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BOUNDING THE OPEN k-MONOPOLY NUMBER OF STRONG PRODUCT GRAPHS

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Abstract

Let $G = (V, E)$ be a simple graph without isolated vertices and minimum degree δ, and let $k \in \{1 - \lceil \delta/2 \rceil, \ldots, \lfloor \delta/2 \rfloor\}$ be an integer. Given a set $M \subset V$, a vertex v of G is said to be k-controlled by M if $\delta_M(v) \geq \frac{\delta_G(v)}{2} + k$, where $\delta_M(v)$ represents the number of neighbors of v in M and $\delta_G(v)$ the degree of v in G. A set M is called an open k-monopoly if every vertex v of G is k-controlled by M. The minimum cardinality of any open k-monopoly is the open k-monopoly number of G. In this article we study the open k-monopoly number of strong product graphs. We present general lower and upper bounds for the open k-monopoly number of strong product graphs. Moreover, we study in addition the open 0-monopolies of several specific families of strong product graphs.

Keywords: open monopolies, strong product graphs, alliances, domination.

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1. Introduction and Preliminaries

Let $G = (V, E)$ be a simple graph. Given a set $S \subset V$ and a vertex $v \in V$, we denote by $\delta_S(v)$ the number of neighbors of v in S. If $S = V$, then $\delta_V(v)$ is the degree of v and we write $\delta_G(v)$ (or just $\delta(v)$, if there are no misunderstandings with the graph G). The *minimum degree* of G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. If there is no confusion possible, then we use shorter version δ and Δ for minimum and maximum degree of G. Given an integer $k \in \left\{1 - \left\lceil \frac{\delta}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta}{2} \right\rfloor\right\}$ and a set M, a vertex v of G is said to be *k-controlled* by M if $\delta_M(v) \geq \frac{\delta_G(v)}{2} + k$ (in this case we say that M *k-controls* V). The set M is called an open *k-monopoly* if it k-controls every vertex v of G. The minimum cardinality of any open k-monopoly is the open *k-monopoly number* and is denoted by $\mathcal{M}_k(G)$. An open monopoly of cardinality $\mathcal{M}_k(G)$ is called an $\mathcal{M}_k(G)$-set. In particular, notice that for a graph with a leaf (vertex of degree one), there exist only open 0-monopolies and the neighbor of every leaf is in each $\mathcal{M}_0$-set. The degree $\delta_G(v)$ in the definition of an open k-monopoly can clearly be replaced by the cardinality of its open neighborhood $|N_G(v)|$. Open k-monopolies in graphs were introduced in [12] as a natural contrast to closed monopolies which use close neighborhood instead of open (and $k = 0$). Closed monopolies were studied earlier in [13], called just monopolies there. Other studies about closed monopolies in graphs and some of its applications can be found in [5, 8, 14, 15, 19]. Since we are only interested in the open monopolies, we skip the term open in what follows.

Several applications to practical problems have been described for (open and closed) monopolies in graphs. A great part of these applications are related to the notion of overcomes and failures, considering the fact that monopolies have frequent applications in the notion of majorities: consensus problems [3], diagnosis problems [17] or voting systems [6], among other applications and references. Monopolies in graphs are also closely related to different parameters in graphs. According to several connections which exist between monopolies, global alliances and signed domination in graphs (see [12]), it is known that the complexity of computing the k-monopoly number of a graph is an NP-hard problem for any suitable k (see [12, 16]). In this sense, it is desirable to study the k-monopoly number of some particular families of graphs. In this article we obtain general and particular bounds for the k-monopoly number of strong product graphs.

Studies about graph products have been appearing very frequently in the last few decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. the new book [7]. The most studied graph products are the Cartesian product, the strong product, the direct product, and the lexicographic product, which are also called *standard products*. A standard approach to graph products is to deduce properties of the product with respect to the same or different properties of its factors. In this sense, the topic which we deal with has been recently studied for the direct product [10] and for the lexicographic product [11] of graphs. It is now our goal to study the strong product graph with respect to its monopolies.

An equivalent definition for a k-monopoly in G can be described in the following way. Let $\overline{M}$ be the *complement* of a set M. Hence, a set of vertices M is a k-monopoly in G if and only if for every vertex v of G, $\delta_M(v) \geq \delta_{\overline{M}}(v) + 2k$.

From now on, given a graph G, we use the standard notations $N_G(g)$ for the *open neighborhood* $\{g' : gg' \in E(G)\}$ and $N_G[g]$ for the *closed neighborhood* $N_G(g) \cup \{g\}$. The *strong product* $G \boxtimes H$ of graphs G and H is a graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \boxtimes H$ whenever ($gg' \in E(G)$ and $h = h'$) or ($g = g'$ and $hh' \in E(H)$) or ($gg' \in E(G)$ and $hh' \in E(H)$). For a fix $h \in V(H)$ we call $G^h = \{(g, h) \in V(G \boxtimes H) : g \in V(G)\}$ a *G-layer* in $G \boxtimes H$. An H-layer ${}^gH$ for a fix $g \in V(G)$ is defined symmetrically. Notice that a subgraph induced by G^h and ${}^gH$ are isomorphic to G and H, respectively. The commutativity of the strong product follows from the symmetry of the definition of adjacency, while for associativity, see [7]. The closed neighborhoods of vertices in strong product graphs are nicely connected to closed neighborhoods of projections to the factors

$$N_{G \boxtimes H}[g, h] = N_G[g] \times N_H[h].$$

Notice that the same situation does not occur with open neighborhoods in strong product.

