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ON TWO GENERALIZED CONNECTIVITIES OF GRAPHS

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Abstract

The concept of generalized k-connectivity $\kappa_k(G)$, mentioned by Hager in 1985, is a natural generalization of the path-version of the classical connectivity. The pendant tree-connectivity $\tau_k(G)$ was also introduced by Hager in 1985, which is a specialization of generalized k-connectivity but a generalization of the classical connectivity. Another generalized connectivity of a graph G, named k-connectivity $\kappa'_k(G)$, introduced by Chartrand *et al.* in 1984, is a generalization of the cut-version of the classical connectivity.

In this paper, we get the lower and upper bounds for the difference of $\kappa'_k(G)$ and $\tau_k(G)$ by showing that for a connected graph G of order n, if $\kappa'_k(G) \neq n-k+1$ where $k \geq 3$, then $1 \leq \kappa'_k(G) - \tau_k(G) \leq n-k$; otherwise, $1 \leq \kappa'_k(G) - \tau_k(G) \leq n-k+1$. Moreover, all of these bounds are sharp. We get a sharp upper bound for the 3-connectivity of the Cartesian product of any two connected graphs with orders at least 5. Especially, the exact values for some special cases are determined. Among our results, we also study the pendant tree-connectivity of Cayley graphs on Abelian groups of small degrees and obtain the exact values for $\tau_k(G)$, where G is a cubic or 4-regular Cayley graph on Abelian groups, $3 \leq k \leq n$.

Keywords: k-connectivity, pendant tree-connectivity, Cartesian product, Cayley graph.

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1. Introduction

We refer to a book [5] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$ and $E(G)$ be the set of vertices and the set of edges of G, respectively. For $S \subseteq V(G)$, we denote by $G \setminus S$ the subgraph obtained by deleting from G the vertices of S together with the edges incident with them. We use P_n, C_m and K_ℓ to denote a path of order n, a cycle of order m and a complete graph of order ℓ, respectively.

Connectivity is one of the most basic concepts in graph theory, both in a combinatorial sense and in an algorithmic sense. The classical connectivity has two equivalent definitions. The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set $S \subseteq V(G)$ such that $G \setminus S$ is disconnected or has only one vertex. This definition is called the *cut-version* definition of the connectivity. A well-known theorem of Menger provides an equivalent definition, which can be called the *path-version* definition of the connectivity. For any two distinct vertices x and y in G, the local connectivity $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x, y)|x, y \in V(G), x \neq y\}$ is defined to be the connectivity of G.

Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings and so people tried to generalize this concept. For the cut-version definition of the connectivity, we find that the above minimum vertex set does not regard to the number of components of $G \setminus S$. Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n-1}$ and the path P_n $(n \geq 3)$ are both trees of order n and therefore have connectivity 1, but the deletion of a cut-vertex from $K_{1,n-1}$ produces a graph with $n-1$ components while the deletion of a cut-vertex from P_n produces only two components. Chartrand *et al.* [6] generalized the cut-version definition of the connectivity as follows: For an integer $k \geq 2$ and a graph G of order $n \geq k$, the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G produces a graph with at least k components or a graph with fewer than k vertices. By definition, we clearly have $\kappa'_2(G) = \kappa(G)$. Thus, the concept of the k-connectivity could be seen as a generalization of the classical connectivity. For more details about this topic, we refer to [6, 8, 28, 29, 36, 38].

Another generalized connectivity of a graph G, mentioned by Hager in 1985 [14], is a natural generalization of the path-version definition of the connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is such a subgraph

T of G that is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *internally disjoint* if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. The *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* (or *k-tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa_G(S) |\ S \subseteq V(G), |S| = k\}$. Thus, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all the k-subsets of $V(G)$. By definition, we clearly have $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. There are many results on the generalized k-connectivity, such as [7, 11, 14, 20, 21, 23, 24, 26, 32–35, 37–39].

The concept of pendant tree-connectivity, introduced by Hager in 1985 [14], is a specialization of generalized k-connectivity but a generalization of the classical connectivity. For an S-Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pendant S-Steiner tree*. Two pendant S-trees T_1 and T_2 are said to be *internally disjoint* if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. The *local pendant tree-connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pendant S-trees in G. For an integer k with $2 \leq k \leq n$, the *pendant tree-connectivity* is defined as $\tau_k(G) = \min\{\tau_G(S) |\ S \subseteq V(G), |S| = k\}$. Thus, $\tau_k(G)$ is the minimum value of $\tau_G(S)$ when S runs over all the k-subsets of $V(G)$. By definition, we clearly have $\tau_2(G) = \kappa(G)$.

In addition to being a natural combinatorial measure, both the pendant tree-connectivity and the generalized k-connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one wants to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI [12,13,30] and computer communication networks [9]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then the number of totally independent ways to connect them is a measure for this purpose. The generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G. For the topic of generalized connectivities and their applications, the reader is referred to a survey [22] and a monograph [25].

In [38], Sun and Li compared the k-connectivity $\kappa'_k(G)$ and the generalized k-connectivity $\kappa_k(G)$ of a graph G and obtained sharp lower bounds and upper bounds for $\kappa'_k(G) - \kappa_k(G)$, where $3 \leq k \leq n$. Note that the k-connectivity $\kappa'_k(G)$ and the pendant tree-connectivity $\tau_k(G)$ of a graph are also different. For example, just consider the cycle C_n with $n \geq 6$, we clearly have $\kappa'_3(G) = 3$ and $\tau_3(G) = 0$. Hence, we want to find the difference between these two parameters and the following problem is very interesting.

Problem 1.1. Give nice bounds for $\kappa'_k(G) - \tau_k(G)$, where $3 \leq k \leq n$.

In this paper, we will answer Problem 1.1 by giving sharp lower and upper bounds for $\kappa'_k(G) - \tau_k(G)$ with $3 \le k \le n$ (Theorem 8).

