

*Discussiones Mathematicae*  
*Graph Theory* 38 (2018) 275–285  
doi:10.7151/dmgt.2006

## HEREDITARY EQUALITY OF DOMINATION AND EXPONENTIAL DOMINATION

MICHAEL A. HENNING

*Department of Pure and Applied Mathematics*  
*University of Johannesburg*  
*Auckland Park, 2006, South Africa*

**e-mail:** mahenning@uj.ac.za

SIMON JÄGER

AND

DIETER RAUTENBACH

*Institute of Optimization and Operations Research*  
*Ulm University, Ulm, Germany*

**e-mail:** simon.jaeger@uni-ulm.de  
dieter.rautenbach@uni-ulm.de

### Abstract

We characterize a large subclass of the class of those graphs  $G$  for which the exponential domination number of  $H$  equals the domination number of  $H$  for every induced subgraph  $H$  of  $G$ .

**Keywords:** domination, exponential domination, hereditary class.

**2010 Mathematics Subject Classification:** 05C69.

### 1. INTRODUCTION

Domination in graphs is an important area within graph theory, and an astounding variety of different domination parameters are known [6]. Essentially all of these parameters involve merely local conditions, which makes them amenable to similar approaches and arguments. In [5] Dankelmann *et al.* introduce a truly non-local variant of domination, the so-called exponential domination, where the influence of vertices extends to any arbitrary distance within the graph but decays exponentially with that distance. There is relatively few research concerning

exponential domination [1–4], and even apparently basic results require new and careful arguments.

As follows easily from the precise definitions given below, the exponential domination number of any graph is at most its domination number. Bessy *et al.* [4] show that computing the exponential domination number is APX-hard for subcubic graphs and describe an efficient algorithm for subcubic trees, but the complexity for general trees is unknown. It is not even known how to decide efficiently for a given tree  $T$  whether its exponential domination number  $\gamma_e(T)$  equals its domination number  $\gamma(T)$ . In [8] we study relations between the different parameters of exponential domination and domination. Next to several bounds, we obtain a constructive characterization of the subcubic trees  $T$  with  $\gamma_e(T) = \gamma(T)$ . In view of the efficient algorithms to determine both parameters for such trees, the existence of a constructive characterization is not surprising, but, as said a few lines above, already for general trees all techniques from [3, 4, 8] completely fail.

Note that, since adding a universal vertex to any graph results in a graph  $G$  with  $\gamma_e(G) = \gamma(G)$ , the class of all graphs  $G$  that satisfy  $\gamma_e(G) = \gamma(G)$  is not hereditary, and does not have a simple structure. The difficulty to decide whether  $\gamma_e(G) = \gamma(G)$  for a given graph  $G$  motivates the study of the hereditary class  $\mathcal{G}$  of graphs that satisfy this equality, that is,  $\mathcal{G}$  is the set of those graphs  $G$  such that  $\gamma_e(H) = \gamma(H)$  for every induced subgraph  $H$  of  $G$ . As for the well-known class of perfect graphs, the class  $\mathcal{G}$  can be characterized by minimal forbidden induced subgraphs.

In the present paper we obtain such a characterization for a large subclass of  $\mathcal{G}$ , and pose several related conjectures.

Before we proceed to our results, we collect some notation. We consider finite, simple, and undirected graphs, and use standard terminology. The vertex set and the edge set of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. The order  $n(G)$  of  $G$  is the number of vertices of  $G$ . For a vertex  $u$  of  $G$ , the neighborhood of  $u$  in  $G$  and the degree of  $u$  in  $G$  are denoted by  $N_G(u)$  and  $d_G(u)$ , respectively. The distance  $\text{dist}_G(X, Y)$  between two sets  $X$  and  $Y$  of vertices in  $G$  is the minimum length of a path in  $G$  between a vertex in  $X$  and a vertex in  $Y$ . If no such path exists, then let  $\text{dist}_G(X, Y) = \infty$ .

Let  $D$  be a set of vertices of a graph  $G$ . The set  $D$  is a dominating set of  $G$  [6] if every vertex of  $G$  not in  $D$  has a neighbor in  $D$ . The domination number  $\gamma(G)$  of  $G$  is the minimum size of a dominating set of  $G$ . For two vertices  $u$  and  $v$  of  $G$ , let  $\text{dist}_{(G,D)}(u, v)$  be the minimum length of a path  $P$  in  $G$  between  $u$  and  $v$  such that  $D$  contains exactly one endvertex of  $P$  but no internal vertex of  $P$ . If no such path exists, then let  $\text{dist}_{(G,D)}(u, v) = \infty$ . Note that, if  $u$  and  $v$  are distinct vertices in  $D$ , then  $\text{dist}_{(G,D)}(u, u) = 0$  and  $\text{dist}_{(G,D)}(u, v) = \infty$ . For a vertex  $u$  of  $G$ , let

$$w_{(G,D)}(u) = \sum_{v \in D} \left(\frac{1}{2}\right)^{\text{dist}_{(G,D)}(u,v)-1},$$

where  $(\frac{1}{2})^\infty = 0$ . Dankelmann *et al.* [5] define the set  $D$  to be an *exponential dominating set* of  $G$  if  $w_{(G,D)}(u) \geq 1$  for every vertex  $u$  of  $G$ , and the *exponential domination number*  $\gamma_e(G)$  of  $G$  as the minimum size of an exponential dominating set of  $G$ . Note that  $w_{(G,D)}(u) = 2$  for  $u \in D$ , and that  $w_{(G,D)}(u) \geq 1$  for every vertex  $u$  that has a neighbor in  $D$ , which implies  $\gamma_e(G) \leq \gamma(G)$ .

