]$. A vertex $v \in V$ is called *doubly simplicial* if it is a simplicial vertex and it has a maximum neighbor in G. An ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of V is a *doubly perfect elimination ordering* (DPEO) of G if v_i is a doubly simplicial vertex of the induced subgraph $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for every i, $1 \leq i \leq n$. It is well known that a graph is *chordal* (*doubly chordal*) if and only if G has a PEO (DPEO); see [11] and [4].

To obtain our main results, we need the following result due to Cockayne *et al.* [10].

Proposition 1 ([10]). *If S is a secure dominating set of a graph G, then the subgraph induced by $epn_G(v, S)$ is complete for each $v \in S$.*

3. NP-COMPLETENESS RESULTS OF SECURE DOMINATION PROBLEM

In this section, we shall prove that the decision version of the secure domination problem is NP-complete for star convex bipartite graphs and doubly chordal graphs. Let SDOM denote the decision version of the secure domination problem.

Secure Domination Problem (SDOM)
Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Does G have a secure dominating set of cardinality at most k?

To show that SDOM is NP-complete for star convex bipartite graphs, we will make use of the well-known domination problem (DOM) which is NP-complete for bipartite graphs [3].

Domination Problem (DOM)
Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Does G have a dominating set of cardinality at most k?

Theorem 2. *The SDOM problem is NP-complete for star convex bipartite graphs.*

Proof. The SDOM problem is clearly in NP. We now describe a polynomial transformation from the DOM problem for bipartite graphs to the SDOM problem for star convex bipartite graphs.

Given a bipartite graph $G = (X, Y, E)$ with $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$, we construct the graph $G' = (X', Y', E')$ as follows. Let $X' = X \cup \{x_0, x\}$, $Y' = Y \cup \{y_0, y\}$, and $E' = E \cup \{xy_i \mid 0 \leq i \leq t\} \cup \{yx_i \mid 0 \leq i \leq s\} \cup \{xy\}$. Note that the graph G' is a star convex bipartite graph with an associated star $T = (X', F)$, where $F = \{xx_i \mid 0 \leq i \leq s\}$. Moreover, the construction of G' can be finished in polynomial time. We next show that G has a dominating set of cardinality at most k if and only if G' has a secure dominating set of cardinality at most $k + 2$.

Suppose first G has a dominating set D with $|D| \leq k$. Then it is easy to verify that $D \cup \{x, y\}$ is a secure dominating set of G' of cardinality at most $k+2$.

On the other hand, assume that S is a secure dominating set of G' with $|S| \leq k + 2$. Let $D = S \cap V(G)$. Clearly, D is not empty. If $x \notin S$ and $y \notin S$, then $x_0 \in S$ and $y_0 \in S$ to dominate x_0 and y_0 in G'. It is not hard to see that D is a dominating set of G with $|D| \leq k$, and so we are done. In what follows, we may assume that $|S \cap \{x, y\}| \geq 1$. We consider the following cases depending on the value of $|S \cap \{x, y\}|$.

Case 1. $|S \cap \{x, y\}| = 2$.

Case 1.1. $|S \cap \{x_0, y_0\}| = 2$. If D is a dominating set of G, then we are finished. Thus we assume that D is not a dominating set of G. Let $W \subseteq V(G) \setminus D$ be the set of vertices not dominated by D. Furthermore, we write $W_X = W \cap X$ and $W_Y = W \cap Y$. If $W_X \neq \emptyset$, then each vertex of W_X is only dominated by y in G'. Hence $epn_{G'}(y, S) = W_X$. Note that X is an independent set. By Proposition 1, $|W_X| = 1$. This implies that all vertices in $X \setminus W_X$ are dominated by D. Similarly, if $W_Y \neq \emptyset$, then we can deduce that $|W_Y| = 1$ and all vertices in $Y \setminus W_Y$ are also dominated by D. So we have $|W| \leq 2$. Let $D' = D \cup W$. Then D' is a dominating set of G with $|D'| \leq |D| + 2 \leq k$.

Case 1.2. $|S \cap \{x_0, y_0\}| = 1$. Without loss of generality, suppose that $x_0 \notin S$ and $y_0 \in S$. Recall that S is a secure dominating set of G'. Then $(S \setminus \{y\}) \cup \{x_0\}$ is a dominating set of G'. Thus all vertices in X are dominated by D. If D is a dominating set of G, then we are done. Hence we assume that D is not a dominating set of G. Let $W \subseteq V(G) \setminus D$ be the set of vertices in which every vertex is not dominated by D. Obviously, $W \subseteq Y \setminus D$. By a similar argument that used in the proof of Case 1.1, we can obtain that $|W| = 1$ and all vertices in $Y \setminus W$ are dominated by D. Let $D' = D \cup W$. Then D' is a dominating set of G with $|D'| \leq |D| + 1 \leq k$.

