Discussiones Mathematicae
Graph Theory 38 (2018) 217–231
doi:10.7151/dmgt.2010

MATCHINGS EXTEND TO HAMILTONIAN CYCLES IN 5-CUBE [1]

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Abstract

Ruskey and Savage asked the following question: Does every matching in a hypercube Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Fink confirmed that every perfect matching can be extended to a Hamiltonian cycle of Q_n, thus solved Kreweras' conjecture. Also, Fink pointed out that every matching can be extended to a Hamiltonian cycle of Q_n for $n \in \{2, 3, 4\}$. In this paper, we prove that every matching in Q_5 can be extended to a Hamiltonian cycle of Q_5.

Keywords: hypercube, Hamiltonian cycle, matching.

2010 Mathematics Subject Classification: 05C38, 05C45.

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[1]This work is supported by NSFC (grant nos. 11501282 and 11261019) and the science and technology project of Jiangxi Provincial Department of Education (grant No. 20161BAB201030).

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1. Introduction

Let $[n]$ denote the set $\{1, \ldots, n\}$. The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary strings of length n, i.e., $V(Q_n) = \{u : u = u^1 \cdots u^n \text{ and } u^i \in \{0,1\} \text{ for every } i \in [n]\}$, with two vertices being adjacent whenever the corresponding strings differ in just one position.

The hypercube Q_n is one of the most popular and efficient interconnection networks. It is well known that Q_n is Hamiltonian for every $n \geq 2$. This statement dates back to 1872 [9]. Since then, the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention [2, 3, 4, 6, 12].

A set of edges in a graph G is called a *matching* if no two edges have an endpoint in common. A matching is *perfect* if it covers all vertices of G. A cycle in a graph G is a *Hamiltonian cycle* if every vertex in G appears exactly once in the cycle.

Ruskey and Savage [11] asked the following question: Does every matching in Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Kreweras [10] conjectured that every perfect matching of Q_n for $n \geq 2$ can be extended to a Hamiltonian cycle of Q_n. Fink [5, 7] confirmed the conjecture to be true. Let $K(Q_n)$ be the complete graph on the vertices of the hypercube Q_n.

Theorem 1.1 [5, 7]**.** *For every perfect matching M of $K(Q_n)$, there exists a perfect matching F of Q_n, $n \geq 2$, such that $M \cup F$ forms a Hamiltonian cycle of $K(Q_n)$.*

Also, Fink [5] pointed out that the following conclusion holds.

Lemma 1.2 [5]**.** *Every matching in Q_n can be extended to a Hamiltonian cycle of Q_n for $n \in \{2, 3, 4\}$.*

Gregor [8] strengthened Fink's result and obtained that given a partition of the hypercube into subcubes of nonzero dimensions, every perfect matching of the hypercube can be extended on these subcubes to a Hamiltonian cycle if and only if the perfect matching interconnects these subcubes.

The present authors [14] proved that every matching of at most $3n-10$ edges in Q_n can be extended to a Hamiltonian cycle of Q_n for $n \geq 4$.

In this paper, we consider Ruskey and Savage's question and obtain the following result.

Theorem 1.3. *Every matching in Q_5 can be extended to a Hamiltonian cycle of Q_5.*

It is worth mentioning that Ruskey and Savage's question has been recently done for $n = 5$ independently by a computer search [15]. In spite of this, a direct proof is still necessary, as it may serve in a possible solution of the general question.

2. Preliminaries and Lemmas

Terminology and notation used in this paper but undefined below can be found in [1]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a set $F \subseteq E(G)$, let $G - F$ denote the resulting graph after removing all edges in F from G. Let H and H' be two subgraphs of G. We use $H + H'$ to denote the graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. For $F \subseteq E(G)$, we use $H + F$ to denote the graph with the vertex set $V(H) \cup V(F)$ and edge set $E(H) \cup F$, where $V(F)$ denotes the set of vertices incident with F.

The *distance* between two vertices u and v is the number of edges in a shortest path joining u and v in G, denoted by $d_G(u, v)$, with the subscripts being omitted when the context is clear.

Let $j \in [n]$. An edge in Q_n is called an *j-edge* if its endpoints differ in the jth position. The set of all j-edges in Q_n is denoted by E_j. Thus, $E(Q_n) = \bigcup_{i=1}^{n} E_i$. Let $Q^0_{n-1,j}$ and $Q^1_{n-1,j}$, with the superscripts j being omitted when the context is clear, be the $(n-1)$-dimensional subcubes of Q_n induced by the vertex sets $\{u \in V(Q_n) : u^j = 0\}$ and $\{u \in V(Q_n) : u^j = 1\}$, respectively. Thus, $Q_n - E_j = Q^0_{n-1} + Q^1_{n-1}$. We say that Q_n *splits* into two $(n-1)$-dimensional subcubes Q^0_{n-1} and Q^1_{n-1} at position j; see Figure 1 for example.

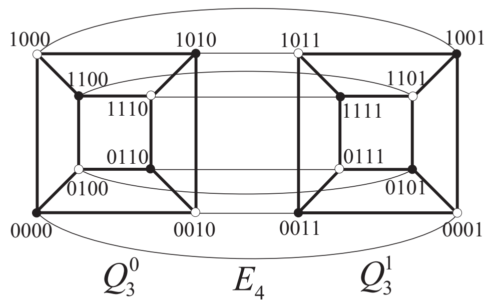

Figure 1. Q_4 splits into two 3-dimensional subcubes Q^0_3 and Q^1_3 at position 4.

The *parity* $p(u)$ of a vertex u in Q_n is defined by $p(u) = \sum_{i=1}^{n} u^i (\text{mod } 2)$. Then there are 2^{n-1} vertices with parity 0 and 2^{n-1} vertices with parity 1 in Q_n. Vertices with parity 0 and 1 are called black vertices and white vertices, respectively. Observe that Q_n is bipartite and vertices of each parity form bipartite sets of Q_n. Thus, $p(u) \neq p(v)$ if and only if $d(u, v)$ is odd.