The following Figure 1 contains forbidden induced subgraphs that relate to the considered subclasses of  $\mathcal{G}$ . Recall that  $P_k$  and  $C_k$  denote the path and cycle of order  $k$ , respectively.

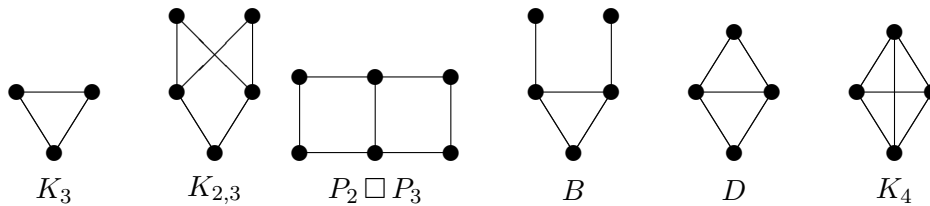


Figure 1. The graphs  $K_3$ ,  $K_{2,3}$ ,  $P_2 \square P_3$ ,  $B$  (bull),  $D$  (diamond), and  $K_4$ .

Our main result is the following.

**Theorem 1.** *If  $G$  is a  $\{B, D, K_4, K_{2,3}, P_2 \square P_3\}$ -free graph, then  $\gamma(H) = \gamma_e(H)$  for every induced subgraph  $H$  of  $G$  if and only if  $G$  is  $\{P_7, C_7, F_1, \dots, F_5\}$ -free (cf. Figure 2).*

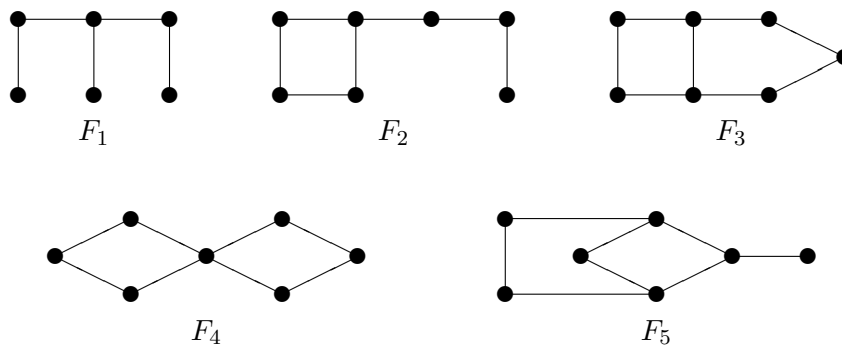


Figure 2. The graphs  $F_1, \dots, F_5$ .

Since all graphs in  $\{B, D, K_4, K_{2,3}, P_2 \square P_3\} \cup \{F_2, \dots, F_5\}$  have girth at most 4, where the girth of a graph is the minimum length of a cycle in it, Theorem 1 has the following immediate corollary.

**Corollary 2.** *If  $G$  is a graph of girth at least 5, then  $\gamma(H) = \gamma_e(H)$  for every induced subgraph  $H$  of  $G$  if and only if  $G$  is  $\{P_7, C_7, F_1\}$ -free.*

For the trees in  $\mathcal{G}$ , we achieve a complete characterization.

**Corollary 3.** *If  $T$  is a tree, then  $\gamma(F) = \gamma_e(F)$  for every induced subgraph  $F$  of  $T$  if and only if  $T$  is  $\{P_7, F_1\}$ -free.*

All proofs and our conjectures are postponed to the next section.

## 2. PROOFS AND CONJECTURES

We split the proof of Theorem 1 into the triangle-free case and the non-triangle-free case. The triangle-free case is considered in the following lemma.

**Lemma 4.** *If  $G$  is a  $\{K_3, K_{2,3}, P_2 \square P_3\}$ -free graph, then  $\gamma(H) = \gamma_e(H)$  for every induced subgraph  $H$  of  $G$  if and only if  $G$  is  $\{P_7, C_7, F_1, \dots, F_5\}$ -free.*

**Proof.** Since  $\gamma(H) > \gamma_e(H)$  for every graph  $H$  in  $\{P_7, C_7, F_1, \dots, F_5\}$ , necessity follows. In order to prove sufficiency, suppose that  $G$  is a  $\{K_3, K_{2,3}, P_2 \square P_3\} \cup \{P_7, C_7, F_1, \dots, F_5\}$ -free graph with  $\gamma(G) > \gamma_e(G)$  of minimum order. By the choice of  $G$ , we have  $\gamma(H) = \gamma_e(H)$  for every proper induced subgraph  $H$  of  $G$ . Clearly,  $G$  is connected. Since  $\gamma_e(G) = 1$  if and only if  $\gamma(G) = 1$ , we obtain  $\gamma_e(G) \geq 2$  and  $\gamma(G) \geq 3$ . Since  $G$  is  $\{P_7, C_7\}$ -free, either  $G$  is a tree or the girth  $g$  of  $G$  is at most 6.

Suppose that  $G$  is a tree. If  $G$  has at most one vertex of degree at least 3, then, since  $G$  is  $\{P_7, F_1\}$ -free, it arises from a path  $P : u_1 \cdots u_\ell$  with  $\ell \leq 6$  by attaching further endvertices to  $u_2$ . Since  $\ell \leq 6$ , the set  $\{u_2, u_{\ell-1}\}$  is a dominating set of  $G$ , which contradicts  $\gamma(G) \geq 3$ . Hence,  $G$  has at least two vertices of degree at least 3. Let  $P : u_1 \cdots u_\ell$  be a shortest path in  $G$  between two such vertices. Since  $G$  is  $F_1$ -free, it arises from  $P$  by attaching at least two further endvertices to  $u_1$  and at least two further endvertices to  $u_\ell$ . Since  $G$  is  $P_7$ -free, we obtain  $\ell \leq 4$ . This implies that the set  $\{u_1, u_\ell\}$  is a dominating set of  $G$ , which contradicts  $\gamma(G) \geq 3$ . Hence, we may assume that  $G$  is not a tree. Let  $C : x_1 x_2 x_3 \cdots x_g x_1$  be a shortest cycle of  $G$ , where we consider the indices modulo  $g$ . Let  $R = V(G) \setminus V(C)$ .

