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REQUIRING THAT MINIMAL SEPARATORS INDUCE COMPLETE MULTIPARTITE SUBGRAPHS

Terry A. McKee

Department of Mathematics and Statistics
Wright State University
Dayton, Ohio 45435 USA

e-mail: terry.mckee@wright.edu

Abstract

Complete multipartite graphs range from complete graphs (with every partite set a singleton) to edgeless graphs (with a unique partite set). Requiring minimal separators to all induce one or the other of these extremes characterizes, respectively, the classical chordal graphs and the emergent unichord-free graphs. New theorems characterize several subclasses of the graphs whose minimal separators induce complete multipartite subgraphs, in particular the graphs that are 2-clique sums of complete, cycle, wheel, and octahedron graphs.

Keywords: minimal separator, complete multipartite graph, chordal graph, unichord-free graph.

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1. Introduction and Terminology

Define a *complete-multipartite-separator graph* to be a graph in which every minimal separator (as defined later in this section) induces a complete multipartite subgraph. As one special case, the graphs in which every minimal separator induces a complete graph are precisely the *chordal graphs*, a classic graph class with many characterizations, the most common being that every cycle of length 4 or more has at least one chord; see [1, 7]. At the other extreme, the graphs for which every minimal separator induces an edgeless subgraph are precisely the *unichord-free graphs*, a recent graph class whose name comes from the characterization that no cycle has exactly one chord; see [2, 3, 4, 6, 8].

Section 2 will characterize the complete-multipartite-separator graphs, which include all complete multipartite graphs, all chordal graphs, and all unichord-free graphs. But this characterization fails to generalize the existing characterizations of chordal graphs and unichord-free graphs. Section 3 will remedy this, along with generalizing these previously studied classes to increasingly larger subclasses of complete-multipartite-separator graphs.

For any set S of vertices of a graph G, let $G[S]$ denote the subgraph of G induced by S and let $G - S$ denote $G[V(G) - S]$. Let $\overline{G}$ denote the graph complement of G and, for every graph H, define G to be *H-free* if no induced subgraph of G is isomorphic to H. A *chord* of a cycle C is an edge vw with $v, w \in V(C)$ and yet $vw \notin E(C)$. Let C_n and P_n denote, respectively, the cycle and path of order n (so P_n has length $n-1$). For any x-to-y path π, let $\pi^\circ = V(\pi) - \{x, y\}$ be the set of *internal vertices* of π.

For nonadjacent vertices v and w in a connected graph G, a *v,w-separator* of G is a set $S \subseteq V(G) - \{v, w\}$ such that v and w are in different *components* (maximal connected subgraphs) of $G - S$, and a *vertex separator* of G is a v,w-separator for some $v, w \in V(G)$. A *minimal v,w-separator* of G is a v,w-separator that is not a proper subset of another v,w-separator, and a *minimal separator* of G is a minimal v,w-separator for some $v, w \in V(G)$. This means that one minimal separator can be contained in another one, since a minimal v, w-separator might be contained in a minimal v', w'-separator. See [1] for more about minimal separators, including that a vertex separator S of G is a minimal separator of G if and only if, for some two components G_1 and G_2 of $G - S$, each vertex in S has a neighbor in each G_i.

A *complete k-partite graph* G has $V(G)$ partitioned into $k \geq 1$ nonempty *partite sets* $V_1, \ldots, V_k$ where $E(G) = \{xy : (x, y) \in V_i \times V_j \text{ with } i \neq j\}$; denote G by $K(n_1, \ldots, n_k)$ where each $n_i = |V_i|$ and $1 \leq n_1 \leq \cdots \leq n_k$. A *complete multipartite graph* is a complete k-partite graph for some $k \geq 1$. Therefore—as will be used several times in the following sections—a graph is complete multipartite if and only if it has no induced subgraph $H \cong \overline{P_3} = K_1 \cup K_2$ (if, say, $V(H) = \{x, y, z\}$ with $E(H) = \{xy\}$, then x and y would have to be in the same partite set as z in a complete multipartite graph, but then they would not be adjacent to each other.) The two extremes among complete multipartite graphs are the complete graphs $K_n = K(1, \ldots, 1)$ (n-partite with each $n_i = 1$) and the edgeless graphs $\overline{K_n} = K(n)$ (1-partite with the unique $n_i = n_1 = n$).

