MATCHINGS EXTEND TO HAMILTONIAN CYCLES IN 5-CUBE\(^1\)

**Fan Wang**\(^2\)

*School of Sciences*
*Nanchang University*
*Nanchang, Jiangxi 330000, P.R. China*
*e-mail: wangfan620@163.com*

**Weisheng Zhao**

*Institute for Interdisciplinary Research*
*Jianghan University*
*Wuhan, Hubei 430056, P.R. China*
*e-mail: weishengzhao101@aliyun.com*

**Abstract**

Ruskey and Savage asked the following question: Does every matching in a hypercube \(Q_n\) for \(n \geq 2\) extend to a Hamiltonian cycle of \(Q_n\)? Fink confirmed that every perfect matching can be extended to a Hamiltonian cycle of \(Q_n\), thus solved Kreweras’ conjecture. Also, Fink pointed out that every matching can be extended to a Hamiltonian cycle of \(Q_n\) for \(n \in \{2, 3, 4\}\). In this paper, we prove that every matching in \(Q_5\) can be extended to a Hamiltonian cycle of \(Q_5\).

**Keywords:** hypercube, Hamiltonian cycle, matching.

**2010 Mathematics Subject Classification:** 05C38, 05C45.

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\(^1\)This work is supported by NSFC (grant nos. 11501282 and 11261019) and the science and technology project of Jiangxi Provincial Department of Education (grant No. 20161BAB201030).

\(^2\)Corresponding author.
1. Introduction

Let \([n]\) denote the set \(\{1, \ldots, n\}\). The \(n\)-dimensional hypercube \(Q_n\) is a graph whose vertex set consists of all binary strings of length \(n\), i.e., \(V(Q_n) = \{u : u = u^1 \cdots u^n \text{ and } u^i \in \{0, 1\} \text{ for every } i \in [n]\}\), with two vertices being adjacent whenever the corresponding strings differ in just one position.

The hypercube \(Q_n\) is one of the most popular and efficient interconnection networks. It is well known that \(Q_n\) is Hamiltonian for every \(n \geq 2\). This statement dates back to 1872 [9]. Since then, the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention [2, 3, 4, 6, 12].

A set of edges in a graph \(G\) is called a matching if no two edges have an endpoint in common. A matching is perfect if it covers all vertices of \(G\). A cycle in a graph \(G\) is a Hamiltonian cycle if every vertex in \(G\) appears exactly once in the cycle.

Ruskey and Savage [11] asked the following question: Does every matching in \(Q_n\) for \(n \geq 2\) extend to a Hamiltonian cycle of \(Q_n\)? Kreweras [10] conjectured that every perfect matching of \(Q_n\) for \(n \geq 2\) can be extended to a Hamiltonian cycle of \(Q_n\). Fink [5, 7] confirmed the conjecture to be true. Let \(K(Q_n)\) be the complete graph on the vertices of the hypercube \(Q_n\).

**Theorem 1.1** [5, 7]. For every perfect matching \(M\) of \(K(Q_n)\), there exists a perfect matching \(F\) of \(Q_n\), \(n \geq 2\), such that \(M \cup F\) forms a Hamiltonian cycle of \(K(Q_n)\).

Also, Fink [5] pointed out that the following conclusion holds.

**Lemma 1.2** [5]. Every matching in \(Q_n\) can be extended to a Hamiltonian cycle of \(Q_n\) for \(n \in \{2, 3, 4\}\).

Gregor [8] strengthened Fink’s result and obtained that given a partition of the hypercube into subcubes of nonzero dimensions, every perfect matching of the hypercube can be extended on these subcubes to a Hamiltonian cycle if and only if the perfect matching interconnects these subcubes.

The present authors [14] proved that every matching of at most \(3n - 10\) edges in \(Q_n\) can be extended to a Hamiltonian cycle of \(Q_n\) for \(n \geq 4\).

In this paper, we consider Ruskey and Savage’s question and obtain the following result.

**Theorem 1.3.** Every matching in \(Q_5\) can be extended to a Hamiltonian cycle of \(Q_5\).

It is worth mentioning that Ruskey and Savage’s question has been recently done for \(n = 5\) independently by a computer search [15]. In spite of this, a direct proof is still necessary, as it may serve in a possible solution of the general question.
2. Preliminaries and Lemmas

Terminology and notation used in this paper but undefined below can be found in [1]. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. For a set $F \subseteq E(G)$, let $G - F$ denote the resulting graph after removing all edges in $F$ from $G$. Let $H$ and $H'$ be two subgraphs of $G$. We use $H + H'$ to denote the graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. For $F \subseteq E(G)$, we use $H + F$ to denote the graph with the vertex set $V(H) \cup V(F)$ and edge set $E(H) \cup F$, where $V(F)$ denotes the set of vertices incident with $F$.

The distance between two vertices $u$ and $v$ is the number of edges in a shortest path joining $u$ and $v$ in $G$, denoted by $d_G(u, v)$, with the subscripts being omitted when the context is clear.