Suppose  $g = 6$ . Since  $\gamma(C_6) = \gamma_e(C_6) = 2$ , some vertex  $y$  in  $R$  has a neighbor  $x_i$  on  $C$ . Since  $g = 6$ , the vertex  $y$  has no further neighbor on  $C$ , implying that  $G[\{y, x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}] = F_1$ , contradicting the fact that  $G$  is  $F_1$ -free. Hence,  $g < 6$ .

Suppose  $g = 5$ . This implies that no vertex in  $R$  has more than one neighbor on  $C$ . If some vertex  $z$  has distance 2 from  $V(C)$  in  $G$  and  $x_i y z$  is a path in  $G$ , then  $G[\{z, y, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}] = F_1$ , which is a contradiction. Hence, every vertex in  $R$  has a unique neighbor on  $C$ . Suppose that there is some  $i \in [5]$  such that  $x_i$  has a neighbor  $y_i$  in  $R$  and  $x_{i+1}$  has a neighbor  $y_{i+1}$  in  $R$ . Since  $g = 5$ , we note that  $y_i \neq y_{i+1}$  and that the vertex  $y_i$  is not adjacent to  $y_{i+1}$ , implying that  $G[\{x_{i-2}, x_{i-1}, x_i, x_{i+1}, y_i, y_{i+1}\}] = F_1$ , which is a contradiction. This implies the existence of some index  $i \in [5]$  such that  $\{x_i, x_{i+2}\}$  is a dominating set of  $G$ , which contradicts  $\gamma(G) \geq 3$ . Hence,  $g \leq 4$ . Since  $G$  is  $K_3$ -free, this implies that  $g = 4$ .

Since  $G$  is  $\{K_3, K_{2,3}\}$ -free, no vertex in  $R$  has more than one neighbor on  $C$ , and since  $G$  is  $F_2$ -free, no vertex in  $R$  has distance more than 2 from  $V(C)$ .

Suppose that some vertex  $z$  has distance 2 from  $V(C)$ . Let  $x_1 y z$  be a path in  $G$ . Suppose that  $x_2$  has a neighbor  $u$  in  $R$ . Recall that  $u$  is not adjacent to any other vertex on  $C$ . Since  $G$  is  $P_2 \square P_3$ -free, the vertex  $u$  is not adjacent to  $y$ . If  $u$  is not adjacent to  $z$ , then  $G[\{u, x_1, x_2, x_4, y, z\}] = F_1$ , which is a contradiction. If  $u$  is adjacent to  $z$ , then  $G[V(C) \cup \{u, y, z\}] = F_3$ , which is a contradiction. Hence, by symmetry, we obtain  $d_G(x_2) = d_G(x_4) = 2$ .

Suppose that  $x_1$  has a neighbor  $u$  in  $R \setminus \{y\}$ . Since  $G$  is  $\{K_3, K_{2,3}\}$ -free, the vertex  $u$  is not adjacent to any vertex in  $\{x_2, x_3, x_4, y\}$ . If  $u$  is not adjacent to  $z$ , then  $G[\{x_1, x_2, x_3, u, y, z\}] = F_1$ , which is a contradiction. If  $u$  is adjacent to  $z$ , then  $G[V(C) \cup \{u, y, z\}] = F_4$ , which is a contradiction. Hence, we obtain  $d_G(x_1) = 3$ .

Since  $\{x_3, y\}$  is not a dominating set of  $G$ , and no vertex in  $R$  has distance more than 2 from  $V(C)$ , the degrees of  $x_1, x_2$ , and  $x_4$  imply the existence of a path  $x_3 u v$ , where  $v$  has distance 2 to  $V(C)$ , and  $v$  is not adjacent to  $y$ . Since  $G[\{v, u, x_3, x_2, x_1, y, z\}]$  is neither  $P_7$  nor  $C_7$ , the vertex  $u$  is adjacent to  $y$  or  $z$ . If  $u$  is adjacent to  $z$ , then, because  $G$  is  $K_3$ -free,  $G[\{u, v, y, z, x_3, x_2\}] = F_1$ , which is a contradiction. Hence, the vertex  $u$  is adjacent to  $y$ . If  $v$  is adjacent to  $z$ , then  $G[\{u, v, y, z, x_1, x_2, x_3\}] = F_3$ , which is a contradiction. Hence, the vertex  $v$  is not adjacent to  $z$ , and  $G[\{u, v, y, z, x_3, x_2\}] = F_1$ , which is a contradiction. Hence, every vertex in  $R$  has a unique neighbor on  $C$ .

Since  $\gamma(G) > 2$ , we may assume that  $x_i$  has a neighbor  $y_i$  in  $R$  for  $i \in [3]$ . Since  $G$  is  $P_2 \square P_3$ -free, the vertex  $y_2$  is not adjacent to  $y_1$  or  $y_3$ . Since  $G$  is  $F_5$ -free, the vertex  $y_1$  is not adjacent to  $y_3$ . Now,  $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1$ , which is a contradiction, and completes the proof. ■

With Lemma 4 at hand, we now proceed to the proof of Theorem 1.