Products of graphs occur naturally in discrete mathematics as tools in combinatorial constructions, they give rise to important classes of graphs and deep structural problems. Cartesian product is one of the most important graph products and plays a key role in design and analysis of networks. Many researchers have investigated the topic of Cartesian product graphs in the past several decades, the reader is referred to three important books [15, 17, 18]. In this paper, we will continue the research on the topic of k-connectivity and study the 3-connectivity of the Cartesian product graphs by getting a sharp upper bound for $\kappa'_3(G\Box H)$, where G and H are any two connected graphs with orders at least five (Theorem 10). Among our results, we will also obtain the precise values for $\kappa'_3(G\Box H)$, where G and H belong to some special graph classes (Propositions 9, 12, 13, 14 and 15).

Let X be a finite group, with operation denoted additively, and A be a subset of $X \setminus \{0\}$ such that $a \in A$ implies $-a \in A$, where 0 is the identity element of X. The *Cayley graph* $Cay(X, A)$ is defined to have vertex set X such that there is an edge between x and y if and only if $x - y \in A$. It is clear that $Cay(X, A)$ is connected if and only if A is a generating set of X. Cayley graphs have been important objects of study in algebraic graph theory over many years [2, 4]. In particular, mathematicians and computer scientists recommend (e.g. [1, 16, 40]) Cayley graphs as models for interconnection networks because they exhibit many properties that ensure high performance. In fact, a number of networks of both theoretical and practical importance, including hypercubes, butterflies, cube-connected cycles, star graphs and their generalizations, are Cayley graphs. The reader is referred to the survey papers [16, 19] for results pertaining to Cayley graphs as models for interconnection networks. Due to the importance of Cayley graphs in network design and the significance of reliability of networks, it is of interest to understand the generalized k-connectivity and the pendant tree-connectivity of Cayley graphs. The generalized k-connectivity of Cayley graphs on Abelian groups of small degrees was studied by Sun and Zhou [39]. In this paper we will focus on the pendant tree-connectivity of Cayley graphs on Abelian groups of small degrees and obtain the exact values for $\tau_k(G)$, where G is a cubic or 4-regular Cayley graph on Abelian groups (Theorems 17 and 22).

2. Bounds for $\kappa'_k(G) - \tau_k(G)$

For a general graph G, the following result concerns the bounds for $\tau_k(G)$.

Proposition 1 [27]. *Let k, n be two integers with $3 \le k \le n$, and let G be a graph of order n. Then $0 \le \tau_k(G) \le n - k$.*

From [27], we know that $\tau_k(K_n) = n - k$. By the definition of $\tau_k(G)$, we clearly have the following observation.

Observation 2. *$\tau_k(G) = 0$ for $k \geq \max\{\delta(G) + 1, 3\}$.*

The following two observations were introduced in [38].

Observation 3 [38]**.** *If H is a spanning subgraph of G, then $\kappa'_k(H) \leq \kappa'_k(G)$.*

Observation 4 [38]**.** *For a connected graph G of order n, we have $\kappa'_k(G) \leq n - \alpha(G)$, where $\alpha(G)$ is the independence number of G.*

For two integers k, n with $1 \leq k \leq n-1$, we define a class of graphs $G(k)$ as follows: For each graph $G \in G(k)$, there exists a cut vertex x such that $G \setminus \{x\}$ contains at least k components. By definition, for any $k_1 < k_2$, we have that $G(k_2)$ is a subclass of $G(k_1)$.

Proposition 5 [38]**.** *Let k, n be two integers with $2 \leq k \leq n$. For a connected graph G of order n, $1 \leq \kappa'_k(G) \leq n - k + 1$. Moreover, $\kappa'_k(G) = 1$ if and only if $k = n$ or $G \cong G(k)$, and $\kappa'_k(G) = n - k + 1$ if and only if $\alpha(G) \leq k - 1$.*

For example, for $2 \leq k < n$ we know $K_{1,n-1} \in G(k)$, and we clearly have $\kappa'_k(K_{1,n-1}) = 1$ and $\kappa'_k(K_n) = n-k+1$. For a general $k \geq 3$, if $\kappa'_k(G) \neq n-k+1$, then we can get sharp lower and upper bounds for $\kappa'_k(G) - \tau_k(G)$.

Lemma 6. *For a connected graph G of order n, if $\kappa'_k(G) \neq n-k+1$ with $k \geq 3$, then $1 \leq \kappa'_k(G) - \tau_k(G) \leq n - k$. Moreover, the bounds are sharp.*

Proof. Since $\kappa'_k(G) \neq n-k+1$, we have $\kappa'_k(G) \leq n-k$ by Proposition 5, and then $n \geq \kappa'_k(G) + k$. By definition, there exists a set $S \subseteq V(G)$ with $|S| = \kappa'_k(G)$ such that $G \setminus S$ contains ℓ components, say $G_1, G_2, \ldots, G_\ell$, where $\ell \geq k$. We choose $S' = \{u\} \cup \{u_i | \ 1 \leq i \leq k-1\}$, where $u \in S$ and $u_i \in V(G_i)$ for $1 \leq i \leq k-1$. By definitions of the local pendant tree-connectivity $\tau_G(S')$ and the pendant tree-connectivity $\tau_k(G)$, we can deduce that $\tau_k(G) \leq \tau_G(S') \leq |S| - 1 = \kappa'_k(G) - 1$, and then $\kappa'_k(G) - \tau_k(G) \geq 1$. For the sharpness of this bound, we consider the graph $G \in G(k)$. By the definition of $G(k)$, it is not hard to show that $\kappa'_k(G) = 1$ and $\tau_k(G) = 0$, then $\kappa'_k(G) - \tau_k(G) = 1$.

Since $\tau_k(G) \geq 0$ and $\kappa'_k(G) \leq n-k$, we have $\kappa'_k(G) - \tau_k(G) \leq n-k$. For the sharpness of this bound, we consider the following example. Let G be a connected graph with vertex set $V(G) = A \cup B$ such that $A = \{u_i | \ 1 \leq i \leq n-k\}$ is a clique, $B = \{v_j | \ 1 \leq j \leq k\}$ is an independent set, and $u_1v_1, u_iv_j \in E(G)$ where $1 \leq i \leq n-k, 2 \leq j \leq k$. Since G is connected and $\delta(G) = 1$, we have $\tau_k(G) = 0$ by Observation 2. Clearly, $\alpha(G) = k$ and so $\kappa'_k(G) \leq n-k$ by Observation 4. It is also not hard to show that for any set $S \subseteq V(G)$ with $|S| < n-k$, the subgraph $G \setminus S$ contains at most two components, and so we have $\kappa'_k(G) \geq n-k$. Thus, $\kappa'_k(G) = n-k$, and then $\kappa'_k(G) - \tau_k(G) = n-k$. ■

The following result concerns the case that $\kappa'_k(G) = n - k + 1$.