2. General Bounds

To begin the description of our results we give some general bounds for the k-monopoly number of strong product graphs in terms of the order and the monopoly number of the factor graphs. Some of these results are related to the alliances of the factor graphs. Alliances were presented first in [9], and for more information about alliances in graphs we suggest the recent survey [18].

For $k \in \{-\Delta, \ldots, \Delta\}$, a nonempty set $A \subseteq V$ is a *defensive k-alliance* in G if for every $v \in A$ it follows that

$$\delta_A(v) \geq \delta_{\overline{A}}(v) + k. \tag{1}$$

Moreover, for $k \in \{2-\Delta, \ldots, \Delta\}$, a nonempty set $A \subseteq V$ is an *offensive k-alliance* in G if for every $v \in \partial A$ it follows that

$$\delta_A(v) \geq \delta_{\overline{A}}(v) + k, \tag{2}$$

where ∂A is the set of all vertices not in A which are adjacent to at least one vertex of A.

For $k \in \{-\Delta, \ldots, \Delta-2\}$, if A is a defensive k-alliance as well as an offensive $(k+2)$-alliance, then A is a *powerful k-alliance*. A powerful k-alliance is called *global* if it is a dominating set. The *global powerful k-alliance number* of G, denoted by $\gamma_k^p(G)$, is defined as the minimum cardinality of a global powerful k-alliance in G. For a global powerful k-alliance A, it is easy to see that $\partial A = \overline{A}$. Hence $\delta_A(v) \geq \delta_{\overline{A}}(v) + k$ holds for every $v \in A$ and $\delta_A(v) \geq \delta_{\overline{A}}(v) + k + 2$ holds for every $v \in \overline{A}$.

Theorem 1. *Let G and H be two graphs and let $\ell = \min\{\delta(G), \delta(H)\}$. If $k \in \left\{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\right\}$, then*

$$\mathcal{M}_k(G \boxtimes H) \leq \min\left\{\gamma_k^p(G)|V(H)|, |V(G)|\gamma_k^p(H)\right\}.$$

Proof. Let (g,h) be an arbitrary vertex of $G \boxtimes H$ and let A_G and A_H be a $\gamma_k^p(G)$-set and a $\gamma_k^p(H)$-set, respectively. Hence, if $g \in A_G$, then $\delta_{A_G}(g) \geq \delta_{\overline{A_G}}(g) + k$, and if $g \notin A_G$, then $\delta_{A_G}(g) \geq \delta_{\overline{A_G}}(g) + k + 2$. On the other hand, if $h \in A_H$, then $\delta_{A_H}(h) \geq \delta_{\overline{A_H}}(h) + k$, and if $h \notin A_H$, then $\delta_{A_H}(h) \geq \delta_{\overline{A_H}}(h) + k + 2$. Hence, we can split the neighborhood $N_{G\boxtimes H}(g,h)$ of a vertex (g,h) of $G \boxtimes H$ to the following disjoint union of sets

$$(A \times C) \cup (B \times C) \cup (A \times D) \cup (B \times D) \cup (\{g\} \times C) \cup (\{g\} \times D) \cup (A \times \{h\}) \cup (B \times \{h\}),$$

where $A = N_G(g) \cap A_G$, $B = N_G(g) \cap \overline{A_G}$, $C = N_H(h) \cap A_H$ and $D = N_H(h) \cap \overline{A_H}$. Clearly, $|A \times C| = \delta_{A_G}(g)\delta_{A_H}(h)$, $|B \times C| = \delta_{\overline{A_G}}(g)\delta_{A_H}(h)$, $|A \times D| = \delta_{A_G}(g)\delta_{\overline{A_H}}(h)$, $|B \times D| = \delta_{\overline{A_G}}(g)\delta_{\overline{A_H}}(h)$, $|\{g\} \times C| = \delta_{A_H}(h)$, $|\{g\} \times D| = \delta_{\overline{A_H}}(h)$, $|A \times \{h\}| = \delta_{A_G}(g)$ and $|B \times \{h\}| = \delta_{\overline{A_G}}(g)$.