**Proof of Theorem 1.** Necessity follows as above. In order to prove sufficiency, suppose that  $G$  is a  $\{B, D, K_4, K_{2,3}, P_2 \square P_3\} \cup \{P_7, C_7, F_1, \dots, F_5\}$ -free graph with  $\gamma(G) > \gamma_e(G)$  of minimum order. By the choice of  $G$ , we have  $\gamma(H) = \gamma_e(H)$

for every proper induced subgraph  $H$  of  $G$ . Clearly,  $G$  is connected. Since  $\gamma_e(G) = 1$  if and only if  $\gamma(G) = 1$ , we obtain  $\gamma_e(G) \geq 2$  and  $\gamma(G) \geq 3$ .

By Lemma 4,  $G$  is not  $K_3$ -free, that is, the girth of  $G$  is 3.

We proceed with a series of claims. Let  $F$  be the graph that is obtained from the triangle  $x_1x_2x_3$  and the path  $y_1y_2y_3$  by adding the edge  $x_1y_1$ .

**Claim 1.**  $F$  is not an induced subgraph of  $G$ .

**Proof.** Suppose that  $F$  is an induced subgraph of  $G$ . Since  $\{x_1, y_2\}$  is not a dominating set of  $G$ , there is a vertex  $u$  at distance 2 from the set  $\{x_1, y_2\}$  in  $G$ .

We proceed with three subclaims.

**Claim 1.1.** The vertex  $u$  is not adjacent to  $x_2$  or  $x_3$ .

**Proof.** Suppose that  $u$  is adjacent to  $x_2$ . Since  $G$  is  $D$ -free, the vertex  $u$  is not adjacent to  $x_3$ , and, since  $G$  is  $B$ -free,  $u$  is adjacent to  $y_1$ . If  $u$  is not adjacent to  $y_3$ , then  $G[\{u, x_1, x_3, y_1, y_2, y_3\}] = F_1$ , which is a contradiction. If  $u$  is adjacent to  $y_3$ , then  $G[\{u, x_1, x_2, y_1, y_2, y_3\}] = P_2 \square P_3$ , which is a contradiction. Hence, by symmetry, we obtain that  $u$  is not adjacent to  $x_2$  or  $x_3$ .  $\square$

**Claim 1.2.** The vertex  $u$  is not adjacent to  $y_1$ .

**Proof.** Suppose that  $u$  is adjacent to  $y_1$ . Since  $G$  is  $F_1$ -free, the vertex  $u$  is adjacent to  $y_3$ . Since  $\{x_1, y_3\}$  is not a dominating set of  $G$ , there is a vertex  $v$  at distance 2 from the set  $\{x_1, y_3\}$ . Suppose that  $v$  is adjacent to  $x_2$ . Since  $G$  is  $D$ -free, the vertex  $v$  is not adjacent to  $x_3$ , and, since  $G$  is  $B$ -free,  $v$  is adjacent to  $y_1$ . If  $v$  is adjacent to  $u$ , then  $G[\{u, v, x_2, y_1, y_3\}] = B$ , which is a contradiction. Hence,  $v$  is not adjacent to  $u$ , and, by symmetry,  $v$  is also not adjacent to  $y_2$ . Therefore,  $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = F_4$ , which is a contradiction. Thus, by symmetry,  $v$  is not adjacent to  $x_2$  or  $x_3$ . Next, suppose that  $v$  is adjacent to  $y_1$ . Since  $G$  is  $F_1$ -free, the vertex  $v$  is adjacent to both  $u$  and  $y_2$ , which yields the contradiction  $G[\{u, v, y_1, y_2\}] = D$ . Thus,  $v$  is not adjacent to  $y_1$ . Suppose that  $v$  is adjacent to  $u$ . If  $v$  is adjacent to  $y_2$ , then  $G[\{u, v, y_1, y_2, y_3\}] = K_{2,3}$ , which is a contradiction. If  $v$  is not adjacent to  $y_2$ , then  $G[\{u, v, x_1, x_2, y_1, y_2\}] = F_1$ , which is a contradiction. Thus, by symmetry,  $v$  is not adjacent to  $u$  or  $y_2$ , implying that  $v$  is at distance 2 from the set  $\{u, x_1, x_2, x_3, y_1, y_2, y_3\}$ .

Since the vertex  $v$  is at distance 2 from the set  $\{x_1, y_3\}$  in  $G$ , there is a neighbor  $v'$  of  $v$ , that is adjacent to  $x_1$  or to  $y_3$  or to both  $x_1$  and  $y_3$ . First, suppose that  $v'$  is not adjacent to  $x_1$ , implying that  $v'$  is adjacent to  $y_3$ . Suppose that  $v'$  is adjacent to  $y_1$ . If  $v'$  is adjacent to  $u$ , then  $G[\{u, v', y_1, y_3\}] = D$ , which is a contradiction. Thus, by symmetry,  $v'$  is adjacent to neither  $u$  nor  $y_2$ , implying that  $G[\{u, v', y_1, y_2, y_3\}] = K_{2,3}$ , which is a contradiction. Thus,  $v'$  is not adjacent to  $y_1$ . Since  $G$  is  $B$ -free,  $v'$  is not adjacent to  $u$  or  $y_2$ , which yields the contradiction  $G[\{v, v', x_1, x_2, y_1, y_2, y_3\}] = P_7$ . Therefore,  $v'$  is adjacent to  $x_1$ .