Lemma 7. *For a connected graph G of order n, if $\kappa'_k(G) = n - k + 1$ where $k \geq 3$, then $1 \leq \kappa'_k(G) - \tau_k(G) \leq n - k + 1$. Moreover, the bounds are sharp.*

Proof. The bounds are deduced from Proposition 1 and the assumption that $\kappa'_k(G) = n-k+1$. For the sharpness of the lower bound, just consider the graph K_n, since $\tau_k(K_n) = n-k$ and $\kappa'_k(K_n) = n-k+1$, we have $\kappa'_k(K_n) - \tau_k(K_n) = 1$. For the sharpness of the upper bound, we consider the following example. Let G be a graph with $V(G) = \{u_i|\ 1 \leq i \leq n\}$ such that $u_1u_n \in E(G)$ and $V' = \{u_i|\ 1 \leq i \leq n-1\}$ is a clique. By Observation 2, $\tau_k(G) = 0$ since $\delta(G) = 1$. We know $\alpha(G) = 2 \leq k-1$, so $\kappa'_k(G) = n-k+1$ by Proposition 4, and then we have $\kappa'_k(G) - \tau_k(G) = n-k+1$. ■

By Lemmas 6 and 7, we now give sharp lower and upper bounds for $\kappa'_k(G) - \tau_k(G)$.

Theorem 8. *For a connected graph G of order n, if $\kappa'_k(G) \neq n - k + 1$ where $k \geq 3$, then $1 \leq \kappa'_k(G) - \tau_k(G) \leq n-k$; otherwise, $1 \leq \kappa'_k(G) - \tau_k(G) \leq n-k+1$. Moreover, all of these bounds are sharp.*

3. Cartesian Product Graphs

The *Cartesian product* of two graphs G and H, denoted by $G\Box H$, is defined to have vertex set $V(G) \times V(H)$ such that (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. Note that this product is commutative, that is, $G\Box H = H\Box G$. Furthermore, the Cartesian product of two graphs is connected if and only if these two graphs are both connected [17]. Let G and H be two graphs with $V(G) = \{u_i|\ 1 \leq i \leq n\}$ and $V(H) = \{v_j|\ 1 \leq j \leq m\}$, then $V(G\Box H) = \{(u_i, v_j)|\ 1 \leq i \leq n, 1 \leq j \leq m\}$. For $1 \leq j \leq m$, we use $G(v_j)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v_j)|\ 1 \leq i \leq n\}$; for $1 \leq i \leq n$, we use $H(u_i)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v_j)|\ 1 \leq j \leq m\}$. Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. For example, see the graphs in Figure 1, $G(v_j) \cong G \cong P_3$ for $1 \leq j \leq 4$ and $H(u_i) \cong H \cong K_4$ for $1 \leq i \leq 3$.

The following result will be used in our proof of Theorem 10.

Proposition 9. *For $n, m \geq 5$, we have $\kappa'_3(K_n\Box K_m) = 2(n + m - 3)$.*

Proof. Let $G \cong K_n$, $H \cong K_m$, $V(G) = \{u_i|\ 1 \leq i \leq n\}$, $V(H) = \{v_j|\ 1 \leq j \leq m\}$ and $S' = \{(u_1, v_j)|\ 2 \leq j \leq m\} \cup \{(u_2, v_j)|\ 3 \leq j \leq m\} \cup \{(u_i, v_1)|\ 2 \leq i \leq n\} \cup \{(u_i, v_2)|\ 3 \leq i \leq n\}$. Clearly, $W^* = (G\Box H) \setminus S'$ contains exactly three components. Among these components, there are two trivial components and one nontrivial component with $(n-2)(m-2)$ vertices. Note that $|S'| = 2(n+m-3)$ and $|V(W^*)| = 2 + (n-2)(m-2)$. Hence, $\kappa'_3(K_n \square K_m) \le 2(n+m-3)$. In the following, we will show that $\kappa'_3(K_n \square K_m) \ge 2(n+m-3)$.

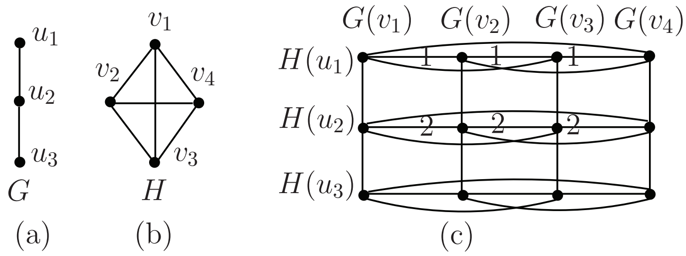

Figure 1. Graphs G, H and their Cartesian product.

Let S be any subset of $G \square H$ such that $(G \square H) \setminus S$ contains ℓ components, say $D_1, D_2, \ldots, D_\ell$, where $\ell \ge 3$. Since $G \cong K_n$, $H \cong K_m$, we know that for any $1 \le \ell_1 \ne \ell_2 \le \ell$, D_{ℓ_1} and D_{ℓ_2} must satisfy the following property: if $(u_{i_1}, v_{j_1}) \in D_{\ell_1}, (u_{i_2}, v_{j_2}) \in D_{\ell_2}$, then $i_1 \ne i_2$ and $j_1 \ne j_2$.