By the commutativity of the strong product, it is enough to show that $\mathcal{M}_k(G \boxtimes H)$ is bounded from above by $|V(G)|\gamma_k^p(H)$. Let $M = V(G) \times A_H$. Clearly $|M| = |V(G)|\gamma_k^p(H)$. Also, $A \times C$, $B \times C$ and $\{g\} \times C$ are subsets of M, while $A \times D$, $B \times D$ and $\{g\} \times D$ are subsets of $\overline{M}$. Sets $A \times \{h\}$ and $B \times \{h\}$ are subsets of M whenever $h \in A_H$ and subsets of $\overline{M}$ whenever $h \in \overline{A_H}$. We will show that $\delta_M(g,h) \geq \frac{\delta_{G\boxtimes H}(g,h)}{2} + k$ for every vertex (g,h) and every $k \in \left\{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\right\}$, where $\ell = \min\{\delta(G), \delta(H)\}$. We consider the following cases.

Case 1. If $h \in A_H$, then, since A_H is a global defensive k-alliance, we have that

$$\begin{aligned}
\delta_M(g,h) - \delta_{\overline{M}}(g,h) &= \delta_{A_G}(g)\delta_{A_H}(h) + \delta_{\overline{A_G}}(g)\delta_{A_H}(h) + \delta_{A_H}(h) + \delta_{A_G}(g) \\
&\quad + \delta_{\overline{A_G}}(g) - \delta_{A_G}(g)\delta_{\overline{A_H}}(h) - \delta_{\overline{A_G}}(g)\delta_{\overline{A_H}}(h) - \delta_{\overline{A_H}}(h) \\
&= \delta_{A_G}(g)(\delta_{A_H}(h) - \delta_{\overline{A_H}}(h) + 1) \\
&\quad + \delta_{\overline{A_G}}(g)(\delta_{A_H}(h) - \delta_{\overline{A_H}}(h) + 1) + \delta_{A_H}(h) - \delta_{\overline{A_H}}(h)
\end{aligned}$$

$$\geq \delta_{A_G}(g)(k+1) + \delta_{\overline{A_G}}(g)(k+1) + k = (k+1)\delta_G(g) + k \geq 2k+1.$$

Now, since $\delta_{G\boxtimes H}(g,h) = \delta_M(g,h) + \delta_{\overline{M}}(g,h)$, from the above we have that

$$\delta_M(g,h) \geq \frac{\delta_{G\boxtimes H}(g,h) + 2k + 1}{2} = \frac{\delta_{G\boxtimes H}(g,h)}{2} + k + \frac{1}{2}. \tag{3}$$

Case 2. If $h \in \overline{A_H}$, then, since A_H is a global offensive $(k+2)$-alliance, we have

$$\begin{aligned}
\delta_M(g,h) - \delta_{\overline{M}}(g,h) &= \delta_{A_G}(g)\delta_{A_H}(h) + \delta_{\overline{A_G}}(g)\delta_{A_H}(h) + \delta_{A_H}(h) - \delta_{A_G}(g) - \\
&\quad + \delta_{\overline{A_G}}(g) - \delta_{A_G}(g)\delta_{\overline{A_H}}(h) - \delta_{\overline{A_G}}(g)\delta_{\overline{A_H}}(h) - \delta_{\overline{A_H}}(h) \\
&= \delta_{A_G}(g)(\delta_{A_H}(h) - \delta_{\overline{A_H}}(h) - 1) \\
&\quad + \delta_{\overline{A_G}}(g)(\delta_{A_H}(h) - \delta_{\overline{A_H}}(h) - 1) + \delta_{A_H}(h) - \delta_{\overline{A_H}}(h) \\
&\geq (k+1)\delta_{A_G}(g) + (k+1)\delta_{\overline{A_G}}(g) + k + 2 \\
&= (k+1)\delta_G(g) + k + 2 \geq 2k+3.
\end{aligned}$$

Similarly to the Case 1, we have that

$$\delta_M(g,h) \geq \frac{\delta_{G\boxtimes H}(g,h) + 2k + 3}{2} \geq \frac{\delta_{G\boxtimes H}(g,h)}{2} + k + 1. \tag{4}$$

As a consequence of both cases we obtain that M is a k-monopoly and the proof is complete. ■

If we consider two graphs G and H such that all its vertices have even degree, then, since k is an integer, it follows from the expressions (3) and (4) that every (g,h) of $V(G\boxtimes H)$ satisfies $\delta_M(g,h) \geq \frac{\delta_{G\boxtimes H}(g,h)}{2} + k + 1$ and so, the following result.

Theorem 2. *Let G and H be two graphs graphs without vertices and let $\ell = \min\{\delta(G), \delta(H)\}$. If G and H are Eulerian graphs, then for $k \in \left\{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\right\}$*

$$\mathcal{M}_{k+1}(G\boxtimes H) \leq \min\left\{\gamma_k^p(G)|V(H)|, |V(G)|\gamma_k^p(H)\right\}.$$

To give some lower bounds for the k-monopoly number of strong product graphs we need some additional terminology. Given a graph G and a set $S \subset V(G)$, we say that closed neighborhoods over S form a *closed subpartition* for G if $N_G[u] \cap N_G[v] = \emptyset$ for every different $u, v \in S$. A closed subpartition S for G is a *maximum closed subpartition*, if $V(G) \setminus \bigcup_{v\in S} N_G[v]$ has the minimum cardinality among all closed subpartitions S' for G. Notice that, if S forms a maximum closed subpartition which is also a partition of $V(G)$, then S is called a *perfect code* [2] or an *efficient dominating set* [1] of G. We have the following result, which yields a corollary that is important for this work.