Since  $G$  is  $\{B, K_4\}$ -free, the vertex  $v'$  is not adjacent to  $x_2$  or  $x_3$ . Since  $G$  is  $B$ -free, the vertex  $v'$  is not adjacent to  $y_1$ . Suppose that  $v'$  is not adjacent to  $y_3$ . If  $v'$  is not adjacent to  $u$ , then  $G[\{u, v, v', x_1, x_2, y_1\}] = F_1$ , which is a contradiction. Thus, by symmetry,  $v'$  is adjacent to both  $u$  and  $y_2$ , which yields the contradiction  $G[\{u, v', y_1, y_2, y_3\}] = K_{2,3}$ . Thus,  $v'$  is adjacent to  $y_3$ . Since  $G$  is  $F_1$ -free, the vertex  $v'$  is adjacent to both  $u$  and  $y_2$ , implying that  $G[\{u, v, v', y_1, y_3\}] = B$ , which is a contradiction. Therefore,  $u$  is not adjacent to  $y_1$ .  $\square$

**Claim 1.3.** *The vertex  $u$  is not adjacent to  $y_3$ .*

**Proof.** Suppose that  $u$  is adjacent to  $y_3$ . Since  $\{x_1, y_3\}$  is not a dominating set of  $G$ , there is a vertex  $v$  at distance 2 from the set  $\{x_1, y_3\}$  in  $G$ .

Suppose that  $v$  is adjacent to  $x_2$ . Since  $G$  is  $D$ -free, the vertex  $v$  is not adjacent to  $x_3$ . Since  $G$  is  $B$ -free,  $v$  is adjacent to  $y_1$ . If  $v$  is not adjacent to  $y_2$ , then  $G[\{v, x_1, x_3, y_1, y_2, y_3\}] = F_1$ , which is a contradiction. If  $v$  is adjacent to  $y_2$ , then we get the contradiction  $G[\{x_2, v, y_1, y_2, y_3\}] = B$ . Therefore, by symmetry,  $v$  is not adjacent to  $x_2$  or  $x_3$ .

Next, suppose that  $v$  is adjacent to  $y_1$ . If  $v$  is not adjacent to  $y_2$ , then  $G[\{v, x_1, x_2, y_1, y_2, y_3\}] = F_1$ , which is a contradiction. If  $v$  is adjacent to  $y_2$ , then  $G[\{v, x_1, y_1, y_2, y_3\}] = B$ , which is a contradiction. Thus,  $v$  is not adjacent to  $y_1$ .

Next, suppose that  $v$  is adjacent to  $y_2$ . If  $v$  is not adjacent to  $u$ , then  $G[\{u, v, x_1, y_1, y_2, y_3\}] = F_1$ , which is a contradiction. If  $v$  is adjacent to  $u$ , then  $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = F_2$ , which is a contradiction. Therefore,  $v$  is not adjacent to  $y_2$ , implying that  $v$  is at distance 2 from the set  $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ .

If  $v$  is adjacent to  $u$ , then  $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = P_7$ , which is a contradiction. Hence,  $v$  is not adjacent to  $u$ . Since the vertex  $v$  is at distance 2 from the set  $\{x_1, y_3\}$  in  $G$ , there is a neighbor  $v'$  of  $v$  that is adjacent to  $x_1$  or to  $y_3$  or to both  $x_1$  and  $y_3$ . Note that  $v' \neq u$ .

First, suppose that  $v'$  is not adjacent to  $x_1$ , implying that  $v'$  is adjacent to  $y_3$ . If  $v'$  is not adjacent to  $y_2$ , then analogous arguments as in Claim 1.1 and Claim 1.2 (with the vertex  $u$  replaced by the vertex  $v'$ ) show that  $y_3$  is the only vertex in the set  $\{x_1, x_2, x_3, y_1, y_2, y_3\}$  that is adjacent to  $v'$ . This in turn implies that  $G[\{v, v', x_1, x_2, y_1, y_2, y_3\}] = P_7$ , which is a contradiction. Hence,  $v'$  is adjacent to  $y_2$ . If  $v'$  is adjacent to  $y_1$ , then  $G[\{v', y_1, y_2, y_3\}] = D$ , which is a contradiction. Thus,  $v'$  is not adjacent to  $y_1$ , implying that  $G[\{v, v', y_1, y_2, y_3\}] = B$ , which is a contradiction. Therefore,  $v'$  is adjacent to  $x_1$ .

Since  $G$  is  $\{D, K_4\}$ -free, the vertex  $v'$  is not adjacent to  $x_2$  or  $x_3$ . If  $v'$  is adjacent to  $y_1$ , then  $G[\{v, v', x_1, x_2, y_1\}] = B$ , which is a contradiction. Thus,  $v'$  is not adjacent to  $y_1$ . If  $v'$  is not adjacent to  $y_2$ , then  $G[\{v, v', x_1, x_2, y_1, y_2\}] = F_1$ , which is a contradiction. Thus,  $v'$  is adjacent to  $y_2$ . If  $v'$  is not adjacent to  $y_3$ , then  $G[\{v, v', x_1, x_2, y_2, y_3\}] = F_1$ , which is a contradiction. Thus,  $v'$  is adjacent to  $y_3$ , implying that  $G[\{v, v', y_1, y_2, y_3\}] = B$ , which is a contradiction. Therefore,  $u$  is not adjacent to  $y_3$ .  $\square$