By relabeling the subscripts of vertices of G, H and $G \square H$, we can make sure that these components $D_1, D_2, \ldots, D_\ell$ satisfy the following two properties:

(i) $|V(D_1)| \le |V(D_2)| \le \cdots \le |V(D_\ell)|$;

(ii) for any two components D_{ℓ_1}, D_{ℓ_2}, where $1 \le \ell_1, \ell_2 \le \ell$, and for any two vertices $(u_{i_1}, v_{j_1}) \in D_{\ell_1}, (u_{i_2}, v_{j_2}) \in D_{\ell_2}$, if $\ell_1 < \ell_2$, then $i_1 < i_2$ and $j_1 < j_2$. The graph $(G \square H) \setminus S$ must belong to one of the following three cases:

(a) $|V(D_1)| \ge 2$;

(b) $|V(D_1)| = 1$ and $|V(D_2)| \ge 2$;

(c) $|V(D_1)| = |V(D_2)| = 1$.

Note that the graph W^* belongs to case (c). For each case, it is not hard to show that $|V((G \square H) \setminus S)| \le |V(W^*)| = 2+(n-2)(m-2)$ and so $|S| \ge 2(m+n-3)$.

For example, see Figure 2, we set $G \cong K_5$, $H \cong K_6$. Now $V(W^*) = \{(u_1, v_1), (u_2, v_2)\} \cup \{(u_i, v_j) |\ 3 \le i \le 5, 3 \le j \le 6\}$. Let $D_1 = \{(u_1, v_1), (u_1, v_2)\}$, $D_2 = \{(u_2, v_3), (u_2, v_4), (u_3, v_3)\}$, $D_3 = \{(u_4, v_5), (u_4, v_6), (u_5, v_5), (u_5, v_6)\}$ and $S = V(G \square H) \setminus (V(D_1) \cup V(D_2) \cup V(D_3))$. Clearly, D_1, D_2, D_3 satisfy the above two properties and $|S| = 21 > 16 = 2(m+n-3)$.

By the above argument and the definition of 3-connectivity, we can deduce that $\kappa'_3(K_n \square K_m) \ge 2(n+m-3)$. Hence, our result holds. ■

For $u \in V(G)$, we use $N_G(u)$ to denote the set of neighbors of u. For a non-complete graph, let $d_2(G) = \min\{d(u) + d(v) - |N_G(u) \cap N_G(v)| |\ u, v \in V(G),$

$uv \notin E(G)\}$. We now give one of our main results in terms of $d_2(G)$ and the minimum degree. Note that we have $1 \le \kappa_3'(H) \le n-2$ for any connected graph H of order n by Proposition 5.

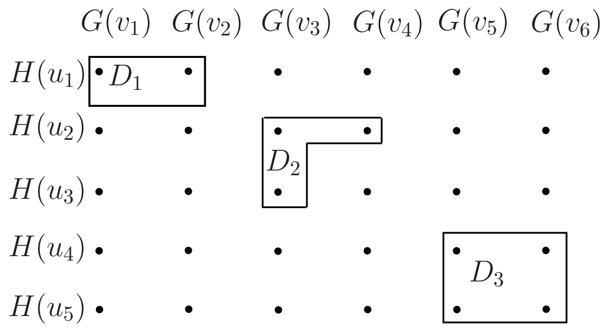

Figure 2. An example for Proposition 9.

Theorem 10. *Let G and H be any two connected graphs with orders n and m, respectively, where $n, m \ge 5$.*

(i) *If G is non-complete, then $\kappa_3'(G\Box H) \le d_2(G) + 2\delta(H)$.*

(ii) *Let G be complete. If $1 \le \kappa_3'(H) \le n-3$, then $\kappa_3'(G\Box H) \le n\kappa_3'(H)$.*

Otherwise, we have $\kappa_3'(G\Box H) \le 2(n+m-3)$. Moreover, all of these bounds are sharp.

Proof. Let $V(G) = \{u_i|\ 1 \le i \le n\}$ and $V(H) = \{v_j|\ 1 \le j \le m\}$. Then $V(G\Box H) = \{(u_i, v_j)|\ 1 \le i \le n, 1 \le j \le m\}$. For $1 \le i \le n$ and $1 \le j \le m$, we use the definitions of $H(u_i)$ and $G(v_j)$ at the beginning of this section. Since G and H are connected, $G\Box H$ is connected.

Firstly, we prove the assertion (i). Without loss of generality, we assume that $d_H(v_1) = \delta(H)$, $u_1u_2 \notin E(G)$ and $d_2(G) = d(u_1) + d(u_2) - |N_G(u_1) \cap N_G(u_2)|$.

Let S_1 be the set of neighbors of (u_1, v_1) and (u_2, v_1) in $G(v_1)$, S_2 be the set of neighbors of (u_1, v_1) in $H(u_1)$, S_3 be the set of neighbors of (u_2, v_1) in $H(u_2)$. Clearly, we have $|S_1| = d_2(G)$ and $|S_2| = |S_3| = \delta(H)$. Let $S = S_1 \cup S_2 \cup S_3$, it is not hard to show that the removal of S in $G\Box H$ produces a graph with at least three components, and among them there are at least two trivial components which contain (u_1, v_1) and (u_2, v_1), respectively. Thus, $\kappa_3'(G\Box H) \le |S| = d_2(G) + 2\delta(H)$.

For the sharpness of this bound, we just let $G \cong P_n, H \cong P_m$, where $n, m \ge 5$. Clearly, $d_2(G) = 2, \delta(H) = 1$, so $\kappa_3'(G\Box H) \le d_2(G) + 2\delta(H) = 4$. Furthermore, it is not hard to show that for any set $S \subseteq V(P_n\Box P_m)$ with $|S| \le 3$, the removal of S in $P_n\Box P_m$ produces a graph with at most two components, so $\kappa_3'(G\Box H) \ge 4$. Thus, $\kappa_3'(G\Box H) = 4$ and the bound is sharp.

Secondly, we prove the assertion (ii). If $\kappa_3'(H) = n-2$, then by Observation 3 and Proposition 9, we have $\kappa_3'(G\Box H) \leq \kappa_3'(K_n\Box K_m) = 2(n+m-3)$ and this bound is sharp.