Proposition 3. *Let G and H be graphs. If S_G and S_H form maximum closed subpartitions for G and H, respectively, then $S = S_G \times S_H$ form a closed subpartition for $G \boxtimes H$.*

Proof. Let S_G and S_H be maximum closed subpartitions for G and H, respectively, and let $S = S_G \times S_H$. Since $N_{G\boxtimes H}[(g,h)] = N_G[g] \times N_H[h]$ holds, we have $N_{G\boxtimes H}[g,h] \cap N_{G\boxtimes H}[g',h'] = \emptyset$ for every different vertices (g,h) and (g',h') from S. ■

Now we present lower bounds on the monopoly number of strong product graphs. One of them will be based on the following observation.

Observation 4. *Let G be a graph. If $S \subset V(G)$ forms a closed subpartition of G, then*

$$\mathcal{M}_k(G) \geq k|S| + \sum_{v \in S} \left\lceil \frac{\delta(v)}{2} \right\rceil .$$

The following theorem follows directly from the above observation, Proposition 3 and the fact that $\delta_{G\boxtimes H}(g,h) = \delta_G(g)\delta_H(h) + \delta(g) + \delta(h)$.

Theorem 5. *Let G and H be graphs without isolated vertices and let $\ell = \delta(G)\delta(H) + \delta(G) + \delta(H)$. If S_G and S_H are maximum closed subpartitions of G and H, respectively, then for $k \in \left\{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\right\}$ we have*

$$\mathcal{M}_k(G \boxtimes H) \geq k|S_G||S_H| + \sum_{(g,h) \in S_G \times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h) + \delta(g) + \delta(h)}{2} \right\rceil .$$

The next result, presented previously in [10], is also useful to obtain a lower bound on the monopoly number of strong product graphs.

Proposition 6 [10]**.** *Let G be a graph of order n, minimum degree δ and maximum degree Δ. Then, for any $k \in \left\{1 - \left\lceil \frac{\delta}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta}{2} \right\rfloor\right\}$*

$$\mathcal{M}_k(G) \geq \left\lceil \frac{n}{\Delta} \left(\left\lceil \frac{\delta}{2} \right\rceil + k \right) \right\rceil .$$

Corollary 7. *Let G and H be two graphs without isolated vertices of order n and m, respectively. For any $k \in \left\{1 - \left\lceil \frac{\delta(G)\delta(H)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(G)\delta(H)}{2} \right\rfloor\right\}$ we have*

$$\mathcal{M}_k(G \boxtimes H) \geq \left\lceil \frac{mn}{\Delta(G)\Delta(H) + \Delta(G) + \Delta(H)} \left(\left\lceil \frac{\delta(G)\delta(H) + \delta(G) + \delta(H)}{2} \right\rceil + k \right) \right\rceil .$$

The best performance of the last bound is achieved when both factors are Eulerian regular graphs.

3. Bounds or Closed Formulaes for Particular Families of Strong Product Graphs

We start with the strong product of two cycles.

Theorem 8. *Let $r, t \geq 3$, $r \leq t$, be two integers. If r and t are even, then $\mathcal{M}_0(C_r \boxtimes C_t) = \frac{rt}{2}$. Moreover, if at least one of integers r or t, say t, is odd, then*

$$\left\lceil \frac{rt}{2} \right\rceil \leq \mathcal{M}_0(C_r \boxtimes C_t) \leq \begin{cases} \frac{rt}{2} + \left\lceil \frac{r}{4} \right\rceil, & \text{if } r \text{ is even,} \\ \left\lfloor \frac{rt}{2} \right\rfloor + \left\lceil \frac{r}{4} \right\rceil + \left\lfloor \frac{t}{4} \right\rfloor, & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Let $V(C_r) = \{u_0, \ldots, u_{r-1}\}$ and $V(C_t) = \{v_0, \ldots, v_{t-1}\}$. From now on, operations with subscripts of vertices of C_r and C_t are done modulo r and modulo t, respectively. Also we assume that u_i, u_{i+1} and v_i, v_{i+1} are adjacent in C_r and C_t, respectively. Clearly, $C_r \boxtimes C_t$ is an 8-regular graph and $\mathcal{M}_0(C_r \boxtimes C_t) \geq \left\lceil \frac{rt}{2} \right\rceil$ holds by Corollary 7.