A *u, v-path* is a path with endpoints u and v, denoted by P_{uv} when we specify a particular such path. We say that a spanning subgraph of G whose components are k disjoint paths is a *spanning k-path* of G. A spanning 1-path thus is simply a spanning or Hamiltonian path. We say that a path P (respectively, a cycle C) passes through a set M of edges if $M \subseteq E(P)$ (respectively, $M \subseteq E(C)$).

Lemma 2.1 [13]**.** *Let u, v, x, y be pairwise distinct vertices in Q_3 with $p(u) = p(v) \neq p(x) = p(y)$ and $d(u, x) = d(v, y) = 1$. If M is a matching in $Q_3 - \{u, v\}$, then there exists a spanning 2-path $P_{ux} + P_{vy}$ in Q_3 passing through M.*

Lemma 2.2 [13]**.** *For $n \in \{3, 4\}$, let $u, v \in V(Q_n)$ be such that $p(u) \neq p(v)$. If M be a matching in $Q_n - u$, then there exists a Hamiltonian path in Q_n joining u and v passing through M.*

3. Proof of Theorem 1.3

Let M be a matching in Q_5. If M is a perfect matching, then the theorem holds by Theorem 1.1. So in the following, we only need to consider the case that M is not perfect. Since Q_5 has 2^5 vertices, we have $|M| \leq 15$.

Choose a position $j \in [5]$ such that $|M \cap E_j|$ is as small as possible. Then $|M \cap E_j| \leq 3$. Without loss of generality, we may assume $j = 5$. Split Q_5 into Q_4^0 and Q_4^1 at position 5. Then $Q_5 - E_5 = Q_4^0 + Q_4^1$. Let $\alpha \in \{0, 1\}$. Observe that every vertex $u_\alpha \in V(Q_4^\alpha)$ has in $Q_4^{1-\alpha}$ a unique neighbor, denoted by $u_{1-\alpha}$. Let $M_\alpha = M \cap E(Q_4^\alpha)$. We distinguish four cases to consider.

Case 1. $M \cap E_5 = \emptyset$. We say that a vertex u is covered by M if $u \in V(M)$. Otherwise, we say that u is uncovered by M. Since M is not perfect in Q_5, there exists a vertex uncovered by M. By symmetry we may assume that the uncovered vertex lies in Q_4^0, and denote it by u_0. In other words, $u_0 \in V(Q_4^0) \setminus V(M)$. First apply Lemma 1.2 to find a hamiltonian cycle C_1 in Q_4^1 passing through M_1. Let v_1 be a neighbor of u_1 on C_1 such that $u_1v_1 \notin M$. Since M is a matching, this is always possible. Since $p(u_0) \neq p(v_0)$ and M_0 is a matching in $Q_4^0 - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0v_0}$ in Q_4^0 passing through M_0. Hence $P_{u_0v_0} + C_1 + \{u_0u_1, v_0v_1\} - u_1v_1$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 2.

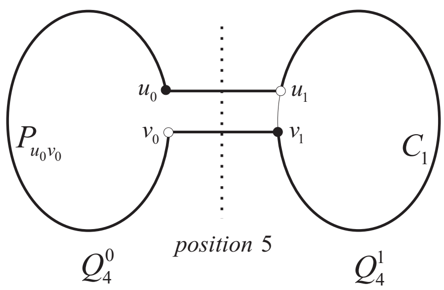

Figure 2. Illustration for Case 1.

Case 2. $|M \cap E_5| = 1$. Let $M \cap E_5 = \{u_0u_1\}$, where $u_\alpha \in V(Q_4^\alpha)$. Let v_α be a neighbor of u_α in Q_4^α for $\alpha \in \{0,1\}$. Then $p(u_0) \neq p(v_0)$ and $p(u_1) \neq p(v_1)$. Since $u_0u_1 \in M \cap E_5$, we have $u_\alpha \notin V(M_\alpha)$ for every $\alpha \in \{0,1\}$. In other words, M_α is a matching in $Q_4^\alpha - u_\alpha$. By Lemma 2.2 there exist Hamiltonian paths $P_{u_\alpha v_\alpha}$ in Q_4^α passing through M_α for every $\alpha \in \{0,1\}$. Hence $P_{u_0v_0} + P_{u_1v_1} + \{u_0u_1, v_0v_1\}$ is a Hamiltonian cycle in Q_5 passing through M.

Case 3. $|M \cap E_5| = 2$. Let $M \cap E_5 = \{u_0u_1, v_0v_1\}$, where $u_\alpha, v_\alpha \in V(Q_4^\alpha)$. If $p(u_0) \neq p(v_0)$, then $p(u_1) \neq p(v_1)$, the proof is similar to Case 2. So in the following we may assume $p(u_0) = p(v_0)$. Now $p(u_1) = p(v_1)$. In Q_4^α, since there are already matched two vertices with the same color, we have $|M_\alpha| \leq 6$ for every $\alpha \in \{0,1\}$. Thus, $\sum_{i\in[4]} |M \cap E_i| = |M_0| + |M_1| \leq 12$ and $|M| \leq 14$.

Choose a position $k \in [4]$ such that $|M \cap E_k|$ is as small as possible. Then $|M \cap E_k| \leq 3$. Without loss of generality, we may assume $k = 4$. Let $\alpha \in \{0,1\}$. Split Q_4^α into $Q_3^{\alpha 0}$ and $Q_3^{\alpha 1}$ at position 4. For clarity, we write $Q_3^{\alpha 0}$ and $Q_3^{\alpha 1}$ as $Q_3^{\alpha L}$ and $Q_3^{\alpha R}$, respectively, see Figure 3. Then $Q_4^\alpha - E_4 = Q_3^{\alpha L} + Q_3^{\alpha R}$. Let $M_{\alpha\delta} = M_\alpha \cap E(Q_3^{\alpha\delta})$ for every $\delta \in \{L, R\}$. Note that every vertex $s_{\alpha L} \in V(Q_3^{\alpha L})$ has in $Q_3^{\alpha R}$ a unique neighbor, denoted by $s_{\alpha R}$, and every vertex $t_{\alpha R} \in V(Q_3^{\alpha R})$ has in $Q_3^{\alpha L}$ a unique neighbor, denoted by $t_{\alpha L}$.