If $1 \leq \kappa_3'(H) \leq n-3$, then there exists a set $S \subseteq V(H)$ such that $H \setminus S$ contains at least three components. Without loss of generality, we assume that $S = \{v_j |\ 1 \leq j \leq \kappa_3'(H)\}$. Let $S' = \{(u_i, v_j)|\ 1 \leq i \leq n, 1 \leq j \leq \kappa_3'(H)\} \subseteq V(G\Box H)$. Clearly, $(G\Box H)\backslash S'$ contains at least three components, so $\kappa_3'(G\Box H) \leq n\kappa_3'(H)$.

For the sharpness of this bound, we just let $G \cong K_n, H \cong K_{1,m-1}$, where n, $m \geq 5$. Clearly, $\kappa_3'(H) = 1$, so $\kappa_3'(G\Box H) \leq n$. For any set $S \subseteq V(G\Box H)$ with $|S| \leq n-1$, we find that there exists an integer $1 \leq i_0 \leq n$ such that $V(H(u_{i_0})) \cap S = \emptyset$, and then the removal of S in $K_n\Box P_m$ produces a connected graph. Thus, $\kappa_3'(G\Box H) \geq n$ and we have $\kappa_3'(G\Box H) = n$. ■

By Theorem 10, the following result clearly holds.

Corollary 11. *For any two connected non-complete graphs G and H with orders at least* 5, *we have*

$$\kappa_3'(G\Box H) \leq \min\{d_2(G) + 2\delta(H), d_2(H) + 2\delta(G)\}.$$

Moreover, the bound is sharp.

Proposition 12. *For $n, m \geq 5$, we have $\kappa_3'(C_n\Box P_m) = 5$.*

Proof. We know that $d_2(C_n) = 3$ and $\delta(P_m) = 1$, then $\kappa_3'(C_n\Box P_m) \leq 5$ by Theorem 10. It is not hard to show that for any set $S \subseteq V(C_n\Box P_m)$ with $|S| \leq 4$, the removal of S in $C_n\Box P_m$ produces a graph with at most two components, so $\kappa_3'(C_n\Box P_m) \geq 5$. Thus, $\kappa_3'(C_n\Box P_m) = 5$. ■

By considering the first example in the proof of Theorem 10, we know that $\kappa_3'(P_n\Box P_m) = 4$ for $n, m \geq 5$. In fact, we can get the following more general result.

Proposition 13. *For two connected graphs G and H with orders at least* 5, *if $d_2(G) = 2$ and $\delta(H) = 1$, then $\kappa_3'(G\Box H) = 4$.*

Proof. By Theorem 10, we clearly have $\kappa_3'(G\Box H) \leq d_2(G) + 2\delta(H) = 4$. It is not hard to show that for any set $S \subseteq V(G\Box H)$ with $|S| \leq 3$, the removal of S in $G\Box H$ produces a graph with at most two components, so $\kappa_3'(G\Box H) \geq 4$. Hence, $\kappa_3'(G\Box H) = 4$. ■

The second example in the proof of Theorem 10 shows that $\kappa_3'(K_n\Box K_{1,m-1}) = n$ for $n, m \geq 5$. The following result implies that $K_{1,m-1}$ can be replaced by any graph H with $\kappa_3'(H) = 1$.

Proposition 14. *Let H be a connected graph with $\kappa_3'(H) = 1$, then $\kappa_3'(K_n \Box H) = n$.*

Proof. By Theorem 10, we clearly have $\kappa_3'(K_n \Box H) \leq n$. For any set $S \subseteq V(K_n \Box H)$ with $|S| \leq n-1$, there must exist an integer $1 \leq i_0 \leq n$ such that $V(H(u_{i_0})) \cap S = \emptyset$, then the removal of S in $K_n \Box H$ produces a connected graph. Thus, $\kappa_3'(K_n \Box H) \geq n$ and so $\kappa_3'(K_n \Box H) = n$. ■

Now we determine the precise value for the 3-connectivity of the Cartesian product of a complete graph and a tree.

Proposition 15. *Let T be a tree with order m. For $n, m \geq 5$, we have*

$$\kappa_3'(K_n \Box T) = \begin{cases} n+1, & \text{if } T \text{ is a path,} \\ n, & \text{otherwise.} \end{cases}$$

Proof. If T is not path, then T contains a vertex with degree at least three, so we clearly have $\kappa_3'(T) = 1$. By Proposition 14, we have $\kappa_3'(K_n \Box T) = n$.

Now suppose T is a path $v_1, v_2, \ldots, v_m$. Let $V(K_n) = \{u_i \mid 1 \leq i \leq n\}$ and $S' = \{(u_1, v_1), (u_1, v_3), (u_i, v_2) \mid 2 \leq i \leq n\} \subseteq V(K_n \Box T)$. Clearly, $|S'| = n+1$ and there are three components in $(K_n \Box T) \setminus S'$, so $\kappa_3'(K_n \Box T) \leq n+1$. Let $S \subseteq V(K_n \Box T)$ with $|S| = n$. If $S \cap H(u_{i_0}) = \emptyset$ for some $1 \leq i_0 \leq n$, then $(K_n \Box T) \setminus S$ is connected. Otherwise, we have $|S \cap H(u_i)| = 1$ for each $1 \leq i \leq n$, then $(K_n \Box T) \setminus S$ contains at most two components. Hence, $\kappa_3'(K_n \Box T) \geq n+1$, and so $\kappa_3'(K_n \Box T) = n+1$ in this case. ■

4. Cayley Graphs on Abelian Groups with Small Degrees

By the definition of $\tau_k(G)$, we clearly have the following observation.

Observation 16. *$\tau_k(G) \leq 1$ for $k = \delta(G)$ and $\tau_k(G) \leq 2$ for $k = \delta(G) - 1$.*

The *d-dimensional cube Q_d* is the Cartesian product of d copies of the path P_2 of two vertices. The Cartesian product $P(2h) = C_h \Box P_2$ of a cycle C_h of length $h \geq 3$ and P_2 is called a *prism*. As shown in Figure 3,

$$V(P(2h)) = \{u_i, v_j \mid 1 \leq i, j \leq h\},$$

$$E(P(2h)) = \{u_i v_i \mid 1 \leq i \leq h\} \cup \{u_i u_{i+1} \mid 1 \leq i \leq h\} \cup \{v_j v_{j+1} \mid 1 \leq j \leq h\},$$

with subscripts modulo h. The *Möbius ladder* $M(2h)$ of order $2h$ is the graph obtained from $P(2h)$ by deleting the edges $u_1 u_h$ and $v_1 v_h$ and adding the edges $u_1 v_h$ and $u_h v_1$.