Let $M = \{(u_i, v_j) : i, j \text{ have different parities}\}$. Notice that the cardinality of M is $\left\lfloor \frac{rt}{2} \right\rfloor$. We suppose first that both r and t are even. Hence, it is easy to check that every vertex $(u_i, v_j) \in V(C_r \boxtimes C_t)$ has exactly four neighbors in M and four neighbors in $\overline{M}$. So, M is a 0-monopoly in $C_r \boxtimes C_t$. Moreover, in every C_r-layer there are exactly $\frac{r}{2}$ vertices of M, which gives all together $\frac{rt}{2}$ vertices in M. Therefore, we have that $\mathcal{M}_0(C_r \boxtimes C_t) = \frac{rt}{2}$.

Now let $t \geq 3$ be an odd integer. We consider first $r \geq 3$ being an even integer. It is straightforward to observe that all the vertices of M belonging to the set $V(C_r \boxtimes C_t) \setminus (C_r^{v_0} \cup C_r^{v_{t-1}})$ are 0-controlled by M (we recall that $C_r^{v_i}$ is the C_r-layer corresponding to the vertex v_i). Vertices from $C_r^{v_0} \cup C_r^{v_{t-1}}$ which are not in M have five neighbors in M and are also 0-controlled. On the other hand, vertices of M in $C_r^{v_0} \cup C_r^{v_{t-1}}$ have only three neighbors in M. Hence, it is necessary to add some extra vertices to M to have a 0-monopoly in $C_r \boxtimes C_t$. That is, $M_1 = M \cup \left\{(u_{4i}, v_{t-1}) : i \in \left\{0, \ldots, \left\lceil \frac{r}{4} \right\rceil - 1\right\}\right\}$ is a 0-monopoly in $C_r \boxtimes C_t$ and, thus, $\mathcal{M}_0(C_r \boxtimes C_t) \leq \frac{rt}{2} + \left\lceil \frac{r}{4} \right\rceil$.

If $r \geq 3$ is an odd integer, then we make an analogue extension of M to a set M_2 as we did above from M to M_1. That is, M_2 is given as the following set.

$$M \cup \left\{(u_{4i}, v_{t-1}) : i \in \left\{0, \ldots, \left\lceil \frac{r}{4} \right\rceil - 1\right\}\right\} \cup \left\{(u_{r-1}, v_{4j+2}) : j \in \left\{0, \ldots, \left\lfloor \frac{t}{4} \right\rfloor - 1\right\}\right\}.$$

Similarly as before, M_2 is a 0-monopoly in $C_r \boxtimes C_t$. Thus, it follows

$$\mathcal{M}_0(C_r \boxtimes C_t) \le \left\lfloor \frac{rt}{2} \right\rfloor + \left\lceil \frac{r}{4} \right\rceil + \left\lfloor \frac{t}{4} \right\rfloor . \qquad \blacksquare$$

By using a similar pattern like in the above result, we obtain the following for the case of strong product of paths.

Proposition 9. *If $r \ge 3$ and $t \ge 3$ are two integers, then*

$$\mathcal{M}_0(P_r \boxtimes P_t) \le \left\lfloor \frac{rt}{2} \right\rfloor + \left\lceil \frac{r+t-2}{2} \right\rceil .$$

Proof. Let $V(P_r) = \{u_1, \ldots, u_r\}$ and $V(P_t) = \{v_1, \ldots, v_t\}$. With the above notation we suppose that two consecutive vertices of $V(P_i)$ are adjacent for $i \in \{r, t\}$. Suppose that r, t have the same parity. Consider the set $M = \{(u_i, v_j) : i, j$ have different parities$\}$. Notice that the cardinality of M is $\lfloor \frac{rt}{2} \rfloor$. First we consider the case that r and t are odd integers. Now, it is easy to see that any vertex $(u_i, v_j) \in V(P_r \boxtimes P_t)$ with $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 8$ has exactly four neighbors in M and four neighbors in $\overline{M}$. Also, if $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 3$, then $\delta_M(u_i, v_j) = 2$, and if $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 5$ and $(u_i, v_j) \notin M$, then $\delta_M(u_i, v_j) = 3$. However, if $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 5$ and $(u_i, v_j) \in M$, then $\delta_M(u_i, v_j) = 2$ which is not enough to satisfy the 0-monopoly condition, and we need to add some additional vertices to M. Notice that there are exactly $2\frac{r-1+t-1}{2} = r + t - 2$ of such vertices, and this is an even number, since r, t have the same parity. Since, the layers $P_r^1, {}^rP_t, P_t^r$ and ${}^1P_t$ form a cycle, say C, and any two consecutive vertices of M lying in C have a common neighbor in C not belonging to M, we can split them into pairs and for each pair we add to M one extra vertex, to obtain a 0-monopoly set in $P_r \boxtimes P_t$. Therefore we have

$$\mathcal{M}_0(P_r \boxtimes P_t) \le \left\lfloor \frac{rt}{2} \right\rfloor + \frac{r+t-2}{2} .$$

Now, we assume that r, t are even integers. We proceed similarly as above, but in this case, there also exist two vertices with $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 3$, for which $\delta_M(u_i, v_j) = 1$ (not satisfying the 0-monopoly condition). Again, if $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 5$ and $(u_i, v_j) \in M$, then $\delta_M(u_i, v_j) = 2$ which is not enough to satisfy the 0-monopoly condition, and we need to add some additional vertices to M. There are exactly $2\frac{r-1+t-1}{2} = r + t - 2$ of such vertices (two with degree three and the rest with degree five). The same lower bound is obtained as above.