2. When Each $G[S]$ is Complete Multipartite

Lemma 1. *Every minimal separator of an induced subgraph of G is contained in a minimal separator of G.*

Proof. Suppose S_0 is a minimal v,w-separator of an induced subgraph H_0 of G such that $H_0 - S_0$ has components $H_0[R_0]$ and $H_0[R'_0]$ where $v \in R_0$, $w \in R'_0$, and each vertex in S_0 has neighbors in both R_0 and R'_0.

Suppose S_0 is not a v,w-separator of G, which means that $G-S_0$ is connected by a v-to-w path π_1 containing some $x_1 \in V(\pi_1^\circ) - V(H_0)$. Thus, $S_0 \cup \{x_1\}$ is contained in a v,w-separator S_1 of an induced subgraph H_1 of G such that H_1-S_1 has components $H_1[R_1]$ and $H_1[R'_1]$ with $R_1 = R_0 \cup \tau^\circ$ where τ is the v-to-x_1 subpath of π_1, and $R'_1 = R'_0 \cup (\tau')^\circ$ where τ' is the x_1-to-w subpath of π_1. As in [1], S_1 is a minimal v,w-separator since each vertex of S_0 has neighbors in both R_0 and R'_0 (and so in both R_1 and R'_1), and each vertex of $S_1 - S_0$ has neighbors in both τ and τ' (and so in both R_1 and R'_1).

Repeat this, sequentially forming larger sets S_i by choosing $x_i \in V(\pi_i^\circ)$ in connected graphs $G - S_{i-1}$. As soon as $G - S_i$ stops being connected, S_i will be a minimal separator of G that contains S_0. ■

If vw is a chord of C and $x, y \in V(C) - \{v, w\}$, say that vw *crosses* $\{x, y\}$ if the four vertices v, x, w, y come in that order around C. For disjoint subsets $S_1, S_2 \subset V(C)$, define an S_1-to-S_2 chord of C to be a chord xy where $x \in S_1$ and $y \in S_2$. A *theta graph* $\Theta = \Theta(u, w; \pi_1, \pi_2, \pi_3)$ of a graph G consists of nonadjacent vertices u and w along with three internally-disjoint chordless u-to-w paths π_1, π_2, and π_3. Define a *chord* of Θ to be a chord of a cycle $\pi_i \cup \pi_j$ with $i \neq j$, and define a *transversal* of Θ to be any $\{z_1, z_2, z_3\}$ where each $z_i \in \pi_i^\circ$. Thus a transversal of Θ is a minimal u, v-separator of Θ (but not necessarily of $G[V(\Theta)]$, because chords of Θ will be edges of $G[V(\Theta)]$). Say that a transversal $\{z_1, z_2, z_3\}$ of Θ is *crossed by a chord* of Θ if some $xy \in E(G)$ has x an internal vertex of some u-to-z_i subpath of π_i and y an internal vertex of some z_j-to-w subpath of $\pi_j \neq \pi_i$.

Recall that complete-multipartite-separator graphs are those in which every minimal separator induces a complete multipartite subgraph. Theorem 2 is equivalent to a result in [5] (which contains a more general discussion of restrictions on minimal separators).

Theorem 2. *A graph G is a complete-multipartite-separator graph if and only if, for every theta subgraph Θ of G with transversal S, if $G[S] \cong \overline{P_3}$, then S is crossed by a chord of Θ.*

Proof. First, suppose G is a complete-multipartite-separator graph containing a theta subgraph $\Theta = \Theta\ (u, w; \pi_1, \pi_2, \pi_3)$ with transversal S where $G[S] \cong \overline{P_3}$. Since complete multipartite graphs are $\overline{P_3}$-free, S cannot be contained in a minimal separator of $G[V(\Theta)]$ by Lemma 1. Therefore, since each π_i is chordless, S is crossed by a chord of Θ.