By symmetry, we may assume $|M_0 \cap E_4| \leq |M_1 \cap E_4|$. Since $|M \cap E_4| = |M_0 \cap E_4| + |M_1 \cap E_4| \leq 3$, we have $|M_0 \cap E_4| \leq 1$. Since $u_0 \in V(Q_4^0)$, without loss of generality we may assume $u_0 \in V(Q_3^{0L})$. Now $u_1 \in V(Q_3^{1L})$. We distinguish two cases to consider.

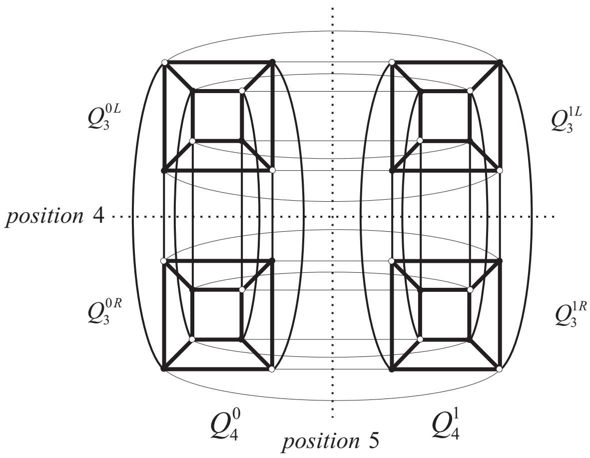

Figure 3. Q_5 splits into four 3-dimensional subcubes Q_3^{0L}, Q_3^{0R}, Q_3^{1L} and Q_3^{1R}.

Subcase 3.1. $v_0 \in V(Q_3^{0R})$. Now $v_1 \in V(Q_3^{1R})$.

Subcase 3.1.1. $M_0 \cap E_4 = \emptyset$. Apply Lemma 1.2 to find a Hamiltonian cycle C_1 in Q_4^1 passing through M_1. Since u_1 has only one neighbor in Q_3^{1R}, we may choose a neighbor x_1 of u_1 on C_1 such that $x_1 \in V(Q_3^{1L})$. Similarly, we may choose a neighbor y_1 of v_1 on C_1 such that $y_1 \in V(Q_3^{1R})$. Since $\{u_0u_1, v_0v_1\} \subseteq M$, we have $\{u_1x_1, v_1y_1\} \cap M = \emptyset$. Since $p(u_0) \neq p(x_0)$ and M_{0L} is a matching in $Q_3^{0L} - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0x_0}$ in Q_3^{0L} passing through M_{0L}. Similarly, there exists a Hamiltonian path $P_{v_0y_0}$ in Q_3^{0R} passing through M_{0R}. Hence $P_{u_0x_0} + P_{v_0y_0} + C_1 + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1\} - \{u_1x_1, v_1y_1\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 4.

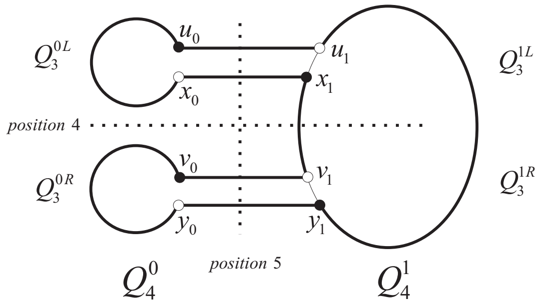

Figure 4. Illustration for Subcase 3.1.1.

Subcase 3.1.2. $|M_0 \cap E_4| = 1$. Now $1 \leq |M_1 \cap E_4| \leq 2$. Let $M_0 \cap E_4 = \{s_{0L}s_{0R}\}$, where $s_{0\delta} \in V(Q_3^{0\delta})$. Since $p(u_0) = p(v_0)$ and $p(s_{0L}) \neq p(s_{0R})$, without loss of generality, we may assume $p(u_0) = p(v_0) = p(s_{0L}) \neq p(s_{0R})$.

First, we claim that there exists a Hamiltonian cycle C_1 in Q_4^1 passing through M_1 such that the two neighbors of v_1 on C_1 both belong to $V(Q_3^{1R})$.

If $|M_1 \cap E_4| = 2$, then $|M \cap E_4| = 3$. So $|M \cap E_i| = 3$ for every $i \in [4]$. Since $|M_\alpha| \leq 6$ for every $\alpha \in \{0, 1\}$ and $|M_0| + |M_1| = \sum_{i \in [4]} |M \cap E_i|$, we have $|M_\alpha| = 6$ for every $\alpha \in \{0, 1\}$. Let $M_1 \cap E_4 = \{a_{1L}a_{1R}, b_{1L}b_{1R}\}$, where $a_{1\delta}, b_{1\delta} \in V(Q_3^{1\delta})$. Then $p(a_{1\delta}) \neq p(b_{1\delta})$ for every $\delta \in \{L, R\}$. (Otherwise, if $p(a_{1\delta}) = p(b_{1\delta})$, then $|M_{1\delta}| \leq 2$. Moreover, either $p(u_1) = p(a_{1L}) = p(b_{1L})$ or $p(v_1) = p(a_{1R}) = p(b_{1R})$, so $|M_{1L}| \leq 1$ or $|M_{1R}| \leq 1$. Thus, $|M_1| \leq 2 + 1 + 2 = 5$, a contradiction).

If $|M_1 \cap E_4| = 1$, let $M_1 \cap E_4 = \{a_{1L}a_{1R}\}$, where $a_{1\delta} \in V(Q_3^{1\delta})$. Since a_{1R} has three neighbors in Q_3^{1R}, we may choose a neighbor b_{1R} of a_{1R} in Q_3^{1R} such that $b_{1R} \neq v_1$. Now $p(a_{1\delta}) \neq p(b_{1\delta})$ for every $\delta \in \{L, R\}$.