On the other hand, without loss of generality, suppose $t \ge 3$ is even and $r \ge 3$ is odd. We consider the same set M, which has the same cardinality as above. Also, only those vertices $(u_i, v_j) \in M$ with $\delta_{P_r \boxtimes P_t}(u_i, v_j) = 5$ and two corner vertices of degree three are not satisfying the 0-monopoly condition, since $\delta_M(u_i, v_j) = 2$ or $\delta_M(u_i, v_j) = 1$, respectively. Thus, we need to add some additional vertices to M. Again, there are exactly $2\frac{r-1+t-1}{2} = r+t-2$ of such vertices, but in this case, this is an odd number, since r, t have different parities. By adding $\left\lceil \frac{r+t-2}{2} \right\rceil$ vertices to M, in a similar way, we obtain a 0-monopoly. Thus

$$\mathcal{M}_0(P_r \boxtimes P_t) \leq \left\lfloor \frac{rt}{2} \right\rfloor + \left\lceil \frac{r+t-2}{2} \right\rceil .$$

■

The next result is a kind of "combination" of the above two results, in the sense we analyze the case when one factor is a path and the second one is a cycle. According to this, the construction of a 0-monopoly set in the proof is very similar to the above ones. So, we omit the proof.

Proposition 10. *If $r \geq 3$ and $t \geq 3$ are two integers, then*

$$\mathcal{M}_0(P_r \boxtimes C_t) \leq \begin{cases} \frac{rt}{2} + 2\left\lceil \frac{t}{4} \right\rceil, & \text{if } t \text{ is even,} \\ \frac{rt}{2} + \left\lfloor \frac{r}{4} \right\rfloor + 2\left\lfloor \frac{t}{4} \right\rfloor, & \text{if } r \text{ is even and } t \text{ is odd,} \\ \frac{rt-1}{2} + \left\lceil \frac{r}{4} \right\rceil + 2\left\lfloor \frac{t}{4} \right\rfloor, & \text{if } r, t \text{ are odd.} \end{cases}$$

The upper bound of k-monopolies of strong product graphs depends on the value of the global powerful k-alliance number of its factors by Theorem 1. For the complete graphs it is as follows.

Lemma 11 [4]**.** *For any K_t of order $t \geq 2$ and $k \in \{2-t, \ldots, t-3\}$,*

$$\gamma_k^p(K_t) = \left\lceil \frac{t+k+1}{2} \right\rceil .$$

Since $C_r \boxtimes K_t$ is a regular graph of degree $3t-1$, from Theorem 1, Corollary 7, Lemma 11, and by making some straightforward calculations, it is possible to observe that for any $r, t \geq 3$, $\left\lceil \frac{rt}{2} \right\rceil \leq \mathcal{M}_0(C_r \boxtimes K_t) \leq r\left\lceil \frac{t+1}{2} \right\rceil$. Nevertheless such results can be improved as we show at next.

Proposition 12. *For integers $r, t \geq 3$ we have*

$$\frac{rt}{2} + \frac{r}{6} \leq \mathcal{M}_0(C_r \boxtimes K_t) \leq r\left\lceil \frac{t+1}{2} \right\rceil - \left\lfloor \frac{r}{3} \right\rfloor .$$

Moreover, if $r \equiv 0 \pmod 3$ and t is odd, then $\mathcal{M}_0(C_r \boxtimes K_t) = \frac{rt}{2} + \frac{r}{6}$.

Proof. Let $V(C_r) = \{u_0, \ldots, u_{r-1}\}$ and $V(K_t) = \{v_1, \ldots, v_t\}$. With the above notation we suppose that two consecutive vertices u_i and u_{i+1} are adjacent (all the operations with the subindexes of u_i are done modulo r). Let M be an $\mathcal{M}_0(C_r \boxtimes K_t)$-set and let $(u_i, v_j) \in M$. Hence, we have that

$$\delta_M(u_i, v_j) \geq \frac{\delta_{C_r \boxtimes K_t}(u_i, v_j)}{2} = \frac{3t-1}{2}. \tag{5}$$