Conversely, suppose G has a minimal separator S such that $G[S]$ is not complete multipartite. Thus, there exists $S_0 = \{x, y, z\} \subseteq S$ that induces a $\overline{P_3}$

subgraph, say with edge xy and isolated vertex z. Let τ_1 and τ_2 be x-to-y paths with τ_1° and τ_2°, respectively, inside distinct components G_1 and G_2 of $G-S$, and let C be the cycle $\tau_1 \cup \tau_2$. Let π be a chordless τ_1°-to-τ_2° path through z with endpoints u and w and with $\pi^\circ \cap V(C) = \emptyset$. If π_1 and π_2 are the two u-to-w subpaths of C and $\pi_3 = \pi$, then S_0 is a transversal of $\Theta\,(u, w; \pi_1, \pi_2, \pi_3)$ of G. But then $G[S_0] \cong \overline{P_3}$, and yet, since S is a minimal separator of G, the transversal S_0 is not crossed by a chord of Θ. ■

Although it follows directly from the $\overline{P_3}$-free characterization of complete multipartite graphs in the final paragraph of Section 1, Theorem 2 fails to display how unichord-free graphs and chordal graphs are the fundamental special cases of the class of complete-multipartite-separator graphs. Theorems 3 and 5 will do this by introducing parameters that stratify this class so that unichord-free graphs and chordal graphs are the parameter-1 cases. Finally, Theorems 7 and 10 will characterize a new graph class that is the conjunction of the parameter-2 cases.

3. When Each $G[S]$ is Independent or is Complete

Motivated by theta graphs (which are sometimes called "3-skeins"), define a *generalized k-skein* $\Theta = \Theta\,(T_1, T_2; \pi_1, \ldots, \pi_k)$ of G to consist of disjoint subtrees T_1 and T_2 of G with no vertex of T_1 adjacent to a vertex of T_2 together with $k \geq 2$ internally-disjoint, chordless T_1-to-T_2 paths $\pi_1, \ldots, \pi_k$ such that each $\pi_i^\circ \neq \emptyset$, each leaf of each T_i is the endpoint of at least two of the paths $\pi_1, \ldots, \pi_k$, and no $v \in V(T_1) \cup V(T_2)$ is adjacent to any internal vertex of any of $\pi_1, \ldots, \pi_k$ except when v is an endpoint of such a path. (The subtrees T_1 and T_2 are not necessarily induced subgraphs of G, and an endpoint of π_i does not have to be a leaf of T_1 or T_2.) Theta graphs are generalized 3-skeins with $V(T_1) = \{u\}$ and $V(T_2) = \{w\}$, and a cycle C with nonconsecutive vertices u and w is a generalized 2-skein with $V(T_1) = \{u\}$ and $V(T_2) = \{w\}$ where $C = \pi_1 \cup \pi_2$.

Define a *chord* of $\Theta = \Theta\,(T_1, T_2; \pi_1, \ldots, \pi_k)$ to be an edge with endpoints in each of π_i° and π_j° where $i \neq j$, and define a *transversal* of Θ to be any set $\{z_1, \ldots, z_k\}$ where each $z_i \in \pi_i^\circ$; thus $\{z_1, \ldots, z_k\}$ is a minimal separator of Θ (but not necessarily of $G[V(\Theta)]$, because chords of Θ will be edges of $G[V(\Theta)]$). Say that a transversal $\{z_1, \ldots, z_k\}$ of Θ is *crossed by a chord* of Θ if some chord xy of Θ has x an internal vertex of the T_1-to-z_i subpath of π_i and y an internal vertex of the z_j-to-T_2 subpath of $\pi_j \neq \pi_i$.

A simple result from [4] is that a graph is unichord-free if and only if every chord xy of every cycle C has $\{x, y\}$ crossed by a chord of C. This characterization will be the $p = 1$ case of Theorem 3, in Corollary 4.