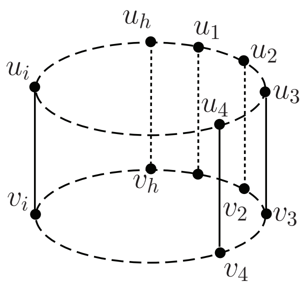

Figure 3. A prism with order $2h$.

Theorem 17. *Let G be a cubic connected Cayley graph on an Abelian group of order n. Then*

$$\tau_k(G) = \begin{cases} 1, & k = 3, \\ 0, & 4 \leq k \leq n. \end{cases}$$

Proof. Let G be a cubic connected Cayley graph on an Abelian group of order n. We clearly have $\tau_k(G) = 0$ for $4 \leq k \leq n$ by Observation 2. We now consider the case that $k = 3$. By Observation 16, we have $\tau_3(G) \leq 1$. It was shown in [31] (and was also restated in [34]) that G is isomorphic to K_4, Q_3, $P(2h)$ or $M(2h)$, then it is not hard to check that for each case, there exists one pendant tree connecting S, where $S \subseteq V(G)$, and so $\tau_3(G) \geq 1$. Thus, $\tau_3(G) = 1$. ■

Consider a Cayley graph $Cay(X, A)$ of degree 4, where $A = \{a, b\}$. As in [3], we call an edge joining x and $x + a$ (respectively, x and $x + b$) an *a-edge* (respectively, *b-edge*). The subgraph of $Cay(X, A)$ induced by the a-edges (respectively, b-edges) is a disjoint union of cycles called *a-cycles* (respectively, *b-cycles*), each with length k_a (respectively, k_b) the order of a (respectively, the order of b). Let α be the number of a-cycles and β the number of b-cycles in $Cay(X, A)$. Then $\alpha k_a = \beta k_b = n$. The authors of [3] introduced a class of simple graphs, denoted by $G(\alpha, \beta)$, with the following properties:

(i) there exist integers α, t, c with $\alpha \geq 1$, $t \geq 3$, $0 \leq c < t$, and $\beta = \gcd(t, c)$;

(ii) the αt vertices of the graph can be labelled (i, j), $1 \leq i \leq \alpha$, $1 \leq j \leq t$, with i taken modulo a and j taken modulo t;

(iii) the edges can be partitioned into two types, such that type 1 edges are of the form $\{(i, j), (i, j+1)\}$, and type 2 edges are of the form $\{(i, j), (i+1, j)\}$, $1 \leq i \leq \alpha - 1$, or $\{(\alpha, j), (1, j + c)\}$.

Thus a graph in this class is constructed by α vertical disjoint cycles C^i $(1 \leq i \leq \alpha)$, $\alpha - 1$ horizontal parallel matchings between the cycles C^i and C^{i+1} $(1 \leq i \leq \alpha)$, and a particular parallel matching between C^1 and C^α, as shown in (a) of Figure 4 when $\alpha \geq 2$ and in (b) of Figure 4 when $\alpha = 1$. Note that in graph (a), we let $(i, j) = V(C^i) \cap V(P^j)$, where $1 \leq i \leq \alpha, 1 \leq j \leq t$.

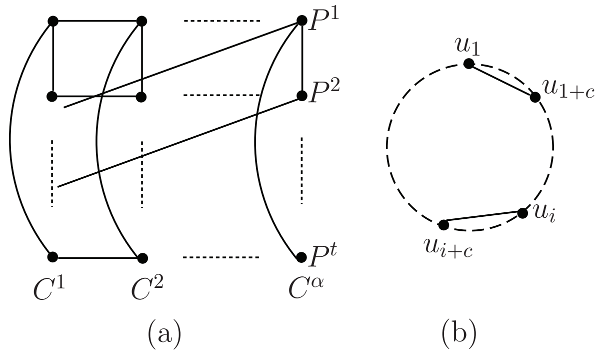

Figure 4. The graphs in $G(\alpha, \beta)$.

Lemma 18 [3]**.** *The class $G(\alpha, \beta)$ consists of the connected Cayley graphs of degree 4 on a finite Abelian group with a generating set $\{a, b\}$, where α is the number of a-cycles and β is the number of b-cycles.*

The following theorem is useful in our argument.

Theorem 19 [3,10]**.** *Every connected Cayley graph of degree 4 on a finite Abelian group can be decomposed into two Hamiltonian cycles.*

Let $C_n = a_1, a_2, \ldots, a_n, a_1$ and $C_m = b_1, b_2, \ldots, b_m, b_1$ be two cycles, and let r be an integer with $0 \leq r \leq m - 1$. The *r-pseudo-cartesian product* [10] of C_n and C_m, denoted by $C_n \Box_r C_m$, is the graph obtained from $C_n \Box C_m$ by replacing the edge set $\{(a_1, b_i)(a_n, b_i) \,|\, 1 \leq i \leq m\}$ by $\{(a_1, b_{i+r})(a_n, b_i) \,|\, 1 \leq i \leq m\}$ with subscripts of b's modulo m. By definition, we have $C_n \Box_r C_m = C_n \Box C_m$ if $r = 0$ or m. Note that by the definition of the graph class $G(\alpha, \beta)$ as shown in Section 4, each graph $G(\alpha, \beta)$ is isomorphic to $C_\alpha \Box_c C_t$.

Consider a Cayley graph $Cay(X, A)$ of degree 4, it is not hard to show that A is one of the following types [10]:

(A) $A = \{a, b\}$ with each element having order at least 3;
(B) $A = \{a, b, c\}$ with two elements of order two;
(C) $A = \{a, b, c, d\}$ with all elements of order 2.

By Lemma 18, we know that each Cayley graph of type (A) is an element of $G(\alpha, \beta)$. Thus we consider graphs of the class $G(\alpha, \beta)$ instead in the sequel.