Case 1.3. $|S \cap \{x_0, y_0\}| = 0$. Then we have $x_0 \notin S$ and $y_0 \notin S$. Notice that S is a secure dominating set of G'. Hence $(S \setminus \{y\}) \cup \{x_0\}$ is a dominating set of G'. This implies that all vertices of X are dominated by D. Moreover, $(S \setminus \{x\}) \cup \{y_0\}$ is also a dominating set of G', implying that all vertices in Y are dominated by D. Therefore D is a dominating set of G of cardinality at most k.

Case 2. $|S \cap \{x, y\}| = 1$. Without loss of generality, suppose that $x \in S$ and $y \notin S$. Then $x_0 \in S$ to dominate x_0. Furthermore, all vertices in X are dominated by D. Note that $|D| \leq k$. If D is a dominating set of G, then we are done. Thus we assume that D is not a dominating set of G. If $y_0 \notin S$, then $(S\setminus \{x\}) \cup \{y_0\}$ is a dominating set of G', since S is a secure dominating set of G'. This derives that each vertex of Y is dominated by D, and hence D is a dominating set of G, a contradiction to our assumption. So $y_0 \in S$ and $|D| \leq k - 1$. Let $W \subseteq V(G) \setminus D$ be the set of vertices not dominated by D. Clearly, $W \subseteq Y \setminus D$. Applying a similar argument that used in the proof of Case 1.1, it follows that $|W| = 1$ and each vertex of $Y \setminus W$ is dominated by D. Let $D' = D \cup W$. So D' is a dominating set of G with $|D'| \leq |D| + 1 \leq k$. This completes the proof of Theorem 2. ∎

Next, we show that SDOM is NP-complete for doubly chordal graphs by proposing a polynomial time reduction from the well-known NP-complete problem, called Exact Cover by 3-Sets (X3C) [12] which is stated below.

Exact Cover by 3-Sets (X3C)
Instance: A finite set X with $|X| = 3q$ and a collection $\mathcal{C}$ of 3-element subsets of X.
Question: Does $\mathcal{C}$ contain an *exact cover* for X, that is, a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that every element in X belongs to exactly one member of $\mathcal{C}'$?

Theorem 3. *The SDOM problem is NP-complete for doubly chordal graphs.*

Proof. Obviously, the SDOM problem belongs to NP. To prove that the SDOM problem for doubly chordal graphs is NP-hard, we present a polynomial time reduction from Exact Cover by 3-Sets (X3C) to it. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an instance I of X3C. We transform I to the instance (G_I, k) of the SDOM problem in which $k = q + 2$ and G_I is the doubly chordal graph formed as follows.

Firstly, we construct a split graph $G = (V, E)$, where V is partitioned into an independent set $X = \bigcup_{i=1}^{3q}\{x_i\}$ and a clique $C = \bigcup_{j=1}^{m}\{c_j\}$. Each vertex $x_i \in X$ is corresponding to an element x_i in X, while each vertex $c_j \in C$ is corresponding to a 3-element subset C_j in $\mathcal{C}$. An edge x_ic_j exists in E if and only if the element x_i belongs to the subset C_j. The graph G_I is obtained from G by adding a path of length three, say $P = uvxy$, and joining the vertex u to each vertex of V. Clearly, G_I is a doubly chordal graph, since it admits a DPEO $\{x_1, x_2, \ldots, x_{3q}, c_1, c_2, \ldots, c_m, u, v, x, y\}$, and the construction of G_I can be completed in polynomial time.

We next show that I has an exact cover of cardinality q if and only if G_I has a secure dominating set of cardinality at most $q + 2$.

Suppose first I has an exact cover $\mathcal{C}'$ of cardinality q . It is not hard to verify that $\{c_j \mid C_j \in \mathcal{C}'\} \cup \{u, x\}$ is a secure dominating set of G_I of cardinality at most $q + 2$.

On the other hand, assume that S is a secure dominating set of G_I with cardinality at most $q + 2$.

Claim 1. $u \in S$.