The *clique number* $\omega = \omega(G)$ of a graph G is the largest order of a complete subgraph of G. Thus, for each $p \geq 1$, saying that a complete multipartite graph

G has clique number $\omega \leq p$ in Theorem 3 is equivalent to saying that G is complete k-partite for some $k \leq p$ (which happens to be how complete p-partite graphs would be defined if the partite sets $V_1, \ldots, V_{p-1}$ had been allowed to be empty). Thus, G is complete multipartite with clique number $\omega \leq p$ if and only if G is $\overline{P_3}$-free (to ensure G is complete multipartite) and K_{p+1}-free (to ensure $\omega(G) \leq p$).

Theorem 3. *Suppose G is a complete-multipartite-separator graph. Every minimal separator of G induces a complete multipartite subgraph with clique number $\omega \leq p$ if and only if, for every generalized $(p+1)$-skein Θ with transversal S, if $G[S] \cong K_{p+1}$, then S is crossed by a chord of Θ.*

Proof. First, suppose every minimal separator of G induces a complete multipartite subgraph with clique number $\omega \leq p$ and $\Theta = \Theta(T_1, T_2; \pi_1, \ldots, \pi_{p+1})$ is a generalized $(p+1)$-skein of G with transversal $S = \{z_1, \ldots, z_k\}$ where $G[S] \cong K_{p+1}$. Since K_{p+1} has clique number $\omega > p$, the transversal S cannot be contained in a minimal separator of $G[V(\Theta)]$ by Lemma 1. Therefore, S is crossed by a chord of Θ (since each π_i is chordless and no vertex of T_1 is adjacent in G to a vertex of T_2).

Conversely, suppose G is a complete-multipartite-separator graph with a minimal separator S' such that $G[S']$ has clique number $\omega > p \geq 1$, say with $S = \{z_1, \ldots, z_{p+1}\} \subseteq S'$ where $G[S] \cong K_{p+1}$. Let σ and τ be z_1-to-z_2 paths with σ° and τ°, respectively, inside distinct components $G[U]$ and $G[W]$ of $G - S'$ such that each $z_i \in S$ has neighbors in both U and W.

Let C be the cycle $\sigma \cup \tau$ with $u_1 \in \sigma^\circ$ and $w_1 \in \tau^\circ$, and let π_1' and π_2' be the u_1-to-w_1 subpaths of C through, respectively, z_1 and z_2. Let $\Pi_2 = \pi_1 \cup \pi_2$ and let π_3 be a chordless u_2-to-w_2 path where $u_2 \in U$ and $w_2 \in W$ are both vertices of Π_2 such that $\pi_3^\circ \cap S' = \{z_3\}$ and $\pi_3^\circ \cap V(\Pi_2) = \emptyset$. Let T_1 be the trivial subtree u_2 of $G[U]$, let T_2 be the trivial subtree w_2 of $G[W]$, and let π_1 and π_2 be the u_2-to-w_2 paths of Π_2. This makes $\Theta\,(u_2, w_2; \pi_1, \pi_2, \pi_3)$ a theta graph and $\Theta\,(T_1, T_2; \pi_1, \pi_2, \pi_3)$ a generalized 3-skein.

For $3 \leq i \leq p$, continue recursively by letting $\Pi_i = \pi_1 \cup \cdots \cup \pi_i$ and letting π_{i+1} be a chordless u_i-to-w_i path where $u_i \in U$ and $w_i \in W$ are vertices of Π_i such that $\pi_{i+1}^\circ \cap S' = \{z_{i+1}\}$ and $\pi_{i+1}^\circ \cap V(\Pi_i) = \emptyset$. Enlarge T_1 to become a minimal subtree of $G[U] \cap \Pi_i$ that contains $\{u_2, \ldots, u_i\}$, and enlarge T_2 to become a minimal subtree of $G[W] \cap \Pi_i$ that contains $\{w_2, \ldots, w_i\}$. This makes $\Theta\,(T_1, T_2; \pi_1, \pi_2, \ldots, \pi_{i+1})$ a generalized $(i+1)$-skein.