We return to the proof of Claim 1. By Claims 1.1, 1.2 and 1.3, the vertex  $u$  is at distance 2 from the set  $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ . Since the vertex  $u$  is at distance 2 from the set  $\{x_1, y_2\}$  in  $G$ , there is a neighbor  $u'$  of  $u$  that is adjacent to  $x_1$  or to  $y_2$  or to both  $x_1$  and  $y_2$ . First, suppose that  $u'$  is adjacent to  $x_1$ . Analogously as above, since  $G$  is  $\{B, D, K_4\}$ -free, the vertex  $u'$  is not adjacent to  $x_2, x_3$  and  $y_1$ . If  $u'$  is not adjacent to  $y_2$ , then  $G[\{u, u', x_1, x_2, y_1, y_2\}] = F_1$ , which is a contradiction. Thus,  $u'$  is adjacent to  $y_2$ . If  $u'$  is adjacent to  $y_3$ , then  $G[\{u, u', y_1, y_2, y_3\}] = B$ , while, if  $u'$  is not adjacent to  $y_3$ , then  $G[\{u, u', x_1, x_2, y_2, y_3\}] = F_1$ . Since both cases produce a contradiction, we deduce that  $u'$  is not adjacent to  $x_1$ , implying that  $u'$  is adjacent to  $y_2$ . Since  $G$  is  $B$ -free,  $u'$  is not adjacent to  $y_1$ . If  $u'$  is not adjacent to  $y_3$ , then  $G[\{u, u', x_1, y_1, y_2, y_3\}] = F_1$ , which is a contradiction. If  $u'$  is adjacent to  $y_3$ , then  $G[\{u, u', y_1, y_2, y_3\}] = B$ , which is a contradiction. This completes the proof of Claim 1.  $\square$

**Claim 2.** *If  $C$  is an arbitrary triangle in  $G$ , then every vertex is within distance 2 from  $V(C)$ .*

**Proof.** Let  $C : x_1x_2x_3$  be a triangle in  $G$ . Suppose that there is a vertex  $y_3$  at distance 3 from  $V(C)$  in  $G$ . Let  $x_1y_1y_2y_3$  be a shortest path in  $G$  from  $y_3$  to  $V(C)$ . Since  $G$  is  $\{D, K_4\}$ -free, the vertex  $y_1$  is adjacent to neither  $x_2$  nor  $x_3$ , implying that  $F$  is an induced subgraph of  $G$ , which contradicts Claim 1.  $\square$

**Claim 3.** *Every triangle contains at least one vertex of degree exactly 2 in  $G$ .*

**Proof.** Let  $C : x_1x_2x_3$  be a triangle in  $G$ . Suppose that every vertex on  $C$  has degree at least 3 in  $G$ . Let  $y_1, y_2, y_3 \in V(G) \setminus V(C)$  be neighbors of  $x_1, x_2, x_3$ , respectively. Since  $G$  is  $\{D, K_4\}$ -free,  $x_i$  is the only neighbor of  $y_i$  in  $V(C)$  for  $i \in [3]$ . Since  $G$  is  $B$ -free, the vertices  $y_1, y_2$  and  $y_3$  induce a triangle  $C'$  in  $G$ . Suppose that there is a vertex  $y \in V(G) \setminus (V(C) \cup V(C'))$  that is adjacent to a vertex on  $C$ , say  $x_1$ . Since  $G$  is  $\{D, K_4\}$ -free,  $x_1$  is the only neighbor of  $y$  on  $C$ , and  $y$  is non-adjacent to some vertex  $y_j$  on  $C'$  with  $j \in \{2, 3\}$ , which implies the contradiction that  $G[\{x_1, x_2, x_3, y, y_j\}] = B$ . Hence, each vertex on  $C$  has degree exactly 3 in  $G$ . By symmetry, each vertex on  $C'$  has degree exactly 3 in  $G$ . Thus,  $G = P_2 \square C_3$ , implying that  $\gamma(G) = \gamma_e(G) = 2$ , which is a contradiction. This completes the proof of Claim 3.  $\square$

**Claim 4.** *Every triangle contains two vertices of degree exactly 2 in  $G$ .*

**Proof.** Let  $C : x_1x_2x_3$  be a triangle in  $G$  and let  $R = V(G) \setminus V(C)$ . By Claim 3, the triangle  $C$  contains at least one vertex of degree exactly 2 in  $G$ . Renaming vertices if necessary, we may assume that  $x_1$  has degree 2 in  $G$ . Suppose that both  $x_2$  and  $x_3$  have degree at least 3 in  $G$ . Since  $G$  is  $D$ -free, the vertices  $x_2$  and  $x_3$  have no common neighbor in  $R$ . Further, since  $G$  is  $B$ -free, every neighbor of



$x_2$  in  $R$  is adjacent to every neighbor of  $x_3$  in  $R$ . Hence, since  $G$  is  $\{D, K_{2,3}\}$ -free, the degrees of  $x_2$  and  $x_3$  are exactly 3 in  $G$ . Let  $y_2$  and  $y_3$  in  $R$  be neighbors of  $x_2$  and  $x_3$ , respectively. Recall that  $\gamma(G) \geq 3$ . Let  $w_2$  be a vertex not dominated by  $\{x_2, y_3\}$ , and let  $w_3$  be a vertex not dominated by  $\{x_3, y_2\}$ . By Claim 2, the vertex  $w_2$  is within distance 2 from  $V(C)$ , implying that  $w_2$  is adjacent to  $y_2$ . Analogously, the vertex  $w_3$  is adjacent to  $y_3$ . Note that  $w_2 \neq w_3$ . If  $w_2$  is adjacent to  $w_3$ , then  $G[\{w_2, w_3, x_2, x_3, y_2, y_3\}] = P_2 \square P_3$ . If  $w_2$  is not adjacent to  $w_3$ , then  $G[\{w_2, w_3, x_1, x_2, y_2, y_3\}] = F_1$ . Both cases produce a contradiction, which completes the proof of Claim 4.  $\square$

Let  $C : x_1x_2x_3$  be a triangle in  $G$ . By Claim 4, we may assume, renaming vertices if necessary, that  $x_2$  and  $x_3$  have degree 2 in  $G$ . Since  $\gamma(G) \geq 3$ , the vertex  $x_1$  does not dominate  $V(G)$ . Let  $D_2 = V(G) \setminus N_G[x_1]$ . Claim 2 implies that every vertex in  $D_2$  is at distance exactly 2 from  $x_1$  in  $G$ . Let  $D_1$  be the set of neighbors in  $V(G) \setminus D_2$  of the vertices in  $D_2$ . Note that  $D_1 \subset N_G(x_1)$ . By Claim 4, the set  $D_1$  is independent.