Let $j \in [n]$. An edge in $Q_n$ is called an $j$-edge if its endpoints differ in the $j$th position. The set of all $j$-edges in $Q_n$ is denoted by $E_j$. Thus, $E(Q_n) = \bigcup_{i=1}^{n} E_i$. Let $Q_{n-1, j}^0$ and $Q_{n-1, j}^1$, with the superscripts $j$ being omitted when the context is clear, be the $(n - 1)$-dimensional subcubes of $Q_n$ induced by the vertex sets \{\(u \in V(Q_n) : w^j = 0\)\} and \{\(u \in V(Q_n) : w^j = 1\)\}, respectively. Thus, $Q_n - E_j = Q_{n-1}^0 + Q_{n-1}^1$. We say that $Q_n$ splits into two $(n - 1)$-dimensional subcubes $Q_{n-1}^0$ and $Q_{n-1}^1$ at position $j$; see Figure 1 for example.

The parity $p(u)$ of a vertex $u$ in $Q_n$ is defined by $p(u) = \sum_{i=1}^{n} u^i \pmod{2}$. Then there are $2^{n-1}$ vertices with parity 0 and $2^{n-1}$ vertices with parity 1 in $Q_n$. Vertices with parity 0 and 1 are called black vertices and white vertices, respectively. Observe that $Q_n$ is bipartite and vertices of each parity form bipartite sets of $Q_n$. Thus, $p(u) \neq p(v)$ if and only if $d(u, v)$ is odd.

A $u, v$-path is a path with endpoints $u$ and $v$, denoted by $P_{uv}$ when we specify a particular such path. We say that a spanning subgraph of $G$ whose components are $k$ disjoint paths is a spanning $k$-path of $G$. A spanning 1-path thus is simply a spanning or Hamiltonian path. We say that a path $P$ (respectively, a cycle $C$) passes through a set $M$ of edges if $M \subseteq E(P)$ (respectively, $M \subseteq E(C)$).

![Figure 1. $Q_4$ splits into two 3-dimensional subcubes $Q_3^0$ and $Q_3^1$ at position 4.](image-url)
Lemma 2.1 [13]. Let \( u, v, x, y \) be pairwise distinct vertices in \( Q_3 \) with \( p(u) = p(v) \neq p(x) = p(y) \) and \( d(u, x) = d(v, y) = 1 \). If \( M \) is a matching in \( Q_3 - \{u, v\} \), then there exists a spanning 2-path \( P_{ux} + P_{vy} \) in \( Q_3 \) passing through \( M \).

Lemma 2.2 [13]. For \( n \in \{3, 4\} \), let \( u, v \in V(Q_n) \) be such that \( p(u) \neq p(v) \). If \( M \) be a matching in \( Q_n - u \), then there exists a Hamiltonian path in \( Q_n \) joining \( u \) and \( v \) passing through \( M \).

3. Proof of Theorem 1.3

Let \( M \) be a matching in \( Q_5 \). If \( M \) is a perfect matching, then the theorem holds by Theorem 1.1. So in the following, we only need to consider the case that \( M \) is not perfect. Since \( Q_5 \) has 2\( n \) vertices, we have \( |M| \leq 15 \).

Choose a position \( j \in [5] \) such that \( |M \cap E_j| \) is as small as possible. Then \( |M \cap E_j| \leq 3 \). Without loss of generality, we may assume \( j = 5 \). Split \( Q_5 \) into \( Q_4^1 \) and \( Q_4^2 \) at position 5. Then \( Q_5 - E_5 = Q_4^1 + Q_4^2 \). Let \( \alpha \in \{0, 1\} \). Observe that every vertex \( u_\alpha \in V(Q_4^\alpha) \) has in \( Q_4^{1-\alpha} \) a unique neighbor, denoted by \( u_{1-\alpha} \). Let \( M_\alpha = M \cap E(Q_4^\alpha) \). We distinguish four cases to consider.

Case 1. \( M \cap E_5 = \emptyset \). We say that a vertex \( u \) is covered by \( M \) if \( u \in V(M) \). Otherwise, we say that \( u \) is uncovered by \( M \). Since \( M \) is not perfect in \( Q_5 \), there exists a vertex uncovered by \( M \). By symmetry we may assume that the uncovered vertex lies in \( Q_4^2 \), and denote it by \( u_0 \). In other words, \( u_0 \in V(Q_4^2) \setminus V(M) \). First apply Lemma 1.2 to find a hamiltonian cycle \( C_1 \) in \( Q_4^1 \) passing through \( M_1 \). Let \( v_1 \) be a neighbor of \( u_1 \) on \( C_1 \) such that \( u_1 v_1 \notin M \). Since \( M \) is a matching, this is always possible. Since \( p(u_0) \neq p(v_0) \) and \( M_0 \) is a matching in \( Q_4^1 - u_0 \), by Lemma 2.2 there exists a Hamiltonian path \( P_{u_0v_0} \) in \( Q_4^0 \) passing through \( M_0 \). Hence \( P_{u_0v_0} + C_1 + \{u_0u_1, v_0v_1\} - u_1 v_1 \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M \), see Figure 2.