For type (B), without loss of generality, assume that $o(b) = o(c) = 2$, where $o(b)$ and $o(c)$ are orders of b and c in the group X, respectively. Let $J = \langle \{b, c\} \rangle = \{0, b, c, b + c\}$ and $X_1 = X/J = \{0, \overline{b}, \ldots, (h-1)\overline{b}\}$. Let $C' = 0,\, b, \ldots, (h-1)b, 0$

and $C'' = 0, b, b+c, c, 0$. Clearly, C' and C'' are two cycles of $Cay(X, A)$. It was shown in [10] that either $Cay(X, A) \cong C' \Box C''$ or $Cay(X, A) \cong C' \Box_2 C''$ or $Cay(X, A)$ is the graph shown in Figure 5. It was also shown in [10] that each graph of type (C) is isomorphic to $C_4 \Box C_4$.

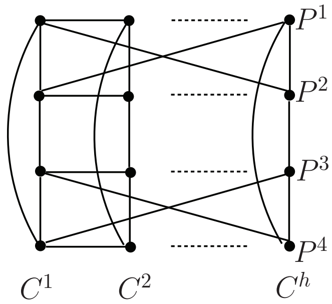

Figure 5. A graph of type (B).

Lemma 20. *Let $G \in G(\alpha, \beta)$. If one of the following cases holds:*

(i) $\alpha \geq 2$,

(ii) $\alpha = 1$ *and* $c \geq 3$,

(iii) $\alpha = 1$, $c = 2$ *and* n *is odd,*

then we have

$$\tau_k(G) = \begin{cases} 2, & k = 3, \\ 1, & k = 4, \\ 0, & 5 \leq k \leq n. \end{cases}$$

Otherwise, that is, $\alpha = 1$, $c = 2$ *and* n *is even, we have*

$$\tau_k(G) = \begin{cases} 1, & k = 3, 4, \\ 0, & 5 \leq k \leq n. \end{cases}$$

Proof. Let $G \in G(\alpha, \beta)$. Since G is 4-regular, we clearly have $\tau_k(G) = 0$ for $5 \leq k \leq n$ by Observation 2. Furthermore, by Observation 16, we have $\tau_4(G) \leq 1$. It is not hard to check that for each case, there exists one pendant tree connecting S, where $S \subseteq V(G)$ and $|S| = 4$, and so $\tau_4(G) \geq 1$. Thus, $\tau_4(G) = 1$.

Now we focus on the case that $k = 3$. By Observation 16, we have $\tau_3(G) \leq 2$. If one of the following cases holds

(i) $\alpha \geq 2$,

(ii) $\alpha = 1$ and $c \geq 3$,

(iii) $\alpha = 1$, $c = 2$ and n is odd,

then it is not hard to check that there exist two internally disjoint pendant S-trees, where $S \subseteq V(G)$ and $|S| = 3$, so $\tau_3(G) \geq 2$. Hence, $\tau_3(G) = 2$. For example, as shown in Figure 6, here we have $\alpha = 1$, $c = 2$ and $n = 7$, $S = \{u_1, u_2, u_3\}$, there are two internally disjoint pendant S-trees, say T_1 and T_2, in G.

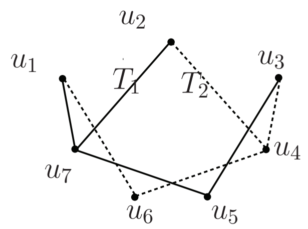

Figure 6. An example for the case that $\alpha = 1$, $c = 2$ and n is odd.

For the case that $\alpha = 1$, $c = 2$ and n is even, we choose $S = \{u_1, u_2, u_3\}$, it is not hard to show that there is exactly one pendant tree connecting S, so $\tau_G(S) = 1$ and then $\tau_3(G) = 1$. ■

With a similar argument to that of Lemma 20, we can obtain the following result.

Lemma 21. *Let G be the graph of Figure 5 or be isomorphic to $C_4 \square C_4$. We have*

$$\tau_k(G) = \begin{cases} 2, & k = 3, \\ 1, & k = 4, \\ 0, & 5 \le k \le n. \end{cases}$$

According to the above arguments, and by Lemmas 20 and 21, we have the following result.

Theorem 22. *Let G be a connected Cayley graph of degree 4 on an Abelian group of order $n \ge 3$. If one of the following cases holds:*

(i) $\alpha \ge 2$,
(ii) $\alpha = 1$ *and* $c \ge 3$,
(iii) $\alpha = 1$, $c = 2$ *and n is odd,*

then we have

$$\tau_k(G) = \begin{cases} 2, & k = 3, \\ 1, & k = 4, \\ 0, & 5 \le k \le n. \end{cases}$$

Otherwise, that is, $\alpha = 1$, $c = 2$ and n is even, we have

$$\tau_k(G) = \begin{cases} 1, & k = 3, 4, \\ 0, & 5 \le k \le n. \end{cases}$$

Note that in the proof of Theorems 17 and 22, we used structural properties of Cayley graphs on Abelian group with degree 3 or 4. Similar properties are not known for general Cayley graphs on Abelian groups with degree greater than 4, and as such we may need to find other approaches in studying their pendant tree-connectivities.

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References

[1] S.B. Akers and B. Krishnamurthy, *A group-theoretic model for symmetric interconnection networks*, IEEE Trans. Comput. **38** (1989) 555–566. doi:10.1109/12.21148

[2] L. Babai, *Automorphism groups, isomorphism, reconstruction*, in: Handbook of Combinatorics, R.L. Graham *et al.* (Ed(s)), (Elsevier, Amsterdam, 1995) 1447–1540.

[3] J.-C. Bermond, O. Favaron and M. Maheo, *Hamiltonian decomposition of Cayley graphs of degree* 4, J. Combin. Theory Ser. B **46** (1989) 142–153. doi:10.1016/0095-8956(89)90040-3

[4] N. Biggs, Algebraic Graph Theory (Cambridge University Press, New York, 1992).

[5] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244 (Springer, Berlin, 2008).

[6] G. Chartrand, S.F. Kappor, L. Lesniak and D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. **2** (1984) 1–6.