For the above two cases, since $M_{1\delta}$ is a matching in $Q_3^{1\delta} - a_{1\delta}$, by Lemma 2.2 there exist Hamiltonian paths $P_{a_{1\delta}b_{1\delta}}$ in $Q_3^{1\delta}$ passing through $M_{1\delta}$ for every $\delta \in \{L, R\}$. Let $C_1 = P_{a_{1L}b_{1L}} + P_{a_{1R}b_{1R}} + \{a_{1L}a_{1R}, b_{1L}b_{1R}\}$. In the former case, since $\{v_0v_1, a_{1L}a_{1R}, b_{1L}b_{1R}\} \subseteq M$, we have $v_1 \notin \{a_{1R}, b_{1R}\}$. In the latter case, since $\{v_0v_1, a_{1L}a_{1R}\} \subseteq M$, we have $v_1 \neq a_{1R}$, and therefore, $v_1 \notin \{a_{1R}, b_{1R}\}$. Hence C_1 is a Hamiltonian cycle in Q_4^1 passing through M_1 such that the two neighbors of v_1 on C_1 both belong to $V(Q_3^{1R})$, see Figure 5(1).

Next, choose a neighbor x_1 of u_1 on C_1 such that $x_1 \in V(Q_3^{1L})$ and choose a neighbor y_1 of v_1 on C_1 such that $y_0 \neq s_{0R}$. Since M_{0R} is a matching in $Q_3^{0R} - v_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{v_0 y_0}$ in Q_3^{0R} passing through M_{0R}, see Figure 5(2). Since $s_{0R} \notin \{v_0, y_0\}$, we may choose a neighbor t_{0R} of s_{0R} on $P_{v_0 y_0}$ such that $t_{0L} \neq x_0$. Now u_0, x_0, s_{0L}, t_{0L} are pairwise distinct vertices in Q_3^{0L}, and $p(u_0) = p(s_{0L}) \neq p(x_0) = p(t_{0L})$, and $d(u_0, x_0) = d(s_{0L}, t_{0L}) = 1$. Since M_{0L} is a matching in $Q_3^{0L} - \{u_0, s_{0L}\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0 x_0} + P_{s_{0L} t_{0L}}$ in Q_3^{0L} passing through M_{0L}. Hence $P_{u_0 x_0} + P_{s_{0L} t_{0L}} + P_{v_0 y_0} + C_1 + \{u_0 u_1, x_0 x_1, v_0 v_1, y_0 y_1, s_{0L} s_{0R}, t_{0L} t_{0R}\} - \{u_1 x_1, v_1 y_1, s_{0R} t_{0R}\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 5(2).

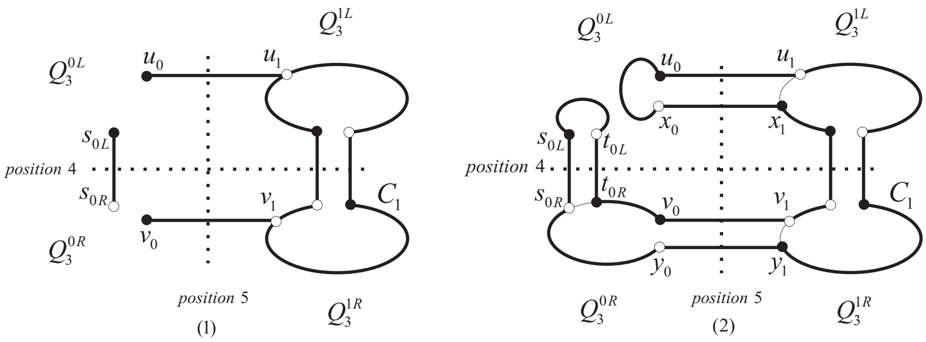

Figure 5. Illustration for Subcase 3.1.2.

Subcase 3.2. $v_0 \in V(Q_3^{0L})$. Now $v_1 \in V(Q_3^{1L})$. Let x_1 be the unique vertex in Q_3^{1L} satisfying $d(x_1, v_1) = 3$. Then $d(x_1, u_1) = 1$. Since M_1 is a matching in $Q_4^1 - u_1$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_1 x_1}$ in Q_4^1 passing through M_1. Since v_1 has only one neighbor in Q_3^{1R}, we may choose a neighbor y_1 of v_1 on $P_{u_1 x_1}$ such that $y_1 \in V(Q_3^{1L})$. Since $d(x_1, v_1) = 3$, we have $y_1 \neq x_1$. Then u_1, x_1, v_1, y_1 are pairwise distinct vertices, and $p(u_1) = p(v_1) \neq p(x_1) = p(y_1)$, and $d(u_1, x_1) = d(v_1, y_1) = 1$, and the same properties also hold for the corresponding vertices u_0, x_0, v_0, y_0. If we can find a spanning 2-path $P'_{u_0 x_0} + P'_{v_0 y_0}$ in Q_4^0 passing through M_0, then $P'_{u_0 x_0} + P'_{v_0 y_0} + P_{u_1 x_1} + \{u_0 u_1, x_0 x_1, v_0 v_1, y_0 y_1\} - v_1 y_1$ is a Hamiltonian cycle in Q_5 passing through M. So in the following, we only need to show that the desired spanning 2-path $P'_{u_0 x_0} + P'_{v_0 y_0}$ exists. We distinguish several cases to consider.