This process ends with a generalized $(p+1)$-skein $\Theta = \Theta\,(T_1, T_2;\, \pi_1, \ldots, \pi_{p+1})$, with $z_i \in \pi_i^\circ$ whenever $1 \leq i \leq p+1$, such that Θ has transversal S. But then $G[S] \cong K_{p+1}$ and yet S is not crossed by a chord of Θ (since S' is a minimal separator of G with T_1 and T_2 in, respectively, the components $G[U]$ and $G[W]$ of $G - S'$). ■

Corollary 4. *A graph is unichord-free if and only if every chord xy in every cycle C has $\{x, y\}$ crossed by a chord of C.*

Proof. Recall that G is unichord-free if and only if every minimal separator induces an edgeless subgraph (a complete multipartite subgraph with clique number $\omega = 1$). The "only if" direction follows from a cycle C with a chord xy corresponding to a generalized 2-skein with each $|V(T_i)| = 1$ of which $S = \{x, y\}$ is a transversal (with $G[S] \cong P_2$). Therefore, by the $p = 1$ case of Theorem 3, $S = \{x, y\}$ is crossed by a chord of C.

For the "if" direction, a graph that is not unichord-free has a cycle C with a unique chord xy, where C corresponds to a generalized 2-skein Θ with transversal $S = \{x, y\}$ that has $G[S] \cong P_2$, and yet S is not crossed by a chord of Θ. ■

Now consider chordal graphs, at the other extreme of complete multipartite graphs from unichord-free graphs. A simple inductive argument on the length of cycles show that a graph is chordal if and only if every two nonadjacent vertices x and y of every cycle C has $\{x, y\}$ crossed by a chord of C. This characterization will be the $q = 1$ case of Theorem 5, in Corollary 6.

The *independence number* $\alpha = \alpha(G)$ of a graph G is the largest order of an edgeless induced subgraph of G. Thus, G is complete multipartite graph with independence number $\alpha \le q$ if and only if G is $\overline{P_3}$-free (to ensure G is complete multipartite) and $\overline{K_{q+1}}$-free (to ensure $\alpha(G) \le q$).

Theorem 5. *Suppose G is a complete-multipartite-separator graph. Every minimal separator of G induces a complete multipartite subgraph with independence number $\alpha \le q$ if and only if, for every generalized $(q+1)$-skein Θ with transversal S, if $G[S] \cong \overline{K_{q+1}}$, then S is crossed by a chord of Θ.*

Proof. This follows by the same proof as Theorem 3, replacing mentions of clique number with independence number, p with q, and K_{p+1} with $\overline{K_{q+1}}$. ■

Corollary 6. *A graph is chordal if and only if every two nonadjacent vertices x and y of every cycle C has $\{x, y\}$ crossed by a chord of C.*

Proof. Recall that G is chordal if and only if every minimal separator induces a complete subgraph (a complete multipartite graph with independence number $\alpha = 1$). The "only if" direction follows as in the proof of Corollary 4, except now $G[S] \cong \overline{P_2}$ and the $q = 1$ case of Theorem 5 is used.

For the "if" direction, a graph that is not chordal has a chordless cycle C of length at least 4, where C corresponds to a generalized 2-skein Θ with (each) transversal $S = \{x, y\}$ that has $G[S] \cong \overline{P_2}$, and yet S is not crossed by a chord of Θ. ■

As an easy joint consequence of the $p = 1$ and $q = 1$ cases of Theorems 3 and 5, a graph G is simultaneously unichord-free and chordal if and only if, for every two nonconsecutive vertices x and y (adjacent or not) of every cycle C of G, there is a chord that crosses $\{x, y\}$—in other words, every pair of vertices of C that might be crossed by a chord of C is crossed by a chord of C. The following two conditions are trivially equivalent to that characterization:

(C1) Every 2-connected subgraph of G is complete (such a G is often called a *block graph*).

(C2) Every minimal separator of G is a singleton.

Section 4 will consider a more interesting joint consequence of Theorems 3 and 5, except now of the $p = 2$ and $q = 2$ cases.