**Claim 5.** *Every vertex in  $D_2$  has exactly one neighbor in  $D_1$ .*

**Proof.** Since  $D_1$  is an independent set, and, since  $G$  is  $K_{2,3}$ -free, every vertex in  $D_2$  has at most two neighbors in  $D_1$ . Suppose that a vertex  $w_1$  in  $D_2$  has two neighbors  $y_1, y_2$  in  $D_1$ . Since  $\{x_1, y_1\}$  is not a dominating set of  $G$ , there is a vertex  $w_2 \in D_2$  that is not adjacent to  $y_1$ .

**Claim 5.1.** *The vertex  $w_2$  is not adjacent to  $y_2$ .*

**Proof.** Suppose that  $w_2$  is adjacent to  $y_2$ . Since  $\{x_1, y_2\}$  is not a dominating set, there is a vertex  $w_3$  in  $D_2$  that is not adjacent to  $y_2$ . Suppose that  $w_3$  is adjacent to  $y_1$ . If  $w_3$  is not adjacent to  $w_2$ , then  $G[\{x_1, x_2, y_1, y_2, w_2, w_3\}] = F_1$ , which is a contradiction. Hence,  $w_3$  is adjacent to  $w_2$ . If  $w_3$  is adjacent to  $w_1$ , then, since  $G$  is  $D$ -free,  $w_1$  is not adjacent to  $w_2$ , implying that  $G[\{x_1, y_1, w_1, w_2, w_3\}] = B$ , which is a contradiction. Thus,  $w_3$  is not adjacent to  $w_1$ . If  $w_1$  is not adjacent to  $w_2$ , then  $G[\{x_1, x_2, y_1, w_1, w_2, w_3\}] = F_1$ , while, if  $w_1$  is adjacent to  $w_2$ , then  $G[\{x_1, y_2, w_1, w_2, w_3\}] = B$ . Since both cases produce a contradiction, we deduce that  $w_3$  is not adjacent to  $y_1$ . Since  $G$  is  $\{D, P_2 \square P_3\}$ -free, the vertex  $w_3$  is therefore adjacent to at most one of  $w_1$  and  $w_2$ .

Let  $y_3$  be a neighbor of  $w_3$  in  $D_1$ . As observed earlier, every vertex in  $D_2$  has at most two neighbors in  $D_1$ . In particular,  $w_1$  is not adjacent to  $y_3$ . If  $w_3$  is not adjacent to  $w_1$ , then  $G[\{x_1, x_2, y_1, y_3, w_1, w_3\}] = F_1$ , which is a contradiction. Thus,  $w_3$  is adjacent to  $w_1$ , implying that  $w_3$  is not adjacent to  $w_2$ . If  $w_2$  is not adjacent to  $y_3$ , then  $G[\{x_1, x_2, y_2, y_3, w_2, w_3\}] = F_1$ , which is a contradiction. Hence,  $w_2$  is adjacent to  $y_3$ . If  $w_1$  and  $w_2$  are not adjacent, then  $G[\{x_1, y_1, y_2, y_3, w_1, w_2\}] = P_2 \square P_3$ , which is a contradiction. Hence,  $w_1$  and  $w_2$

are adjacent, implying that  $G[\{x_1, y_2, w_1, w_2, w_3\}] = B$ , which is a contradiction. Therefore,  $w_2$  is not adjacent to  $y_2$ .  $\square$

Recall that  $w_2$  is not adjacent to  $y_1$ . By Claim 5.1, the vertex  $w_2$  is not adjacent to  $y_2$ . Let  $y_4$  be a neighbor of  $w_2$  in  $D_1$ . Since every vertex in  $D_2$  has at most two neighbors in  $D_1$ , the vertex  $w_1$  is not adjacent to  $y_4$ . If  $w_1$  is not adjacent to  $w_2$ , then  $G[\{x_1, x_2, y_1, y_4, w_1, w_2\}] = F_1$ , which is a contradiction. Hence,  $w_1$  is adjacent to  $w_2$ . As  $\{x_1, w_1\}$  is not a dominating set of  $G$ , there is a vertex  $w_4$  in  $D_2$  that is not adjacent to  $w_1$ . Since  $G$  is  $F_1$ -free, the vertex  $w_4$  is adjacent to  $y_1$  or to  $y_2$  or to both  $y_1$  and  $y_2$ . If  $w_4$  is adjacent to  $y_1$  and  $y_2$ , then  $G[\{x_1, y_1, y_2, w_1, w_4\}] = K_{2,3}$ , which is a contradiction. Hence, by symmetry, we may assume that  $w_4$  is adjacent to  $y_1$ , but not to  $y_2$ . If  $w_4$  is adjacent to  $w_2$ , then we get the contradiction  $G[\{x_1, y_1, y_2, w_1, w_2, w_4\}] = P_2 \square P_3$ . If  $w_4$  is not adjacent to  $w_2$ , then  $G[\{x_1, x_2, y_1, y_4, w_2, w_4\}] = F_1$ , which is a contradiction, and completes the proof of Claim 5.  $\square$