![Figure 2. Illustration for Case 1.](image)

Case 2. \( |M \cap E_5| = 1 \). Let \( M \cap E_5 = \{u_0u_1\} \), where \( u_\alpha \in V(Q_4^\alpha) \). Let \( v_\alpha \) be a
neighbor of \( u_\alpha \) in \( Q_4^\alpha \) for \( \alpha \in \{0, 1\} \). Then \( p(u_0) \neq p(v_0) \) and \( p(u_1) \neq p(v_1) \). Since \( u_0u_1 \in M \cap E_5 \), we have \( u_\alpha \notin V(M_\alpha) \) for every \( \alpha \in \{0, 1\} \). In other words, \( M_\alpha \) is a matching in \( Q_4^\alpha - u_\alpha \). By Lemma 2.2 there exist Hamiltonian paths \( P_{u_\alpha v_\alpha} \) in \( Q_4^\alpha \) passing through \( M_\alpha \) for every \( \alpha \in \{0, 1\} \). Hence \( P_{u_0v_0} + P_{u_1v_1} + \{u_0u_1, v_0v_1\} \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M \).

**Case 3.** \( |M \cap E_5| = 2 \). Let \( M \cap E_5 = \{u_0u_1, v_0v_1\} \), where \( u_\alpha, v_\alpha \in V(Q_4^\alpha) \). If \( p(u_0) \neq p(v_0) \), then \( p(u_1) \neq p(v_1) \), the proof is similar to Case 2. So in the following we may assume \( p(u_0) = p(v_0) \). Now \( p(u_1) = p(v_1) \). In \( Q_4^\alpha \), since there are already matched two vertices with the same color, we have \( |M_\alpha| \leq 6 \) for every \( \alpha \in \{0, 1\} \). Thus, \( \sum_{\alpha \in \{0, 1\}} |M \cap E_\alpha| = |M_0| + |M_1| \leq 12 \) and \( |M| \leq 14 \).

Choose a position \( k \in \{4\} \) such that \( |M \cap E_k| \) is as small as possible. Then \( |M \cap E_k| \leq 3 \). Without loss of generality, we may assume \( k = 4 \). Let \( \alpha \in \{0, 1\} \). Split \( Q_3^\alpha \) into \( Q_3^{0\alpha} \) and \( Q_3^{1\alpha} \) at position 4. For clarity, we write \( Q_3^{0\alpha} \) and \( Q_3^{1\alpha} \) as \( Q_3^{\alpha L} \) and \( Q_3^{\alpha R} \), respectively, see Figure 3. Then \( Q_3^\alpha - E_4 = Q_3^{\alpha L} + Q_3^{\alpha R} \). Let \( M_{\alpha \delta} = M_\alpha \cap E(Q_3^{\alpha \delta}) \) for every \( \delta \in \{L, R\} \). Note that every vertex \( s_{\alpha L} \in V(Q_3^{\alpha L}) \) has in \( Q_3^{\alpha R} \) a unique neighbor, denoted by \( s_{\alpha R} \), and every vertex \( t_{\alpha R} \in V(Q_3^{\alpha R}) \) has in \( Q_3^{\alpha L} \) a unique neighbor, denoted by \( t_{\alpha L} \).

By symmetry, we may assume \( |M_0 \cap E_4| \leq |M_1 \cap E_4| \). Since \( |M \cap E_4| = |M_0 \cap E_4| + |M_1 \cap E_4| \leq 3 \), we have \( |M_0 \cap E_4| \leq 1 \). Since \( u_0 \in V(Q_3^{0\alpha}) \), without loss of generality we may assume \( u_0 \in V(Q_3^{0\alpha}) \). Now \( u_1 \in V(Q_3^{1\alpha}) \). We distinguish two cases to consider.

**Subcase 3.1.** \( v_0 \in V(Q_3^{0\alpha}) \). Now \( v_1 \in V(Q_3^{1\alpha}) \).

**Subcase 3.1.1.** \( M_0 \cap E_4 = \emptyset \). Apply Lemma 1.2 to find a Hamiltonian cycle \( C_1 \) in \( Q_4^1 \) passing through \( M_1 \). Since \( u_1 \) has only one neighbor in \( Q_3^{1\alpha} \), we may choose...
a neighbor $x_1$ of $u_1$ on $C_1$ such that $x_1 \in V(Q^L_3)$. Similarly, we may choose a neighbor $y_1$ of $v_1$ on $C_1$ such that $y_1 \in V(Q^R_3)$. Since \( \{u_0u_1, v_0v_1\} \subseteq M \), we have \( \{u_1x_1, v_1y_1\} \cap M = \emptyset \). Since \( p(u_0) \neq p(x_0) \) and \( M^L_0 \) is a matching in \( Q^L_3 - u_0 \), by Lemma 2.2 there exists a Hamiltonian path \( P_{u_0x_0} \) in \( Q^L_3 \) passing through \( M^L_0 \). Similarly, there exists a Hamiltonian path \( P_{v_0y_0} \) in \( Q^R_3 \) passing through \( M^R_0 \). Hence \( P_{v_0y_0} \cup P_{u_0x_0} \cup C_1 + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1\} - \{u_1x_1, v_1y_1\} \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M \), see Figure 4.