Subcase 3.2.1. $|M_0 \cap E_4| = 1$. Since M_{0L} is a matching in $Q_3^{0L} - \{u_0, v_0\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0 x_0} + P_{v_0 y_0}$ in Q_3^{0L} passing through M_{0L}. Let $M_0 \cap E_4 = \{s_{0L} s_{0R}\}$, where $s_{0\delta} \in V(Q_3^{0\delta})$. Without loss of generality assume $s_{0L} \in V(P_{v_0 y_0})$. Choose a neighbor t_{0L} of s_{0L} on $P_{v_0 y_0}$. Since $s_{0L} s_{0R} \in M$, we have $s_{0L} t_{0L} \notin M$. Since M_{0R} is a matching in $Q_3^{0R} - s_{0R}$, by Lemma 2.2 there exists a Hamiltonian path $P_{s_{0R}t_{0R}}$ in Q_3^{0R} passing through M_{0R}. Let $P'_{u_0x_0} = P_{u_0x_0}$ and $P'_{v_0y_0} = P_{v_0y_0} + P_{s_{0R}t_{0R}} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - s_{0L}t_{0L}$. Then $P'_{u_0x_0} + P'_{v_0y_0}$ is the desired spanning 2-path in Q_4^0, see Figure 6.

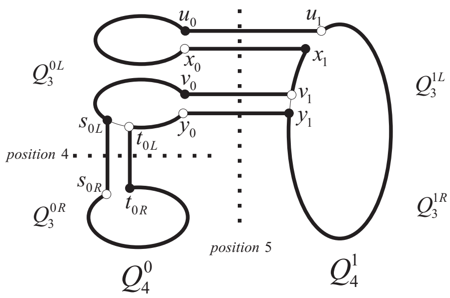

Figure 6. Illustration for Subcase 3.2.1.

Subcase 3.2.2. $M_0 \cap E_4 = \emptyset$. It suffices to consider the case that M_{0L} is maximal in $Q_3^{0L} - \{u_0, v_0\}$ and M_{0R} is maximal in Q_3^{0R}. In Q_3^{0L}, since $p(u_0) = p(v_0)$, we have u_0, v_0 are different in two positions, so there is one possibility of $\{u_0, v_0\}$ up to isomorphism. Since $d(x_0, v_0) = 3$, the vertex x_0 is fixed by v_0. Since $d(y_0, v_0) = 1$, there are two choices of y_0 up to isomorphism. Thus, there are two possibilities of $\{u_0, v_0, x_0, y_0\}$ up to isomorphism, see Figure 7$(a)(b)$. When $\{u_0, v_0, x_0, y_0\}$ is the case (a), since M_{0L} is a maximal matching in $Q_3^{0L} - \{u_0, v_0\}$, there are three possibilities of M_{0L} up to isomorphism, see Figure 7(1)–(3). When $\{u_0, v_0, x_0, y_0\}$ is the case (b), there are seven possibilities of M_{0L}, see Figure 7(4)–(10). In Q_3^{0R}, there are three non-isomorphic maximal matchings, denoted by P_1, P_2 and P_3, see Figure 8.

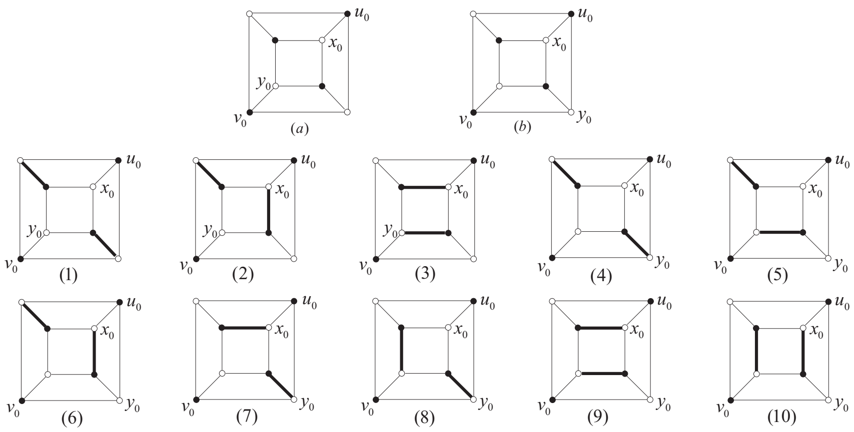

Figure 7. All possibilities of $\{u_0, v_0, x_0, y_0, M_{0L}\}$ up to isomorphism.

Before the proof, we point out that if M_{0R} is isomorphic to the matching P_1 or P_2, then there exists a Hamiltonian cycle in Q_3^{0R} passing through $M_{0R} \cup \{e\}$ for any $e \notin M_{0R}$, see Figure 9.

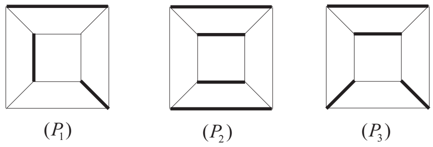

Figure 8. Three non-isomorphic maximal matchings in Q_3^{0R}.

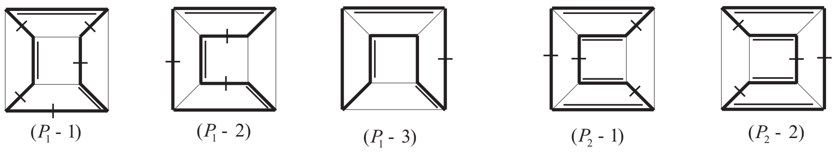

Figure 9. Hamiltonian cycles passing through $M_{0R} \cup \{e\}$ for any $e \notin M_{0R}$ in Q_3^{0R} when M_{0R} is isomorphic to P_1 or P_2.

First, suppose that M_{0R} is isomorphic to P_1. Since M_{0L} is a matching in $Q_3^{0L} - \{u_0, v_0\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L}. Since $|E(P_{u_0x_0} + P_{v_0y_0})| = 6 > |M_{0L}| + |M_{0R}|$, there exists an edge $s_{0L}t_{0L} \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ such that $s_{0R}t_{0R} \notin M_{0R}$. Choose a Hamiltonian cycle C_{0R} in Q_3^{0R} passing through $M_{0R} \cup \{s_{0R}t_{0R}\}$. Hence $P_{u_0x_0} + P_{v_0y_0} + C_{0R} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{s_{0L}t_{0L}, s_{0R}t_{0R}\}$ is the desired spanning 2-path in Q_4^0. (Note that the construction is similar to Subcase 3.2.1, so the readers may refer to the construction in Figure 6.)