Proof. Suppose on the contrary that $u \notin S$. Since S is a secure dominating set of G_I, we have $|S \cap \{v, x, y\}| \geq 2$. Then $|S \cap (X \cup C)| \leq q$. Let $|S \cap X| = s$. So $|S \cap C| \leq q - s$. If $s \geq 1$, then $S \cap (X \cup C)$ dominates at most $3(q - s) + s \leq 3q - 2$ vertices of X, a contradiction. Hence $s = 0$, and so $|S \cap C| = q$ in order to dominate all vertices in X. Further, each vertex $x_i \in X$ is dominated by only one vertex of $S \cap C$, say c_j. Then $(S \setminus \{c_j\}) \cup \{x_i\}$ is not a dominating set of G_I, which is a contradiction. This completes the proof of Claim 1. □

To dominate y, $|S \cap \{x, y\}| \geq 1$. By Claim 1, $|S \cap (X \cup C)| \leq q$.

Claim 2. $|S \cap X| = 0$.

Proof. Otherwise, suppose $|S \cap X| = t \geq 1$. Then $|S \cap C| \leq q - t$. Hence $S \cap (X \cup C)$ dominates at most $3(q - t) + t \leq 3q - 2$ vertices of X. This means that at least two vertices of X are only dominated by u. So $|epn_{G_I}(u, S) \cap X| \geq 2$. By Proposition 1, the subgraph induced by $epn_{G_I}(u, S)$ is complete, however, this is impossible as X is an independent set. This completes the proof of Claim 2. □

According to Claim 2, $|S \cap C| \le q$. By using an analogous argument as in the proof of Claim 2, we can show that $|S \cap C| = q$. Moreover, we claim that each vertex of X is also dominated by $S \cap C$ in G_I. Otherwise, assume that there is a vertex $x_i \in X$ such that it is not dominated by $S \cap C$ in G_I. Then x_i is only dominated by u in G_I. Notice that $|S \cap \{x, y\}| \ge 1$, $|S \cap C| = q$ and $|S| \le q + 2$. By Claim 1, we have $|S \cap \{x, y\}| = 1$. If $y \in S$, then $\{x_i, v\} \subseteq epn_{G_I}(u, S)$, which contradicts to Proposition 1. Hence $x \in S$. Since S is a secure dominating set of G_I, it follows that either $(S \setminus \{u\}) \cup \{v\}$ or $(S \setminus \{x\}) \cup \{v\}$ is a dominating set of G_I. However, it is impossible as $vx_i \notin E(G_I)$ and $vy \notin E(G_I)$. Therefore $3q$ vertices in X are dominated by using precisely q vertices in C. This implies that $\mathcal{C}' = \{C_j \mid c_j \in S \cap C\}$ is an exact cover of I. This completes the proof of Theorem 3. ■

4. Inapproximability of Secure Domination Problem

In this section, we investigate the approximation hardness of the secure domination problem. To our aim, we need the following result due to Chlebík and Chlebíková [8].

Theorem 4 ([8]). *If there is some $\varepsilon > 0$ such that a polynomial time algorithm can approximate the domination problem for a general graph $G = (V, E)$ within a ratio of $(1 - \varepsilon) \ln |V|$, then NP$\subseteq$DTIME$\left(|V|^{O(\log \log |V|)}\right)$. The same result is true for split graphs.*

Theorem 5. *If there is some $\varepsilon > 0$ such that a polynomial time algorithm can approximate the secure domination problem for a general graph $G = (V, E)$ within a ratio of $(1 - \varepsilon) \ln |V|$, then NP$\subseteq$DTIME$\left(|V|^{O(\log \log |V|)}\right)$.*

Proof. We establish an approximation preserving reduction from the domination problem to the secure domination problem as follows. Given a graph $G = (V, E)$, we construct the graph $G' = (V', E')$ by adding two vertices $\{a, b\}$ to G and connecting the vertex a to each vertex of $V \cup \{b\}$.

Claim 3. *G has a dominating set of cardinality at most k if and only if G' has a secure dominating set of cardinality at most $k + 1$.*

Proof. Suppose first G has a dominating set D with $|D| \le k$. It is easy to verify that $D \cup \{a\}$ is a secure dominating set of G' of cardinality at most $k + 1$.

Conversely, assume that S is a secure dominating set of G' with $|S| \le k + 1$. Let $D = S \cap V$. To dominate b, $|S \cap \{a, b\}| \ge 1$. Suppose $|S \cap \{a, b\}| = 1$. Thus we have either $a \in S$ and $b \notin S$ or $a \notin S$ and $b \in S$. If $a \in S$ and $b \notin S$, then, by Proposition 1, D is a dominating set of G of cardinality at most k. If $a \notin S$ and $b \in S$, then it is clear that D is a dominating set of G with $|D| \le k$. Suppose

$|S \cap \{a,b\}| = 2$. If $epn_{G'}(a,S) = \emptyset$, then D is a dominating set of G of cardinality at most $k-1$. If $epn_{G'}(a,S) \neq \emptyset$, then $epn_{G'}(a,S)$ is complete according to Proposition 1. Hence $D \cup \{v\}$ for some vertex $v \in epn_{G'}(a,S)$ is a dominating set of G of cardinality at most k. This completes the proof of Claim 3. □

Assume that there exists some (fixed) $\varepsilon > 0$ such that the secure domination problem for graphs with n vertices can be approximated within a ratio of $\alpha = (1-\varepsilon)\ln n$ by using an algorithm A_{SD} that runs in polynomial time. Let $r > 0$ be an integer. We construct the following algorithm.