4. When Each $G[S]$ is an Induced Subgraph of C_4

Define an *induced-sub-C_4-separator graph* to be a graph G in which every minimal separator S of G induces a graph H that is isomorphic to an induced subgraph of $C_4 \cong K(2, 2)$—equivalently, H is one of the five graphs in Figure 1. It is simple to check that H being an induced subgraph of C_4 is also equivalent to every three vertices of H inducing a path; or, alternatively, to H being simultaneously $\overline{P_3}$-free, K_3-free, and $\overline{K_3}$-free. Theorems 7 and 10 will characterize the induced-sub-C_4-separator graphs (with the $G[S] \cong \overline{P_3}$ condition of Theorem 2 becoming $G[S] \ncong P_3$ in clause (2) of Theorem 7).

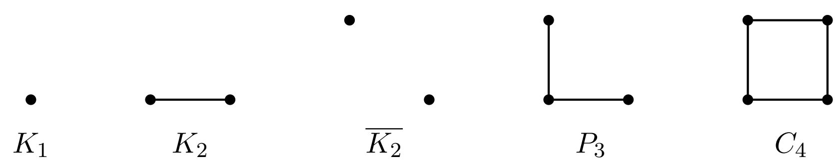

Figure 1. The five induced subgraphs of the complete bipartite graph C_4.

Theorem 7. *Each of the following is equivalent to a graph G being an induced-sub-C_4-separator graph*:

(1) *Every minimal separator of G induces a complete multipartite subgraph with clique number $\omega \leq 2$ and independence number $\alpha \leq 2$.*

(2) *G is a complete-multipartite-separator graph and, for every theta subgraph Θ of G with transversal S, if $G[S] \ncong P_3$, then S is crossed by a chord of Θ.*

Proof. The equivalence with (1) follows from the graphs in Figure 1 being the complete multipartite graphs $K(1)$, $K(1, 1)$, $K(2)$, $K(1, 2)$, and $K(2, 2)$, which are the only complete multipartite graphs that have both $\alpha, \omega \leq 2$.

For the necessity of (2), suppose G is an induced-sub-C_4-separator graph that contains a theta subgraph Θ with transversal S (so $|S| = 3$) such that $G[S] \not\cong P_3$. Theorem 2 implies $G[S] \not\cong \overline{P_3}$, and so $G[S]$ is isomorphic to K_3 or $\overline{K_3}$. Therefore, S is crossed by a chord of Θ using, respectively, the $\omega = p = 2$ case of Theorem 3 or the $\alpha = q = 2$ case of Theorem 5.

For the sufficiency of (2), suppose some minimal separator R of G has $G[R]$ that is not an induced subgraph of C_4. Thus $|R| \geq 3$ (since K_1, K_2, and $\overline{K_2}$ are induced subgraphs of C_4) and there exists an $S = \{x, y, z\} \subseteq R$ with $G[S] \not\cong P_3$. Let τ_1 and τ_2 be x-to-y paths with τ_1° and τ_2° inside different components of $G - R$, let π_3 be a chordless τ_1°-to-τ_2° path through z with $\pi_3^\circ \cap V(\tau_1 \cup \tau_2) = \emptyset$ (using that R is a minimal separator of G), let u and w be the endpoints of π_3, and let π_1 and π_2 be the two u-to-w subpaths of $\tau_1 \cup \tau_2$. Thus S is a transversal of the theta subgraph $\Theta(u, w; \pi_1, \pi_2, \pi_3)$ of G and $G[S] \not\cong P_3$, and yet S is not crossed by a chord of Θ (since R is a minimal separator of G). ■

Lemma 8 will characterize the 3-connected induced-sub-C_4-separator graphs in the style of condition (C1) at the end of Section 3. A *wheel* graph consists of a chordless cycle and a vertex that is adjacent to every vertex of that cycle.

Lemma 8. *A 3-connected graph is an induced-sub-C_4-separator graph if and only if it is either complete, a wheel, or the octahedron $K(2, 2, 2)$.*

Proof. First, suppose G is a 3-connected induced-sub-C_4-separator graph. Thus, every minimal separator S of G has $|S| \geq 3$ and so induces a P_3 or C_4 subgraph. Also suppose G is not complete.