![Figure 4](image.png)

Figure 4. Illustration for Subcase 3.1.1.

**Subcase 3.1.2.** \( |M_0 \cap E_4| = 1 \). Now \( 1 \leq |M_1 \cap E_4| \leq 2 \). Let \( M_0 \cap E_4 = \{ s_0Ls_0R \} \), where \( s_0 \in V(Q^L_3) \). Since \( p(u_0) = p(x_0) \) and \( p(s_0L) \neq p(s_0R) \), without loss of generality, we may assume \( p(u_0) = p(x_0) = p(s_0L) \neq p(s_0R) \).

First, we claim that there exists a Hamiltonian cycle \( C_1 \) in \( Q^L_3 \) passing through \( M_1 \) such that the two neighbors of \( v_1 \) on \( C_1 \) both belong to \( V(Q^R_3) \).

If \( |M_1 \cap E_4| = 2 \), then \( |M \cap E_4| = 3 \). So \( |M \cap E_i| = 3 \) for every \( i \in [4] \). Since \( |M_\alpha| \leq 6 \) for every \( \alpha \in \{0, 1\} \) and \( |M_0| + |M_1| = \sum_{i \in [4]} |M \cap E_i| \), we have \( |M_\alpha| = 6 \) for every \( \alpha \in \{0, 1\} \). Let \( M_1 \cap E_4 = \{a_1L, a_1R, b_1L, b_1R\} \), where \( a_1, b_1 \in V(Q^L_3) \).

Then \( p(a_1\delta) \neq p(b_1\delta) \) for every \( \delta \in \{L, R\} \). (Otherwise, if \( p(a_1\delta) = p(b_1\delta) \), then \( |M_\delta| \leq 2 \). Moreover, either \( p(u_1) = p(a_1L) = p(b_1L) \) or \( p(v_1) = p(a_1R) = p(b_1R) \), so \( |M_L| \leq 1 \) or \( |M_R| \leq 1 \). Thus, \( |M_1| \leq 2 + 1 = 5 \), a contradiction.)

If \( |M_1 \cap E_4| = 1 \), let \( M_1 \cap E_4 = \{a_1L, a_1R\} \), where \( a_1 \in V(Q^L_3) \). Since \( a_1 \) has three neighbors in \( Q^L_3 \), we may choose a neighbor \( b_1 \) of \( a_1 \) in \( Q^L_3 \) such that \( b_1 \neq v_1 \). Now \( p(a_1\delta) \neq p(b_1\delta) \) for every \( \delta \in \{L, R\} \).

For the above two cases, since \( M_\delta \) is a matching in \( Q^L_3 - a_1\delta \), by Lemma 2.2 there exist Hamiltonian paths \( P_{a_1\delta} \) in \( Q^L_3 \) passing through \( M_\delta \) for every \( \delta \in \{L, R\} \). Let \( C_1 = P_{a_1L}b_1L + P_{a_1R}b_1R + \{a_1L, a_1R, b_1L, b_1R\} \). In the former case, since \( \{v_0v_1, a_1L, a_1L, b_1L, b_1R\} \subseteq M \), we have \( v_1 \notin \{a_1R, b_1R\} \). In the latter case, since \( \{v_0v_1, a_1L, a_1R\} \subseteq M \), we have \( v_1 \neq a_1R \), and therefore, \( v_1 \notin \{a_1R, b_1R\} \). Hence \( C_1 \) is a Hamiltonian cycle in \( Q^L_3 \) passing through \( M_1 \) such that the two neighbors of \( v_1 \) on \( C_1 \) both belong to \( V(Q^R_3) \), see Figure 5(1).
Next, choose a neighbor \( x_1 \) of \( u_1 \) on \( C_1 \) such that \( x_1 \in V(Q_3^{1L}) \) and choose a neighbor \( y_1 \) of \( v_1 \) on \( C_1 \) such that \( y_0 \neq s_{0R} \). Since \( M_{0R} \) is a matching in \( Q_3^{0R} - v_0 \), by Lemma 2.2 there exists a Hamiltonian path \( P_{v_0y_0} \) in \( Q_3^{0R} \) passing through \( M_{0R} \), see Figure 5(2). Since \( s_{0R} \notin \{v_0, y_0\} \), we may choose a neighbor \( t_{0L} \) of \( s_{0R} \) on \( P_{v_0y_0} \) such that \( t_{0L} \neq x_0 \). Now \( u_0, x_0, s_{0L}, t_{0L} \) are pairwise distinct vertices in \( Q_3^{0L} \), and \( p(u_0) = p(s_{0L}) \neq p(x_0) = p(t_{0L}) \), and \( d(u_0, x_0) = d(s_{0L}, t_{0L}) = 1 \). Since \( M_{0L} \) is a matching in \( Q_3^{0L} - \{u_0, s_{0L}\} \), by Lemma 2.1 there exists a spanning 2-path \( P_{u_0x_0} \) in \( Q_3^{0L} \) passing through \( M_{0L} \). Hence \( P_{u_0x_0} + P_{s_{0L}t_{0L}} + P_{v_0y_0} + C_1 + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1, s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{u_1x_1, v_1y_1, s_{0R}t_{0R}\} \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M \), see Figure 5(2).