[7] G. Chartrand, F. Okamoto and P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks **55** (2010) 360–367. doi:10.1002/net.20339

[8] D.P. Day, O.R. Oellermann and H.C. Swart, *The ℓ-connectivity function of trees and complete multipartite graphs*, J. Combin. Math. Combin. Comput. **10** (1991) 183–192.

[9] D. Du and X. Hu, Steiner Tree Problems in Computer Communication Networks (World Scientific, 2008). doi:10.1142/6729

[10] C. Fan, D.R. Lick and J. Liu, *Pseudo-cartesian product and hamiltonian decompositions of Cayley graphs on abelian groups*, Discrete Math. **158** (1996) 49–62. doi:10.1016/0012-365X(95)00035-U

[11] R. Gu, X. Li and Y. Shi, *The generalized* 3*-connectivity of random graphs*, Acta Math. Sinica (Chin. Ser.) **57** (2014) 321–330, in Chinese.

[12] M. Grötschel, A. Martin and R. Weismantel, *The Steiner tree packing problem in VLSI design*, Math. Program. **78** (1997) 265–281. doi:10.1007/BF02614374

[13] M. Grötschel, A. Martin and R. Weismantel, *Packing Steiner trees: a cutting plane algorithm and commputational results*, Math. Program. **72** (1996) 125–145. doi:10.1007/BF02592086

[14] M. Hager, *Pendant tree-connectivity*, J. Combin. Theory Ser. B **38** (1985) 179–189. doi:10.1016/0095-8956(85)90083-8

[15] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, Second Edition (CRC Press, Boca Raton, 2011).

[16] M.C. Heydemann, *Cayley graphs and interconnection networks*, in: Graph Symmetry, G. Hahn and G. Sabidussi (Ed(s)), (Kluwer Academic Publishers, Dordrecht, 1997) 167–224. doi:10.1007/978-94-015-8937-6_5

[17] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition (Wiley, New York, 2000).

[18] W. Imrich, S. Klavžar and D.F. Rall, Topics in Graph Theory: Graphs and Their Cartesian Product (A K Peters, Ltd., Wellesley, 2008).

[19] S. Lakshmivarahan, J.-S. Jwo and S.K. Dhall, *Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey*, Parallel Comput. **19** (1993) 361–407. doi:10.1016/0167-8191(93)90054-O

[20] H. Li, X. Li, Y. Mao and Y. Sun, *Note on the generalized connectivity*, Ars Combin. **114** (2014) 193–202.

[21] H. Li, X. Li and Y. Sun, *The generalized* 3*-connectivity of Cartesian product graphs*, Discrete Math. Theor. Comput. Sci. **14** (2012) 43–54.

[22] X. Li and Y. Mao, *A survey on the generalized connectivity of graphs.* arXiv:1207.1838[math.CO]

[23] X. Li and Y. Mao, *On extremal graphs with at most ℓ internally disjoint Steiner trees connecting any $n-1$ vertices*, Graphs Combin. **31** (2015) 2231–2259. doi:10.1007/s00373-014-1500-7

[24] X. Li and Y. Mao, *The generalized* 3*-connectivity of lexicographic product graphs*, Discrete Math. Theor. Comput. Sci. **16** (2014) 339–354.

[25] X. Li and Y. Mao, Generalized Connectivity of Graphs (SpringerBriefs in Mathematics, Springer, Switzerland, 2016).

[26] X. Li, Y. Mao and Y. Sun, *On the generalized* (*edge-*)*connectivity of graphs*, Australas. J. Combin. **58** (2014) 304–319.

[27] Y. Mao, *On the pedant tree-connectivity of graphs.* arXiv:1508.07149v1 [math.CO]

[28] O.R. Oellermann, *On the ℓ-connectivity of a graph*, Graphs Combin. **3** (1987) 285–291. doi:10.1007/BF01788551

[29] O.R. Oellermann, *A note on the ℓ-connectivity function of a graph*, Congr. Numer. **60** (1987) 181–188.

[30] N.A. Sherwani, Algorithms for VLSI Physical Design Automation, 3rd Edition (Kluwer Academic Publishers, London, 1999).

[31] W. Shiu, On 3-Regular and 4-Regular Cayley Graphs of Abelian Groups (Technical Report, Dept. of Mathematics, Hong Kong Baptist University, 1995).

[32] Y. Sun, *Generalized* 3*-edge-connectivity of Cartesian product graphs*, Czechoslovak Math. J. **65** (2015) 107–117.
doi:10.1007/s10587-015-0162-9

[33] Y. Sun, *Generalized* 3*-connectivity and* 3*-edge-connectivity for the Cartesian products of some graph classes*, J. Combin. Math. Combin. Comput. **94** (2015) 215–225.

[34] Y. Sun, *Maximum generalized local connectivities of cubic Cayley graphs on Abelian groups*, J. Combin. Math. Combin. Comput. **94** (2015) 227–236.

[35] Y. Sun, *Sharp upper bounds for generalized edge-connectivity of product graphs*, Discuss. Math. Graph Theory **36** (2016) 833–843.
doi:10.7151/dmgt.1924

[36] Y. Sun, *On the maximum and minimum sizes of a graph with given k-connectivity*, Discuss. Math. Graph Theory **37** (2017), in press.
doi:10.7151/dmgt.1941

[37] Y. Sun, *A sharp lower bound for the generalized* 3*-edge-connectivity of strong product graphs*, Discuss. Math. Graph Theory (2017), in press.
doi:10.7151/dmgt.1982

[38] Y. Sun and X. Li, *On the difference of two generalized connectivities of a graph*, J. Comb. Optim. **33** (2017) 283–291.
doi:10.1007/s10878-015-9956-9

[39] Y. Sun and S. Zhou, *Tree connectivities of Cayley graphs on Abelian groups with small degrees*, Bull. Malays. Math. Sci. Soc. **39** (2016) 1673–1685.
doi:10.1007/s40840-015-0147-8

[40] S. Zhou, *A class of arc-transitive Cayley graphs as models for interconnection networks*, SIAM J. Discrete Math. **23** (2009) 694–714.
doi:10.1137/06067434X

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