Case 1. $S = \{a, b, c\}$ is a minimal separator of G with $G[S] \cong P_3$ having the edges ab and bc. Let G_1 and G_2 be components of $G - S$ such that every vertex of S has a neighbor in each of them, and let τ_1 and τ_2 be chordless a-to-c paths with each τ_i° in G_i. Say $\tau_1 = v_0, v_1, v_2, \ldots, v_t$ with $v_0 = a$ and $v_t = c$ has length $t \geq 2$, and let a' be the neighbor of a in τ_2. If $t \geq 3$, then $\tau_1^\circ \subset N(b)$ (to prevent, when $1 \leq j \leq t - 1$, either $\{a, b, v_j\}$ being in a minimal c, v_{j-1}-separator of $G[S \cup \tau_1^\circ \cup \tau_2^\circ]$ or $\{b, c, v_j\}$ being in a minimal a, v_{j+1}-separator of $G[S \cup \tau_1^\circ \cup \tau_2^\circ]$, either of which would, by Lemma 1, be in a minimal separator of G and induce a $\overline{P_3}$ subgraph of G). If, instead, $t = 2$, then $\tau_1^\circ \subset N(b)$ (to prevent $\{b, v_1, a'\}$ from being a minimal a, c-separator of $G[S \cup \tau_1^\circ \cup \tau_2^\circ]$ that would, by Lemma 1, be in a minimal separator of G and induce a $\overline{P_3}$ or a $\overline{K_3}$ subgraph of G, depending on whether or not a' is adjacent to b). Either way, $\tau_1^\circ \subset N(b)$ and, similarly, $\tau_2^\circ \subset N(b)$. Therefore, S is in the wheel $H_S = G[S \cup \tau_1^\circ \cup \tau_2^\circ]$, centered at b.

If $G - S$ has a third component G_3, then G being 3-connected ensures there exists a chordless a-to-c path τ_3 with τ_3° in G_3, and so $\tau_3^\circ \subset N(b)$ (as in the preceding paragraph) and $x_1, x_2, x_3 \in N(a)$ with each $x_i \in \tau_i^\circ$ would form a minimal a, c-separator of $G[\tau_1 \cup \tau_2 \cup \tau_3]$ that would, by Lemma 1, be in a minimal separator of G and induce a $\overline{K_3}$ subgraph of G. Therefore, $G - S$ has only the two components G_1 and G_2.

If $G \neq H_S$, then there exists $z \in V(G) - V(H_S)$, say with $z \in V(G_1) - \tau_1^\circ$. Form a new graph G^+ by appending one new vertex ν to G such that $N(\nu) = V(\tau_1)$. Since G^+ is also 3-connected, Menger's Theorem ensures that G^+ has three internally-disjoint ν-to-z paths that intersect τ_1 at three vertices in a minimal ν, z-separator S' of G^+; moreover, since $|S'| \geq 3$, at least one of these three vertices, say v_i, is in τ_1°. But then vertices a', v_i, and b would be internal vertices of, respectively, the a-to-c paths τ_1, τ_2, and the length-2 path a, b, c, along with z being an internal vertex of an additional a-to-c path with internal vertices in G_1 (using that G is 3-connected). Thus $\{a', b, v_i, z\}$ would be in a minimal a, c-separator of G that does not induce a C_4 subgraph. Therefore, G is the wheel H_S.