![Figure 5. Illustration for Subcase 3.1.2.](image)

**Subcase 3.2.** \( v_0 \in V(Q_3^{1L}) \). Now \( v_1 \in V(Q_3^{1L}) \). Let \( x_1 \) be the unique vertex in \( Q_3^{1L} \) satisfying \( d(x_1, v_1) = 3 \). Then \( d(x_1, u_1) = 1 \). Since \( M_1 \) is a matching in \( Q_3^{1L} - \{u_1\} \), by Lemma 2.2 there exists a Hamiltonian path \( P_{u_1x_1} \) in \( Q_3^{1L} \) passing through \( M_1 \). Since \( v_1 \) has only one neighbor in \( Q_3^{1R} \), we may choose a neighbor \( y_1 \) of \( v_1 \) on \( P_{u_1x_1} \) such that \( y_1 \in V(Q_3^{1L}) \). Since \( d(x_1, v_1) = 3 \), we have \( y_1 \neq x_1 \). Then \( u_1, x_1, v_1, y_1 \) are pairwise distinct vertices, and \( p(u_1) = p(v_1) \neq p(x_1) = p(y_1) \), and \( d(u_1, x_1) = d(v_1, y_1) = 1 \), and the same properties also hold for the corresponding vertices \( u_0, x_0, y_0 \). If we can find a spanning 2-path \( P_{u_0x_0} + P_{v_0y_0} \) in \( Q_3^{1L} \) passing through \( M_0 \), then \( P'_{u_0x_0} + P_{v_0y_0} + P_{u_1x_1} + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1\} - v_1y_1 \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M_0 \). So in the following, we only need to show that the desired spanning 2-path \( P'_{u_0x_0} + P'_{v_0y_0} \) exists. We distinguish several cases to consider.

**Subcase 3.2.1.** \( |M_0 \cap E_4| = 1 \). Since \( M_{0L} \) is a matching in \( Q_3^{0L} - \{u_0, v_0\} \), by Lemma 2.1 there exists a spanning 2-path \( P_{u_0x_0} + P_{v_0y_0} \) in \( Q_3^{0L} \) passing through \( M_{0L} \). Let \( M_0 \cap E_4 = \{s_{0L}s_{0R}\} \), where \( s_{0L} \in V(Q_3^{0L}) \). Without loss of generality assume \( s_{0L} \in V(P_{v_0y_0}) \). Choose a neighbor \( t_{0L} \) of \( s_{0L} \) on \( P_{v_0y_0} \). Since \( s_{0L} \in M \), we have \( s_{0L}t_{0L} \notin M \). Since \( M_{0R} \) is a matching in \( Q_3^{0R} - s_{0R} \), by Lemma 2.2 there
exists a Hamiltonian path $P_{s_0t_0r}$ in $Q_3^{0R}$ passing through $M_{0R}$. Let $P'_{u_0x_0} = P_{u_0x_0}$ and $P'_{v_0y_0} = P_{v_0y_0} + P_{s_0t_0r} + \{s_0Ls_0R, t_0Lt_0R\} - s_0Lt_0L$. Then $P'_{u_0x_0} + P'_{v_0y_0}$ is the desired spanning 2-path in $Q_4^0$, see Figure 6.

Subcase 3.2.2. $M_0 \cap E_4 = \emptyset$. It suffices to consider the case that $M_{0L}$ is maximal in $Q_3^{0L} - \{u_0, v_0\}$ and $M_{0R}$ is maximal in $Q_3^{0R}$. In $Q_3^{0L}$, since $p(u_0) = p(v_0)$, we have $u_0, v_0$ are different in two positions, so there is one possibility of $\{u_0, v_0\}$ up to isomorphism. Since $d(x_0, v_0) = 3$, the vertex $x_0$ is fixed by $v_0$. Since $d(y_0, v_0) = 1$, there are two choices of $y_0$ up to isomorphism. Thus, there are two possibilities of $\{u_0, v_0, x_0, y_0\}$ up to isomorphism, see Figure 7(a)(b). When $\{u_0, v_0, x_0, y_0\}$ is the case (a), since $M_{0L}$ is a maximal matching in $Q_3^{0L} - \{u_0, v_0\}$, there are three possibilities of $M_{0L}$ up to isomorphism, see Figure 7(1)–(3). When $\{u_0, v_0, x_0, y_0\}$ is the case (b), there are seven possibilities of $M_{0L}$, see Figure 7(4)–(10). In $Q_3^{0R}$, there are three non-isomorphic maximal matchings, denoted by $P_1, P_2$ and $P_3$, see Figure 8.
Before the proof, we point out that if $M_{0R}$ is isomorphic to the matching $P_1$ or $P_2$, then there exists a Hamiltonian cycle in $Q_3^{0R}$ passing through $M_{0R} \cup \{e\}$ for any $e \notin M_{0R}$, see Figure 9.