Algorithm A_{GD}
Input: A graph $G = (V,E)$.
1. **if** there is a minimum dominating set D of G with $|D| < r$ **then**
   output D;
   else
2. Construct the graph G';
3. Compute a secure dominating set S in G' using the algorithm A_{SD};
4. **if** $|S \cap \{a,b\}| = 1$ or $|S \cap \{a,b\}| = 2$ and $epn_{G'}(a,S) = \emptyset$ **then**
   $D = S \cap V$;
5. **if** $|S \cap \{a,b\}| = 2$ and $epn_{G'}(a,S) \neq \emptyset$ **then**
   $D = (S \cap V) \cup \{v\}$ for some vertex $v \in epn_{G'}(a,S)$;
6. Output D.

Firstly, if there is a minimum dominating set D of G with $|D| < r$, then it can be computed in polynomial time. Secondly, the algorithm A_{SD} runs in polynomial time. Thus the algorithm A_{GD} runs in polynomial time. Note that if D is outputted in step 1 of algorithm A_{GD}, then it must be a minimum dominating set of G of cardinality less than r. In the following, we will analyze the case where D is outputted in next steps of algorithm A_{GD}.

Let D^* and S^* be a minimum dominating set of G and a minimum secure dominating set of G', respectively. Then $|D^*| \geq r$, and $|S^*| = |D^*| + 1$ by Claim 3. Given a graph $G = (V,E)$, the algorithm A_{GD} can compute a dominating set of G of size $|D| \leq |S| - 1 \leq \alpha|S^*| - 1 < \alpha(|D^*| + 1) = \alpha(1 + 1/|D^*|)|D^*| \leq \alpha(1+1/r)|D^*|$. Recall that $\alpha = (1-\varepsilon)\ln n = (1-\varepsilon)\ln(|V|+2) \leq (1-\varepsilon)\ln(3|V|) = (1-\varepsilon)(1+\ln 3/\ln|V|)\ln|V|$. Let $r > 0$ be an integer such that $(1+1/r) < (1+\varepsilon/2)$ for some (fixed) $\varepsilon > 0$. In addition, the term $(1+\ln 3/\ln|V|)$ can be bounded by $(1+\varepsilon/2)$ when the value of $|V|$ is sufficiently large. Hence $\alpha(1+1/r) \leq (1-\varepsilon)(1+1/r)(1+\ln 3/\ln|V|)\ln|V| \leq (1-\varepsilon)(1+\varepsilon/2)^2\ln|V| = (1-\varepsilon')\ln|V|$, where $\varepsilon' = 3\varepsilon^2/4 + \varepsilon^3/4$. Then the algorithm A_{GD} approximates the domination problem within a ratio of $(1-\varepsilon')\ln|V|$. By Theorem 4, the desired result follows. ■

Notice that if the graph G is a split graph, then the constructed graph G' in Theorem 5 is also a split graph. Hence we immediately obtain the corollary below.

Corollary 6. *If there is some $\varepsilon > 0$ such that a polynomial time algorithm can approximate the secure domination problem for a split graph $G = (V, E)$ within a ratio of $(1 - \varepsilon) \ln |V|$, then NP$\subseteq$DTIME$\left(|V|^{O(\log \log |V|)}\right)$.*

5. APX-Completeness of Secure Domination Problem

In this section, we will show that the secure domination problem is APX-complete for graphs with maximum degree 4. Now let us recall how to prove APX-completeness of an NP-optimization problem Π. Firstly, Π must be in APX, i.e., Π can be approximated in polynomial time within a constant ratio. Then it is enough to show that there exists an L-reduction from some APX-complete problem to Π. In [2,22], the notation of L-reduction is formally defined as follows.