Case 2. $S = \{a, b, c, d\}$ is a minimal separator of G with $G[S] \cong C_4$ having the edges ab, bc, cd, and ad. Let G_1 and G_2 be components of $G - S$ such that every vertex of S has a neighbor in each of them, and let τ_1 and τ_2 be chordless a-to-c paths with each τ_i° in G_i. As in the argument for Case 1, $\tau_1^\circ \subset N(b)$ (using $G[\{a, b, c\}] \cong P_3$) and $\tau_1^\circ \subset N(d)$ (using $G[\{a, d, c\}] \cong P_3$). Pick any $x_1 \in \tau_1^\circ$ and let τ_1' be the b-to-d path b, x_1, d in G_1. As in the argument for Case 1, $(\tau_1')^\circ \in N(c)$ (using $G[\{b, c, d\}] \cong P_3$) and $(\tau_1')^\circ \subset N(a)$ (using $G[\{b, a, d\}] \cong P_3$). Thus $S \subseteq N(x_1)$, and similarly $S \subseteq N(x_2)$ for some x_2 in G_2. Therefore, S is in the octahedron $H_S = G[S \cup \{x_1, x_2\}]$.

If $G - S$ has a third component G_3, then G being 3-connected ensures there exists a chordless a-to-c or b-to-d path τ_3 with τ_3° in G_3; without loss of generality, say τ_3 is an a-to-c path. Thus, as in Case 1, x_1, x_2 and any $x_3 \in \tau_3^\circ$ would form a minimal a, c-separator of $G[\tau_1^\circ \cup \tau_2^\circ \cup \tau_3^\circ]$ that would induce a $\overline{K_3}$ subgraph of G. Therefore, $G - S$ has only the two components G_1 and G_2.

If $G \neq H_S$, then there exists $z \in V(G) - V(H_S)$, say with $z \in V(G_1) - \{x_1\}$. Since G is 3-connected, Menger's Theorem requires G to have three internally-disjoint z-to-x_2 paths that intersect S at three vertices in a minimal z, x_2-separator of G; without loss of generality, say these three vertices of S are a, b, c. But then vertices x_1, x_2, and b would be internal vertices of, respectively, the length-2 a-to-c paths a, x_1, c and a, x_2, b and a, b, c, along with z being an internal vertex of an additional a-to-c path with internal vertices in G_1 (using that G is 3-connected). Thus $\{b, x_1, x_2, z\}$ would be in a minimal a, c-separator of G that does not induce a C_4 subgraph. Therefore, G is the octahedron H_S.

Conversely, complete graphs have no minimal separators at all, and each minimal separator of a wheel or octahedron induces, respectively, a P_3 or C_4 subgraph. ∎

Lemma 9. *A* 2*-connected graph in which no minimal separator induces a* K_2 *subgraph is an induced-sub-*C_4*-separator graph if and only if it is either complete, a cycle, a wheel, or the octahedron.*

Proof. First, suppose G is a 2-connected induced-sub-C_4-separator graph in which no minimal separator induces a K_2 subgraph. If G is 3-connected, then G is either complete, a wheel, or the octahedron by Lemma 8.

Hence, assume G is not complete and has a minimal separator $S = \{a, b\}$ with $G[S] \cong \overline{K_2}$. Since a minimal a, b-separator of G must be one of the graphs in Figure 1 and cannot be K_1 (using that G is 2-connected) or any of K_2, P_3 or C_4 (using that S is a minimal separator of G), every minimal a, b-separator of G must induce a $\overline{K_2}$ subgraph of G where $G - S$ has exactly two components G_1 and G_2 and each minimal a, b-separator of each subgraph $G_i^+ = G[V(G_i) \cup S]$ is a singleton. Since no minimal separator of G induces a K_1 or K_2 subgraph, G_1^+ and G_2^+ are internally-disjoint a-to-b paths. Therefore, G is a cycle.

Conversely, each minimal separator of a cycle induces a $\overline{K_2}$ subgraph, and the minimal separators of complete graphs, wheels, and the octahedron are covered by Lemma 8. ■

In Theorem 10, the 2-*clique-sum* of vertex-disjoint graphs H_1 and H_2 results from identifying a clique of order at most 2 from each of H_1 and H_2 (in other words, identifying an edge of H_1 with an edge of H_2, or a vertex of H_1 with a vertex of H_2).

Theorem 10. *A graph is an induced-sub-C_4-separator graph if and only if it can be built from complete graphs, cycles, octahedra, and wheels by repeatedly forming* 2-*clique-sums.*

Proof. This follows from Lemma 9. ■

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