![Figure 8. Three non-isomorphic maximal matchings in $Q_3^{0R}$.](image)

![Figure 9. Hamiltonian cycles passing through $M_{0R} \cup \{e\}$ for any $e \notin M_{0R}$ in $Q_3^{0R}$ when $M_{0R}$ is isomorphic to $P_1$ or $P_2$.](image)

First, suppose that $M_{0R}$ is isomorphic to $P_1$. Since $M_{0L}$ is a matching in $Q_3^{0L} - \{u_0, v_0\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L}$ passing through $M_{0L}$. Since $|E(P_{u_0x_0} + P_{v_0y_0})| = 6 > |M_{0L}| + |M_{0R}|$, there exists an edge $s_0Lt_0L \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ such that $s_0Rt_0R \notin M_{0R}$. Choose a Hamiltonian cycle $C_{0R}$ in $Q_3^{0R}$ passing through $M_{0R} \cup \{s_0Rt_0R\}$. Hence $P_{u_0x_0} + P_{v_0y_0} + s_0Lt_0L + s_0Rt_0R - \{s_0Lt_0L, s_0Rt_0R\}$ is the desired spanning 2-path in $Q_3^{0L}$. (Note that the construction is similar to Subcase 3.2.1, so the readers may refer to the construction in Figure 6.)

Next, suppose that $M_{0R}$ is isomorphic to $P_2$. We say that a set $S$ of edges crosses a position $i$ if $S \cap E_i \neq \emptyset$. If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (2)–(10) in Figure 7, then we may choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L}$ such that the set $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ crosses at least two positions, see Figure 10(2)–(10). Since all the edges in $M_{0R}$ lie in the same position, there exists an edge $s_0Lt_0L \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ such that $s_0Rt_0R \notin M_{0R}$. If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to the case (1) in Figure 7, then we may choose two different spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L}$ such that the two sets $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ cross two different positions, see Figure 10(1–1), (1–2), and therefore, at least one of them is different from the position in which $M_{0R}$ lies. Thus, we may choose a suitable spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ such that there exists an edge $s_0Lt_0L \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ and $s_0Rt_0R \notin M_{0R}$. The remaining construction is similar to the above case.

Last, suppose that $M_{0R}$ is isomorphic to $P_3$. Without loss of generality, we may assume $M_{0R} \subseteq (E_2 \cup E_3)$. 
If \{u_0, v_0, x_0, y_0, M_{0L}\} is isomorphic to the case (5) or (8) in Figure 7, we may choose a spanning 2-path \(P_{u_0x_0} + P_{v_0y_0}\) in \(Q^0_3\) passing through \(M_{0L}\) such that \((E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 \neq \emptyset\), see Figure 11. Let \(s_{0L}t_{0L} \in (E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1\). Then \(s_{0R}t_{0R} \in E_1\). One can verify that there exists a Hamiltonian cycle \(C_{0R}\) in \(Q^0_3\) passing through \(M_{0R} \cup \{s_{0R}t_{0R}\}\). Hence \(P_{u_0x_0} + P_{v_0y_0} + C_{0R} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{s_{0L}t_{0L}, s_{0R}t_{0R}\}\) is the desired spanning 2-path in \(Q^0_4\).

![Figure 10. Spanning 2-paths \(P_{u_0x_0} + P_{v_0y_0}\) in \(Q^0_3\) with the possible edges \(s_{0L}t_{0L}\) lined by \(\backslash\).](image)

If \{u_0, v_0, x_0, y_0, M_{0L}\} is isomorphic to one of the cases (3), (6), (7) or (10) in Figure 7, then choose a spanning 2-path \(P_{u_0x_0} + P_{v_0y_0}\) in \(Q^0_3\) passing through \(M_{0L}\), see Figure 12(3), (6), (7), (10). If \((E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 \neq \emptyset\), then the proof is similar to the above case. If \((E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 = \emptyset\), then the set \(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}\) crosses the positions 2 and 3, and therefore, \(M_{0R}\) has two choices for every case, see Figure 12. Then we can find a spanning 2-path \(P'_{u_0x_0} + P'_{v_0y_0}\) in \(Q^0_4\) passing through \(M_0\), see Figure 12.