Next, suppose that M_{0R} is isomorphic to P_2. We say that a set S of edges crosses a position i if $S \cap E_i \neq \emptyset$. If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (2)–(10) in Figure 7, then we may choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} such that the set $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ crosses at least two positions, see Figure 10(2)–(10). Since all the edges in M_{0R} lie in the same position, there exists an edge $s_{0L}t_{0L} \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ such that $s_{0R}t_{0R} \notin M_{0R}$. If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to the case (1) in Figure 7, then we may choose two different spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} such that the two sets $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ cross two different positions, see Figure 10(1–1), (1–2), and therefore, at least one of them is different from the position in which M_{0R} lies. Thus, we may choose a suitable spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ such that there exists an edge $s_{0L}t_{0L} \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ and $s_{0R}t_{0R} \notin M_{0R}$. The remaining construction is similar to the above case.

Last, suppose that M_{0R} is isomorphic to P_3. Without loss of generality, we may assume $M_{0R} \subseteq (E_2 \cup E_3)$.

If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to the case (5) or (8) in Figure 7, we may choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L} such that $(E(P_{u_0x_0}+P_{v_0y_0})\setminus M_{0L})\cap E_1 \neq \emptyset$, see Figure 11. Let $s_{0L}t_{0L} \in (E(P_{u_0x_0}+P_{v_0y_0})\setminus M_{0L}) \cap E_1$. Then $s_{0R}t_{0R} \in E_1$. One can verify that there exists a Hamiltonian cycle C_{0R} in Q_3^{0R} passing through $M_{0R} \cup \{s_{0R}t_{0R}\}$. Hence $P_{u_0x_0} + P_{v_0y_0} + C_{0R} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{s_{0L}t_{0L}, s_{0R}t_{0R}\}$ is the desired spanning 2-path in Q_4^0.

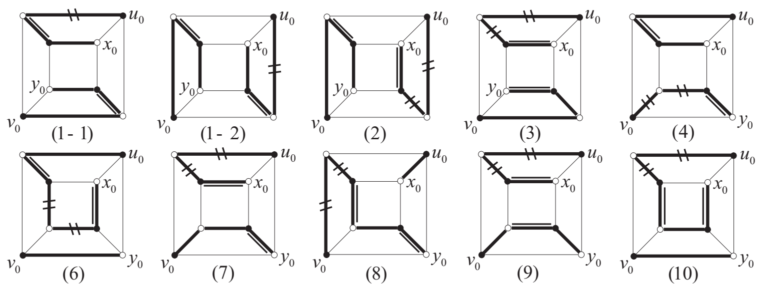

Figure 10. Spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} with the possible edges $s_{0L}t_{0L}$ lined by \\.

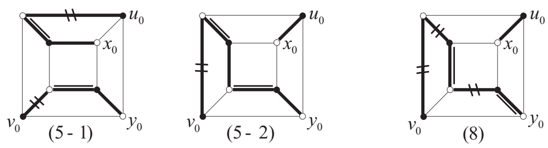

Figure 11. The possible spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ such that $(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 \neq \emptyset$.

If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (3), (6), (7) or (10) in Figure 7, then choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L}, see Figure 12(3), (6), (7), (10). If $(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 \neq \emptyset$, then the proof is similar to the above case. If $(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 = \emptyset$, then the set $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ crosses the positions 2 and 3, and therefore, M_{0R} has two choices for every case, see Figure 12. Then we can find a spanning 2-path $P'_{u_0x_0} + P'_{v_0y_0}$ in Q_4^0 passing through M_0, see Figure 12.

If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (1), (2), (4) or (9) in Figure 7, we observe that there exist two vertices in $V(Q_3^{0L})$ at distance 3, denoted by s_{0L}, t_{0L}, such that there is a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L} + s_{0L}t_{0L}$ passing through $M_{0L} \cup \{s_{0L}t_{0L}\}$, see Figure 13. Next, we can verify that there exists a Hamiltonian path $P_{s_{0R}t_{0R}}$ in Q_3^{0R} passing through M_{0R}. Hence $P_{u_0x_0} + P_{v_0y_0} + P_{s_{0R}t_{0R}} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - s_{0L}t_{0L}$ is the desired spanning 2-path in Q_4^0.

Case 4. $|M \cap E_5| = 3$. Let $M \cap E_5 = \{u_0u_1, v_0v_1, w_0w_1\}$, where $u_\alpha, v_\alpha, w_\alpha \in V(Q_4^\alpha)$. Now $|M \cap E_i| = 3$ for every $i \in [5]$ and $|M| = 15$. Hence there are two vertices of $\{u_\alpha, v_\alpha, w_\alpha\}$ in one partite set and one vertex in the other partite set. Otherwise, if $p(u_\alpha) = p(v_\alpha) = p(w_\alpha)$, then $|M_\alpha| \leq 5$, and therefore, $|M| \leq 13$, a contradiction. Without loss of generality, we may assume $p(u_\alpha) = p(v_\alpha) \neq p(w_\alpha)$.

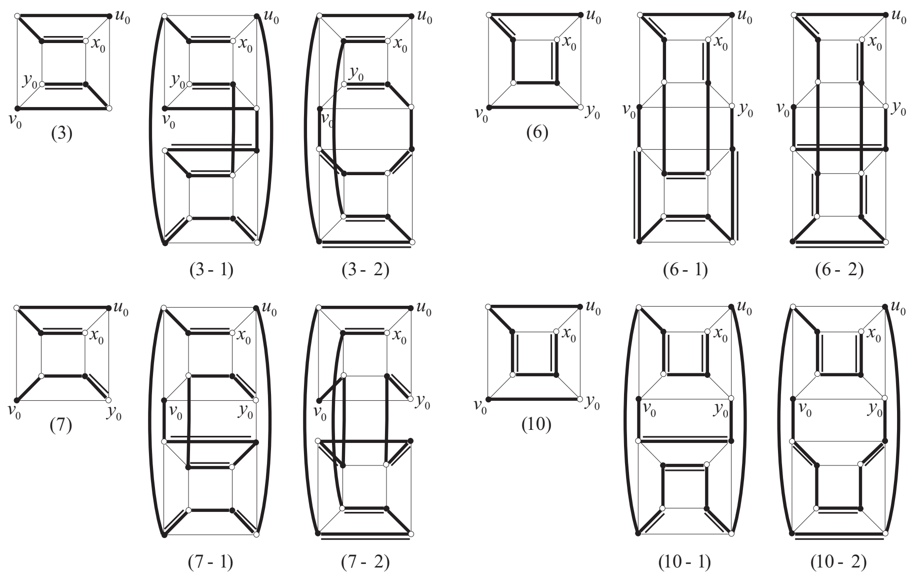

Figure 12. Spanning 2-paths $P'_{u_0x_0} + P'_{v_0y_0}$ in Q_4^0 passing through M_0.