Given two NP-optimization problems Π_1 and Π_2 and a polynomial time transformation f from instances of Π_1 to instances of Π_2, f is called an *L-reduction* if there are two positive constants α and β such that for every instance x of Π_1:
1. $opt_{\Pi_2}(f(x)) \le \alpha \cdot opt_{\Pi_1}(x)$, where opt_{Π_2} and opt_{Π_1} are the optima of $f(x)$ and x, respectively;
2. for every feasible solution y of $f(x)$ with objective value $m_{\Pi_2}(f(x), y) = c_2$, we can in polynomial time find a solution y' of x with $m_{\Pi_1}(x, y') = c_1$ such that $|opt_{\Pi_1}(x) - c_1| \le \beta \cdot |opt_{\Pi_2}(f(x)) - c_2|$.

To show that the secure domination problem is in APX for bounded degree graphs, we need the concept of k-tuple domination introduced by Harary and Haynes [14]. For a fixed positive integer k, a subset D of vertices of a graph G is said to be a *k-tuple dominating set* of G if $|N_G[v] \cap D| \ge k$ for every vertex $v \in V(G)$. The *k-tuple domination number* $\gamma_{\times k}(G)$ of G is the cardinality of a minimum k-tuple dominating set of G. The *k-tuple domination problem* is to find a minimum k-tuple dominating set of a graph. The following theorem is obtained by Klasing and Laforest [18].

Theorem 7 ([18])**.** *The k-tuple domination problem in any graph with maximum degree $\triangle$ can be approximated within an approximation ratio of* $\ln(\Delta + 1) + 1$.

In [20], Merouane and Chellali proved that $\gamma_{\times 2}(G) \le 2\gamma_s(G)$ for a graph G without isolated vertices. Notice that a 2-tuple dominating set (double dominating set) of a graph G is also a secure dominating set of G. By Theorem 7, we immediately have the following result.

Theorem 8. *The secure domination problem in any graph with maximum degree $\triangle$ and minimum degree $\delta \ge 1$ can be approximated within an approximation ratio of* $2(\ln(\Delta + 1) + 1)$.

Theorem 9. *The secure domination problem is APX-complete for graphs with maximum degree* 4.

Proof. Since isolated vertices are trivially in any (secure) dominating set, we may without loss of generality assume that the graphs contain no isolated vertices in the APX-completeness proof. By Theorem 8, the secure domination problem is in APX for bounded degree graphs. It can be found in [1] that the domination problem is APX-complete for graphs with maximum degree 3. To prove our theorem, we present an L-reduction f from instances of the domination problem for graphs with maximum degree 3 to the instances of the secure domination problem for graphs with maximum degree 4. Given a graph $G = (V, E)$ of maximum degree 3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct the graph $G' = (V', E')$ by adding a path of length two, say $P_i = v_i x_i y_i$, to every vertex v_i of G. For notation convenience, let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Thus $V' = V \cup X \cup Y$ and $E' = E \cup (\bigcup_{i=1}^{n} \{v_i x_i, x_i y_i\})$. Also, G' has maximum degree 4.

Claim 4. *G has a dominating set of cardinality at most k if and only if G' has a secure dominating set of cardinality at most $k + n$.*

Proof. Suppose first G has a dominating set D with $|D| \leq k$. It is easy to check that $D \cup X$ is a secure dominating set of G' of cardinality at most $k + n$.

On the other hand, assume that S is a secure dominating set of G' with $|S| \leq k + n$. If $y_i \notin S$ for some $1 \leq i \leq n$, then x_i must belong to S. If $x_i \notin S$ and $y_i \in S$ for some $1 \leq i \leq n$, then $(S \setminus \{y_i\}) \cup \{x_i\}$ is also a secure dominating set of G' with the same size as that of S. Hence we may assume that S contains all vertices of X. Suppose S contains some $y_i \in Y$. Then $(S \setminus \{y_i\}) \cup \{v_i\}$ is also a secure dominating set of G' of cardinality at most $k + n$. Therefore we may assume that none of Y belongs to S. Let $D = S \cap V(G)$. Then we have $|D| \leq k$. By Proposition 1, D is a dominating set of G. This completes the proof of Claim 4. □

Let D^* and S^* be a minimum dominating set of G and a minimum secure dominating set of G', respectively. Then we have $|S^*| = |D^*| + n$ by Claim 4. Since G is of bounded degree 3, each vertex of D^* can dominate at most four vertices of G, and so $|D^*| \geq n/4$. Consequently, $|S^*| = |D^*| + n \leq 5|D^*|$. For the converse, let S be any secure dominating set of G'. From the proof of Claim 4, we can in polynomial time construct a dominating set $D = S \cap V$ of G with cardinality $|D| \leq |S| - n$. Hence $|D| - |D^*| \leq |S| - |S^*|$. This means that f is an L-reduction with $\alpha = 5$ and $\beta = 1$. The proof of Theorem 9 is completed. ■

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