If \{u_0, v_0, x_0, y_0, M_{0L}\} is isomorphic to one of the cases (1), (2), (4) or (9) in Figure 7, we observe that there exist two vertices in \(V(Q^0_3)\) at distance 3, denoted by \(s_{0L}, t_{0L}\), such that there is a spanning 2-path \(P_{u_0x_0} + P_{v_0y_0}\) in \(Q^0_3 + s_{0L}t_{0L}\) passing through \(M_{0L} \cup \{s_{0L}t_{0L}\}\), see Figure 13. Next, we can verify that there exists a Hamiltonian path \(P_{s_{0R}t_{0R}}\) in \(Q^0_3\) passing through \(M_{0R}\). Hence \(P_{u_0x_0} + P_{v_0y_0} + P_{s_{0R}t_{0R}} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - s_{0L}t_{0L}\) is the desired spanning 2-path in \(Q^0_4\).
Case 4. $|M \cap E_5| = 3$. Let $M \cap E_5 = \{u_0u_1, v_0v_1, w_0w_1\}$, where $u_\alpha, v_\alpha, w_\alpha \in V(Q_4^3)$. Now $|M \cap E_i| = 3$ for every $i \in [5]$ and $|M| = 15$. Hence there are two vertices of $\{u_\alpha, v_\alpha, w_\alpha\}$ in one partite set and one vertex in the other partite set. Otherwise, if $p(u_\alpha) = p(v_\alpha) = p(w_\alpha)$, then $|M_\alpha| \leq 5$, and therefore, $|M| \leq 13$, a contradiction. Without loss of generality, we may assume $p(u_\alpha) = p(v_\alpha) \neq p(w_\alpha)$.

Figure 12. Spanning 2-paths $P'_{u_0x_0} + P'_{v_0y_0}$ in $Q_4^0$ passing through $M_0$.

Figure 13. Spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in $Q_4^0 + s_{0L}t_{0L}$ passing through $M_{0L} \cup \{s_{0L}t_{0L}\}$.

Split $Q_4^0$ into two 3-cubes $Q_3^{3L}$ and $Q_3^{3R}$ at some position $k$ such that $u_\alpha \in V(Q_3^{3L})$ and $v_\alpha \in V(Q_3^{3R})$. Without loss of generality, we may assume $k = 4$. Since $p(u_\alpha) = p(v_\alpha) \neq p(w_\alpha)$, by symmetry we may assume $w_\alpha \in V(Q_3^{3L})$. Since $|M_0 \cap E_4| + |M_1 \cap E_4| = |M \cap E_4| = 3$, by symmetry we may assume $|M_0 \cap E_4| \leq 1$. Let $M_{\alpha \delta} = M_\alpha \cap E(Q_3^{3\delta})$ for every $\delta \in \{L, R\}$.

Subcase 4.1. $M_0 \cap E_4 = \emptyset$. Since $p(u_1) \neq p(w_1)$ and $M_1$ is a matching in $Q_4^1 - u_1$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_1w_1}$ in $Q_4^1$ passing
through $M_1$. Since $v_1$ has only one neighbor in $Q_3^{L1}$, we may choose a neighbor $y_1$ of $v_1$ on $P_{u_1w_1}$ such that $y_1 \in V(Q_3^{R1})$. Now $y_0 \in V(Q_3^{R0})$ and $p(u_0) = p(v_0) \neq p(w_0) = p(y_0)$. Since $M_{0L}$ is a matching in $Q_3^{L0} - u_0$ and $M_{0R}$ is a matching in $Q_3^{R0} - v_0$, by Lemma 2.2 there exist Hamiltonian paths $P_{u_0w_0}$ in $Q_3^{L0}$ and $P_{v_0s_0}$ in $Q_3^{R0}$ passing through $M_{0L}$ and $M_{0R}$, respectively. Hence $P_{u_1w_1} + P_{u_0w_0} + P_{v_0s_0} + \{u_0u_1, w_0w_1, v_0v_1, y_0y_1\} - v_1y_1$ is a Hamiltonian cycle in $Q_3$ passing through $M$, see Figure 14.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{Illustration for Subcase 4.1.}
\end{figure}

\textbf{Subcase 4.2.} $|M_0 \cap E_4| = 1$. Now $|M_1 \cap E_4| = 2$. Let $M_0 \cap E_4 = \{w_0s_0R\}$ and $M_1 \cap E_4 = \{a_1L, a_1R, b_1L, b_1R\}$, where $s_0R \in V(Q_3^{L0})$ and $a_1L, b_1R \in V(Q_3^{R1})$. Since $|M| = 15$, $Q_3$ has exactly two vertices uncovered by $M$, one in $Q_3^{L0}$ and the other in $Q_3^{R1}$. Thus, $p(a_1L) \neq p(b_1R)$, and $p(v_0) \neq p(s_0R)$, and $M_{1L}$ is a perfect matching in $Q_3^{L0} - \{u_1, w_1, a_1L, b_1L\}$. Since $p(u_1) \neq p(w_1)$ and $p(a_1L) \neq p(b_1L)$, without loss of generality, we may assume $p(u_1) = p(b_1L) \neq p(w_1) = p(a_1L)$. Thus, $p(v_1) = p(a_1R) \neq p(b_1R)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15.png}
\caption{Illustration for Subcase 4.2.}
\end{figure}