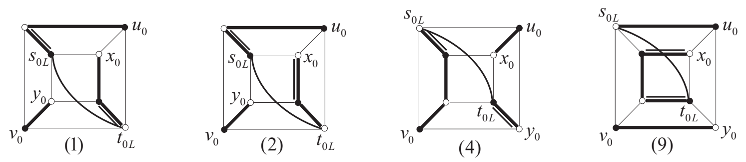

Figure 13. Spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L} + s_{0L}t_{0L}$ passing through $M_{0L} \cup \{s_{0L}t_{0L}\}$.

Split Q_4^α into two 3-cubes $Q_3^{\alpha L}$ and $Q_3^{\alpha R}$ at some position k such that $u_\alpha \in V(Q_3^{\alpha L})$ and $v_\alpha \in V(Q_3^{\alpha R})$. Without loss of generality, we may assume $k = 4$. Since $p(u_\alpha) = p(v_\alpha) \neq p(w_\alpha)$, by symmetry we may assume $w_\alpha \in V(Q_3^{\alpha L})$. Since $|M_0 \cap E_4| + |M_1 \cap E_4| = |M \cap E_4| = 3$, by symmetry we may assume $|M_0 \cap E_4| \leq 1$. Let $M_{\alpha\delta} = M_\alpha \cap E(Q_3^{\alpha\delta})$ for every $\delta \in \{L, R\}$.

Subcase 4.1. $M_0 \cap E_4 = \emptyset$. Since $p(u_1) \neq p(w_1)$ and M_1 is a matching in $Q_4^1 - u_1$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_1w_1}$ in Q_4^1 passing through M_1. Since v_1 has only one neighbor in Q_3^{1L}, we may choose a neighbor y_1 of v_1 on $P_{u_1w_1}$ such that $y_1 \in V(Q_3^{1R})$. Now $y_0 \in V(Q_3^{0R})$ and $p(u_0) = p(v_0) \neq p(w_0) = p(y_0)$. Since M_{0L} is a matching in $Q_3^{0L} - u_0$ and M_{0R} is a matching in $Q_3^{0R} - v_0$, by Lemma 2.2 there exist Hamiltonian paths $P_{u_0w_0}$ in Q_3^{0L} and $P_{v_0y_0}$ in Q_3^{0R} passing through M_{0L} and M_{0R}, respectively. Hence $P_{u_1w_1} + P_{u_0w_0} + P_{v_0y_0} + \{u_0u_1, w_0w_1, v_0v_1, y_0y_1\} - v_1y_1$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 14.

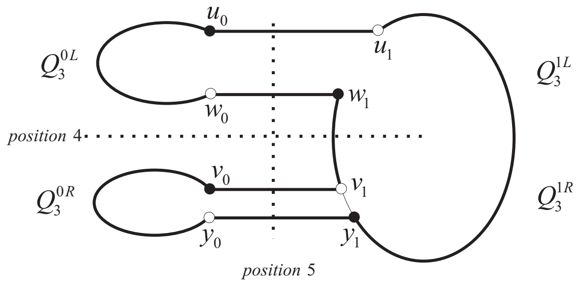

Figure 14. Illustration for Subcase 4.1.

Subcase 4.2. $|M_0 \cap E_4| = 1$. Now $|M_1 \cap E_4| = 2$. Let $M_0 \cap E_4 = \{s_{0L}s_{0R}\}$ and $M_1 \cap E_4 = \{a_{1L}a_{1R}, b_{1L}b_{1R}\}$, where $s_{0\delta} \in V(Q_3^{0\delta})$ and $a_{1\delta}, b_{1\delta} \in V(Q_3^{1\delta})$. Since $|M| = 15$, Q_5 has exactly two vertices uncovered by M, one in Q_3^{0L} and the other in Q_3^{1R}. Thus, $p(a_{1L}) \neq p(b_{1L})$, and $p(v_0) \neq p(s_{0R})$, and M_{1L} is a perfect matching in $Q_3^{1L} - \{u_1, w_1, a_{1L}, b_{1L}\}$. Since $p(u_1) \neq p(w_1)$ and $p(a_{1L}) \neq p(b_{1L})$, without loss of generality, we may assume $p(u_1) = p(b_{1L}) \neq p(w_1) = p(a_{1L})$. Thus, $p(v_1) = p(a_{1R}) \neq p(b_{1R})$.

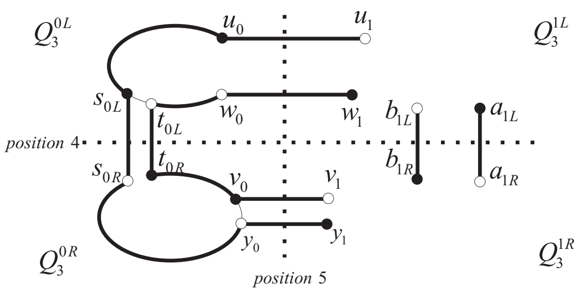

Figure 15. Illustration for Subcase 4.2.