Now, for every $i \in \{0, \ldots, r-1\}$, let $M_i = M \cap (\{u_{i-1}, u_i, u_{i+1}\} \times V(K_t))$. Hence, for every $(x, y) \in M_i$, from (5) we obtain that $|M_i| \geq \delta_{M_i}(x, y) \geq \frac{3t-1}{2}$. On the other hand, it is clear that for every $i \in \{0, \ldots, r-1\}$, $M_i \neq \emptyset$. Also, notice that if there exist $j \in \{0, \ldots, r-1\}$ such that $M \cap (\{u_j\} \times V(K_t)) = \emptyset$, then $M \cap (\{u_l\} \times V(K_t)) \neq \emptyset$ for every $l \in \{j-2, j-1, j+1, j+2\}$. Thus, there exist at least $\frac{r}{3}$ disjoint sets M_i satisfying $|M_i| \geq \frac{3t-1}{2} + 1$, since we need to count one vertex of M_i and its neighbors inside of the set M_i. Therefore, by counting each vertex of M three times, we obtain that

$$|M| \geq \frac{1}{3}\sum_{i=0}^{r-1}|M_i| \geq \frac{1}{3}\sum_{i=0}^{r-1}\left(\frac{3t-1}{2}\right) + \frac{r}{3} = \frac{r(3t-1)}{6} + \frac{r}{3} = \frac{rt}{2} + \frac{r}{6},$$

and the lower bound is proved.

Let S_G be a maximum closed subpartition for C_r. Let A be a set such that if $u_i \in S_{C_r}$, then $\left\lfloor \frac{t}{2} \right\rfloor$ vertices of the ${}^{u_i}K_t$-layer belong to A, otherwise $\left\lfloor \frac{t}{2} \right\rfloor + 1$ vertices of the ${}^{u_i}K_t$-layer belong to A. We will prove that A is a 0-monopoly in $C_r \boxtimes K_t$. Since $C_r \boxtimes K_t$ is a $(3t-1)$-regular graph, we only need to show that any vertex $(u_i, v_j) \in V(C_r \boxtimes K_t)$ has at least $\left\lceil \frac{3t-1}{2} \right\rceil$ neighbors in A. If $(u_i, v_j) \notin A$, then $|N_{C_r \boxtimes K_t}(u_i, v_j) \cap A| \geq 3\left\lfloor \frac{t}{2} \right\rfloor + 2 > \left\lceil \frac{3t-1}{2} \right\rceil$. Otherwise, if $(u_i, v_j) \in A$, then $|N_{C_r \boxtimes K_t}(u_i, v_j) \cap A| \geq 3\left\lfloor \frac{t}{2} \right\rfloor + 1 \geq \left\lceil \frac{3t-1}{2} \right\rceil$. Thus, the 0-monopoly condition holds. Notice that $|A| = r\left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) - |S_{C_r}| = r\left\lceil \frac{t+1}{2} \right\rceil - \left\lfloor \frac{r}{3} \right\rfloor$, which completes the proof of the upper bound.

Now, if $r \equiv 0 (\text{mod } 3)$ and t is odd, then $|S_{C_r}| = \frac{r}{3}$ and we have that

$$\mathcal{M}_0(C_r \boxtimes K_t) \leq r\left\lceil \frac{t+1}{2} \right\rceil - |S_{C_r}| = \frac{r(t+1)}{2} - \frac{r}{3} = \frac{rt}{2} + \frac{r}{6}.$$

Therefore the equality follows for this case. ■

Proposition 13. *For $r, t \geq 3$ we have*

$$\frac{tr}{2} + \frac{r}{6} - \frac{t}{3} \leq \mathcal{M}_0(P_r \boxtimes K_t) \leq \begin{cases} \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3t+1}{2} \right\rceil, & \text{if } r \equiv 0 \pmod 3, \\ \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3t+1}{2} \right\rceil + 1, & \text{if } r \equiv 1 \pmod 3, \\ \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3t+1}{2} \right\rceil + t, & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $V(P_r) = \{u_1, \ldots, u_r\}$ and $V(K_t) = \{v_1, \ldots, v_t\}$. With the above notation we suppose that two consecutive vertices of $V(P_r)$ are adjacent. Let M

be an $\mathcal{M}_0(P_r \boxtimes K_t)$-set and let $(u_i, v_j) \in M \cap (V(P_r) \setminus \{u_1, u_r\}) \times V(K_t)$. Hence we have that

$$\delta_M(u_i, v_j) \geq \frac{\delta_{P_r \boxtimes K_t}(u_i, v_j)}{2} = \frac{3t-1}{2}. \tag{6}$$

Now, for every $i \in \{0, \dots, r+1\}$, let $M_i = M \cap (\{u_{i-1}, u_i, u_{i+1}\} \times V(K_t)$, where u_{-1}, u_0, u_{r+1}, u_{r+2} are imaginary vertices needed for counting three times every vertex from M. It is clear that for every $i \in \{1, \dots, r\}$, $M_i \neq \emptyset$. Hence, for every $(x, y) \in M_i$, from (6) we obtain that $|M_i| \geq \delta_{M_i}(x, y) \geq \frac{3t-1}{2}$. Now, analogously to the proof of Proposition 12, there exist at least $\frac{r}{3}$ disjoint sets M_i satisfying that $|M_i| \geq \frac{3t-1}{2} + 1$. Moreover, we need at least $\frac{2t-1}{2}$ vertices in M_1 and at least $\frac{2t-1}{2}$ in M_r, since $\delta_{P_r \boxtimes K_t}(u_i, v_j) = 2t - 1$ for $i \in \{1, r\}$. Thus, we obtain that