Since $p(u_0) \neq p(w_0)$ and $M_{0L}$ is a matching in $Q_3^{L0} - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0w_0}$ in $Q_3^{L0}$ passing through $M_{0L}$, see Figure 15. Since $s_0L \notin \{u_0, w_0\}$, we may choose a neighbor $t_0L$ of $s_0L$ on $P_{u_0w_0}$ such that $t_0R \neq v_0$. Since $p(s_0R) \neq p(t_0R)$ and $M_{0R}$ is a matching in $Q_3^{R0} - s_0R$,
by Lemma 2.2 there exists a Hamiltonian path \( P_{s_0Rt_0R} \) in \( Q^0_{3R} \) passing through \( M_{0R} \), see Figure 15. Since \( v_0 \notin \{s_0R, t_0R\} \), we may choose a neighbor \( y_0 \) of \( v_0 \) on \( P_{s_0Rt_0R} \) such that \( y_0 \neq b_1R \). Now \( v_1, y_1, a_1R, b_1R \) are pairwise distinct vertices, and \( p(v_1) = p(a_1R) \neq p(y_1) = p(b_1R) \), and \( d(v_1, y_1) = 1 \), and \( M_{1R} \) is a matching in \( Q^1_{3R} - \{v_1, a_1R\} \).

If \( d(a_1R, b_1R) = 1 \), then by Lemma 2.1 there is a spanning 2-path \( P_{v_1y_1} + P_{a_1Rb_1R} \) in \( Q^0_{3R} \) passing through \( M_{1R} \), see Figure 17(1). Since \( M_{1L} \) is a perfect matching in \( Q^1_{3L} - \{u_1, w_1, a_{1L}, b_{1L}\} \), we have \( M_{1L} \cup \{u_1, w_1, a_{1L}, b_{1L}\} \) is a perfect matching in \( K(Q^1_{3L}) \). By Theorem 1.1, there exists a perfect matching \( R \) in \( Q^1_{3L} \) such that \( M_{1L} \cup \{u_1, w_1, a_{1L}, b_{1L}\} \cup R \) forms a Hamiltonian cycle in \( K(Q^1_{3L}) \). Hence \( M_{1L} \cup R \) forms a spanning 2-path in \( Q^1_{3L} \). Note that each path of the spanning 2-path is an \((R, M_{1L})\)-alternating path beginning with an edge in \( R \) and ending with an edge in \( R \). So the number of vertices in each path is even. Since \( Q_5 \) is a bipartite graph, the two endpoints of each path have different parities. Hence one path joins the vertices \( u_1 \) and \( a_{1L} \), and the other path joins the vertices \( w_1 \) and \( b_{1L} \), see Figure 16 for example. Denote the spanning 2-path by \( P_{u_1a_{1L}} + P_{w_1b_{1L}} \). Note that \( s_0L \) is not an endpoint of \( P_{u_1a_{1L}} + P_{w_1b_{1L}} \). Hence \( P_{s_0Lw_0} + P_{s_0Rt_0R} + P_{u_1a_{1L}} + P_{w_1b_{1L}} + P_{v_1y_1} + P_{a_1Rb_1R} + \{u_0, w_0, v_0, w_1, v_1, y_0, y_1, a_{1L}a_{1R}, b_{1L}b_{1R}, s_0Ls_0R, t_0Lt_0R\} - \{v_0y_0, s_0Lt_0R\} \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M \), see Figure 17(1).

Figure 16. The spanning 2-path \( P_{u_1a_{1L}} + P_{w_1b_{1L}} \) (or \( P_{u_1a_{1L}} + P_{a_{1L}b_{1L}} \)) in \( Q^1_{3L} \).

Figure 17. Illustration for Subcase 4.2.
If \( d(a_1 R, b_1 R) = 3 \), then \( d(v_1, b_1 R) = d(a_1 R, y_1) = 1 \). Since \( M_{1 R} \) is a matching in \( Q_{3 R}^1 - \{ v_1, a_1 \} \), by Lemma 2.1 there is a spanning 2-path \( P_{v_1 b_1 R} + P_{a_1 R v_1} \) in \( Q_{3 R}^1 \) passing through \( M_{1 R} \), see Figure 17(2). Since \( M_{1 L} \cup \{ u_1 a_1 L, w_1 b_1 L \} \) is a perfect matching in \( K(Q_{3 L}^1) \), similar to the above case, there is a spanning 2-path \( P_{u_1 w_1} + P_{a_1 L b_1 L} \) in \( Q_{3 L}^1 \) passing through \( M_{1 L} \). Hence \( P_{u_0 w_0} + P_{s_0 t_0 R} + P_{v_1 b_1 R} + P_{y_0 y_1} + P_{a_1 L b_1 L} + \{ u_0 v_1, w_0 w_1, v_0 v_1, y_0 y_1, a_1 L a_1 R, b_1 L b_1 R, s_0 L s_0 R, t_0 L t_0 R \} - \{ v_0 y_0, s_0 L t_0 L \} \) is a Hamiltonian cycle in \( Q_5 \) passing through \( M \), see Figure 17(2).

The proof of Theorem 1.3 is complete.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for their kind suggestions on the original manuscript.

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doi:10.1016/j.jctb.2007.02.007

doi:10.1137/07069288

doi:10.1016/j.ejc.2009.03.007


Received 27 September 2016
Revised 4 November 2016
Accepted 4 November 2016