Since $p(u_0) \neq p(w_0)$ and M_{0L} is a matching in $Q_3^{0L} - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0w_0}$ in Q_3^{0L} passing through M_{0L}, see Figure 15. Since $s_{0L} \notin \{u_0, w_0\}$, we may choose a neighbor t_{0L} of s_{0L} on $P_{u_0w_0}$ such that $t_{0R} \neq v_0$. Since $p(s_{0R}) \neq p(t_{0R})$ and M_{0R} is a matching in $Q_3^{0R} - s_{0R}$, by Lemma 2.2 there exists a Hamiltonian path $P_{s_{0R}t_{0R}}$ in Q_3^{0R} passing through M_{0R}, see Figure 15. Since $v_0 \notin \{s_{0R}, t_{0R}\}$, we may choose a neighbor y_0 of v_0 on $P_{s_{0R}t_{0R}}$ such that $y_1 \neq b_{1R}$. Now v_1, y_1, a_{1R}, b_{1R} are pairwise distinct vertices, and $p(v_1) = p(a_{1R}) \neq p(y_1) = p(b_{1R})$, and $d(v_1, y_1) = 1$, and M_{1R} is a matching in $Q_3^{1R} - \{v_1, a_{1R}\}$.

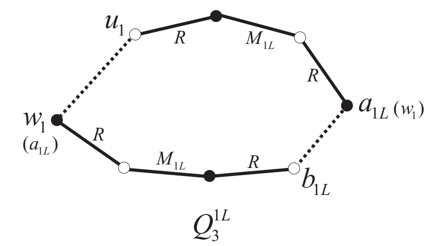

Figure 16. The spanning 2-path $P_{u_1 a_{1L}} + P_{w_1 b_{1L}}$ (or $P_{u_1 w_1} + P_{a_{1L} b_{1L}}$) in Q_3^{1L}.

If $d(a_{1R}, b_{1R}) = 1$, then by Lemma 2.1 there is a spanning 2-path $P_{v_1 y_1} + P_{a_{1R} b_{1R}}$ in Q_3^{1R} passing through M_{1R}, see Figure 17(1). Since M_{1L} is a perfect matching in $Q_3^{1L} - \{u_1, w_1, a_{1L}, b_{1L}\}$, we have $M_{1L} \cup \{u_1 w_1, a_{1L} b_{1L}\}$ is a perfect matching in $K(Q_3^{1L})$. By Theorem 1.1, there exists a perfect matching R in Q_3^{1L} such that $M_{1L} \cup \{u_1 w_1, a_{1L} b_{1L}\} \cup R$ forms a Hamiltonian cycle in $K(Q_3^{1L})$. Hence $M_{1L} \cup R$ forms a spanning 2-path in Q_3^{1L}. Note that each path of the spanning 2-path is an (R, M_{1L})-alternating path beginning with an edge in R and ending with an edge in R. So the number of vertices in each path is even. Since Q_5 is a bipartite graph, the two endpoints of each path have different parities. Hence one path joins the vertices u_1 and a_{1L}, and the other path joins the vertices w_1 and b_{1L}, see Figure 16 for example. Denote the spanning 2-path by $P_{u_1 a_{1L}} + P_{w_1 b_{1L}}$. Note that $s_{0L} t_{0L} \notin M$ and $v_0 y_0 \notin M$. Hence $P_{u_0 w_0} + P_{s_{0R} t_{0R}} + P_{u_1 a_{1L}} + P_{w_1 b_{1L}} + P_{v_1 y_1} + P_{a_{1R} b_{1R}} + \{u_0 u_1, w_0 w_1, v_0 v_1, y_0 y_1, a_{1L} a_{1R}, b_{1L} b_{1R}, s_{0L} s_{0R}, t_{0L} t_{0R}\} - \{v_0 y_0, s_{0L} t_{0L}\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 17(1).

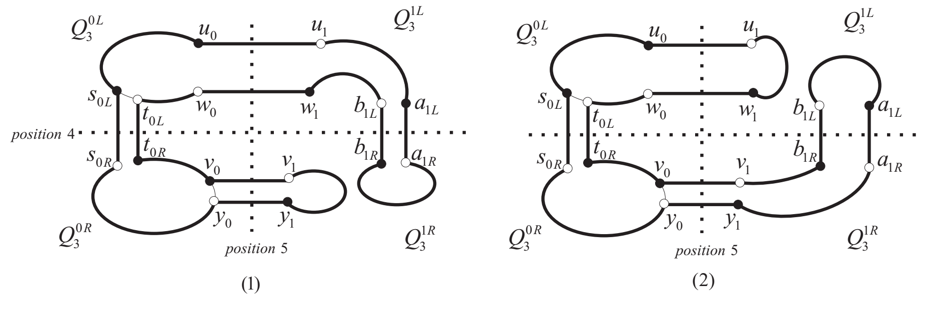

Figure 17. Illustration for Subcase 4.2.

If $d(a_{1R}, b_{1R}) = 3$, then $d(v_1, b_{1R}) = d(a_{1R}, y_1) = 1$. Since M_{1R} is a matching in $Q_3^{1R} - \{v_1, a_{1R}\}$, by Lemma 2.1 there is a spanning 2-path $P_{v_1b_{1R}} + P_{a_{1R}y_1}$ in Q_3^{1R} passing through M_{1R}, see Figure 17(2). Since $M_{1L} \cup \{u_1a_{1L}, w_1b_{1L}\}$ is a perfect matching in $K(Q_3^{1L})$, similar to the above case, there is a spanning 2-path $P_{u_1w_1} + P_{a_{1L}b_{1L}}$ in Q_3^{1L} passing through M_{1L}. Hence $P_{u_0w_0} + P_{s_{0R}t_{0R}} + P_{v_1b_{1R}} + P_{y_1a_{1R}} + P_{u_1w_1} + P_{a_{1L}b_{1L}} + \{u_0u_1, w_0w_1, v_0v_1, y_0y_1, a_{1L}a_{1R}, b_{1L}b_{1R}, s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{v_0y_0, s_{0L}t_{0L}\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 17(2). The proof of Theorem 1.3 is complete.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for their kind suggestions on the original manuscript.

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Received 27 September 2016
Revised 4 November 2016
Accepted 4 November 2016