$$\begin{aligned} |M| &\geq \frac{1}{3}\sum_{i=0}^{r+1} |M_i| \geq \frac{1}{3}\left(|M_0| + |M_1| + |M_r| + |M_{r+1}| + \sum_{i=2}^{r-1} \frac{3t-1}{2}\right) + \frac{r}{3} \\ &\geq \frac{2t-1}{3} + \frac{(r-2)(3t-1)}{6} + \frac{r}{3} = \frac{tr}{2} + \frac{r}{6} - \frac{t}{3}, \end{aligned}$$

and the lower bound is proved. To prove the upper bound we consider the following cases.

Case 1. $r \equiv 0 \pmod 3$. Let A_0 be a set such that if $i \equiv 2 \pmod 3$, then the whole ${}^{u_i}K_t$-layer is a subset of A_0 and, for every $i \equiv 0 \pmod 3$, $\lceil \frac{t+1}{2} \rceil$ vertices of the ${}^{u_i}K_t$-layer belong to A_0. We show that A_0 is a 0-monopoly set in $P_r \boxtimes K_t$. Let (u_i, v_j) be any vertex in $P_r \boxtimes K_t$. We differentiate the following cases.

Case 1.1. $i \equiv 0 \pmod 3$. Then, $\delta_{P_r \boxtimes K_t}(u_r, v_j) = 2t - 1$ and $\delta_{P_r \boxtimes K_t}(u_i, v_j) = 3t - 1$ for $i \neq r$. From the construction of the set A_0, we have that $\delta_{A_0}(u_i, v_j) = t + \lceil \frac{t+1}{2} \rceil = \lceil \frac{3t+1}{2} \rceil$ if $(u_i, v_j) \notin A_0$, and $\delta_{A_0}(u_i, v_j) = t + \lceil \frac{t+1}{2} \rceil - 1 = \lceil \frac{3t-1}{2} \rceil$ if $(u_i, v_j) \in A_0$. Thus, the 0-monopoly condition holds.

Case 1.2. $i \equiv 1 \pmod 3$. Hence, $\delta_{P_r \boxtimes K_t}(u_1, v_j) = 2t - 1$ and $\delta_{P_r \boxtimes K_t}(u_i, v_j) = 3t-1$ for $i \neq 1$. From the construction of the set A_0, we have that $\delta_{A_0}(u_1, v_j) = t$ and $\delta_{A_0}(u_i, v_j) = t + \lceil \frac{t+1}{2} \rceil = \lceil \frac{3t+1}{2} \rceil$, when $i \neq 1$. So, the 0-monopoly condition again holds.

Case 1.3. $i \equiv 2 \pmod 3$. From the construction of the set A_0 we obtain that $\delta_{A_0}(u_i, v_j) = t - 1 + \lceil \frac{t+1}{2} \rceil = \lceil \frac{3t-1}{2} \rceil$. Since $\delta_{P_r \boxtimes K_t}(u_i, v_j) = 3t - 1$, the 0-monopoly condition holds.

Notice that $|A_0| = \lfloor \frac{r}{3} \rfloor \left(t + \lceil \frac{t+1}{2} \rceil\right) = \lfloor \frac{r}{3} \rfloor \cdot \lceil \frac{3t+1}{2} \rceil$, which completes the proof of Case 1.

Case 2. $r \equiv 1 \pmod 3$. Let A_1 be a set such that ${}^{u_{r-1}}K_t \subset A_1$ and, for every $i \equiv 2 \pmod 3$ with $i < r - 1$, it follows ${}^{u_i}K_t \subset A_1$. Moreover, $\lceil \frac{t+3}{2} \rceil$ vertices of the ${}^{u_{r-2}}K_t$-layer belong to A_1, and for every $i \equiv 0 \pmod 3$ such that $i < r-1$, $\lceil \frac{t+1}{2} \rceil$ vertices of the ${}^{u_i}K_t$-layer belong to A_1. By using an analogous procedure, like in Case 1, we can show that A_1 is a 0-monopoly set in $P_r \boxtimes K_t$. Notice that $|A_1| = |A_0| + 1$ and the proof of Case 2 is complete.

Case 3. $r \equiv 2 \pmod 3$. Let A_2 be a set such that if $i \equiv 2 \pmod 3$ and $i < r-2$, then ${}^{u_i}K_t \subset A_2$ and ${}^{u_{r-1}}K_t \subset A_2$. Moreover, for every $i \equiv 0 \pmod 3$, $\lceil \frac{t+1}{2} \rceil$ vertices of the ${}^{u_i}K_t$-layer belong to A_2. An analogous procedure like in Case 1, can be used to show that A_2 is a 0-monopoly set in $P_r \boxtimes K_t$. Notice that $|A_2| = |A_0| + t$, which completes the proof. ∎

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