Let  $D_1 = \{y_1, \dots, y_k\}$ , and, for  $i \in [k]$ , let  $w_i$  be a neighbor of  $y_i$  in  $D_2$ . If  $k = 1$ , then  $\{x_1, y_1\}$  is a dominating set of  $G$ , which is a contradiction. Hence,  $k \geq 2$ . By Claim 5, the vertex  $y_i$  is the only neighbor of  $w_i$  in  $D_1$  for  $i \in [k]$ . Since  $G$  is  $F_1$ -free, the vertices  $w_1, \dots, w_k$  induce a clique in  $G$ . Thus, by Claim 4, we obtain  $k \leq 2$ . This implies  $k = 2$ . Since  $G$  is  $F_1$ -free, each neighbor of  $y_i$  in  $D_2$  is adjacent to every neighbor of  $y_{3-i}$  in  $D_2$  for  $i \in [2]$ . If  $y_1$  and  $y_2$  both have only one neighbor in  $D_2$ , then  $\{x_1, w_1\}$  is a dominating set of  $G$ , which is a contradiction. Hence, by symmetry, we may assume that the vertex  $y_1$  has two neighbors  $w_1$  and  $w'_1$  in  $D_2$ . Both  $w_1$  and  $w'_1$  are adjacent to  $w_2$ . Since  $G$  is  $D$ -free,  $w_1$  and  $w'_1$  are not adjacent. Since  $\{x_1, w_2\}$  is not a dominating set of  $G$ , the vertex  $y_2$  has a neighbor  $w'_2$  in  $D_2$  that is different from  $w_2$  and not adjacent to  $w_2$ . Thus,  $G[\{w_1, w'_1, w_2, w'_2, y_1\}] = K_{2,3}$ , which is a contradiction, and completes the proof of Theorem 1.  $\blacksquare$

We close with a number of conjectures.

**Conjecture 5.** *There is a finite set  $\mathcal{F}$  of graphs such that some graph  $G$  satisfies  $\gamma(H) = \gamma_e(H)$  for every induced subgraph  $H$  of  $G$  if and only if  $G$  is  $\mathcal{F}$ -free.*

**Conjecture 6.** *The set  $\mathcal{F}$  in Conjecture 5 can be chosen such that  $\gamma(F) = 3$  and  $\gamma_e(F) = 2$  for every graph  $F$  in  $\mathcal{F}$ .*

Our proof of Theorem 1 actually implies that every component of a graph that is  $\{B, D, K_4, K_{2,3}, P_2 \square P_3\} \cup \{P_7, C_7, F_1, \dots, F_5\}$ -free has domination number at most 2. Wolk [9] showed that the largest hereditary class of graphs for which every component has domination number 1 is the class of  $\{P_4, C_4\}$ -free graphs. A complete characterization of the largest hereditary class of graphs for which

every component has domination number at most 2 in terms of minimal forbidden induced subgraphs seems to be a challenging and interesting problem, to which our results indirectly contribute.

Similar to the definition of an exponential dominating set, Dankelmann *et al.* [5] define a set  $D$  of vertices of a graph  $G$  to be a *porous exponential dominating set* of  $G$  if  $w_{(G,D)}^*(u) \geq 1$  for every vertex  $u$  of  $G$ , where  $w_{(G,D)}^*(u) = \sum_{v \in D} \left(\frac{1}{2}\right)^{\text{dist}_G(u,v)-1}$ . They define the *porous exponential domination number*  $\gamma_e^*(G)$  of  $G$  as the minimum size of a porous exponential dominating set of  $G$ . Clearly,  $\gamma_e^*(G) \leq \gamma_e(G) \leq \gamma(G)$  for every graph  $G$ .

**Conjecture 7.** *A graph  $G$  satisfies  $\gamma(H) = \gamma_e(H)$  for every induced subgraph  $H$  of  $G$  if and only if  $\gamma(H) = \gamma_e^*(H)$  for every induced subgraph  $H$  of  $G$ .*

## REFERENCES

- [1] M. Anderson, R.C. Brigham, J.R. Carrington, R.P. Vitray and J. Yellen, *On exponential domination of  $C_m \times C_n$* , AKCE Int. J. Graphs Comb. **6** (2009) 341–351.
- [2] A. Aytacı and B. Atay, *On exponential domination of some graphs*, Nonlinear Dyn. Syst. Theory **16** (2016) 12–19.
- [3] S. Bessy, P. Ochem and D. Rautenbach, *Bounds on the exponential domination number*, Discrete Math. **340** (2017) 494–503.  
doi:10.1016/j.disc.2016.08.024
- [4] S. Bessy, P. Ochem and D. Rautenbach, *Exponential domination in subcubic graphs*, Electron. J. Comb. **23** (2016) #P4.42.
- [5] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi and H. Swart, *Domination with exponential decay*, Discrete Math. **309** (2009) 5877–5883.  
doi:10.1016/j.disc.2008.06.040
- [6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).
- [7] M.A. Henning, *Distance domination in graphs*, in: T.W. Haynes, S.T. Hedetniemi and P.J. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998) 321–349.
- [8] M.A. Henning, S. Jäger and D. Rautenbach, *Relating domination, exponential domination, and porous exponential domination*, Discrete Optim. **23** (2017) 81–92.  
doi:0.1016/j.disopt.2016.12.002
- [9] E.S. Wolk, *A note on “the comparability graph of a tree”*, Proc. Amer. Math. Soc. **16** (1965) 17–20.

Received 17 May 2016  
Revised 25 November 2016  
Accepted 25 November 2016