DOMINATION PARAMETERS OF A GRAPH
AND ITS COMPLEMENT

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Abstract

A dominating set in a graph $G$ is a set $S$ of vertices such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$, and the domination number of $G$ is the minimum cardinality of a dominating set of $G$. Placing constraints on a dominating set yields different domination parameters, including total, connected, restrained, and clique domination numbers. In this paper, we study relationships among domination parameters of a graph and its complement.

Keywords: domination, complement, total domination, connected domination, clique domination, restrained domination.

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1. Introduction

The literature on the subject of domination parameters in graphs has been surveyed through 1997 and detailed in the two books [7, 8]. Our aim in this paper
is to study graph relationships involving domination parameters in a graph \( G \) and its complement \( \overline{G} \). We will also study relationships between the domination number of a graph and its total, restrained, clique and connected domination numbers.

For notation and graph theory terminology not defined herein, we refer the reader to [7]. Let \( G = (V, E) \) be a graph with vertex set \( V = V(G) \) of order \( n = |V| \) and edge set \( E = E(G) \) of size \( m = |E| \), and let \( v \) be a vertex in \( V \). The open neighborhood of \( v \) is \( N_G(v) = \{ u \in V \mid uv \in E \} \), and the closed neighborhood of \( v \) is \( N_G[v] = \{ v \} \cup N_G(v) \). We denote the complement of a graph \( G \) by \( \overline{G} \). For any vertex \( v \), we call the subgraph of \( G \) induced by \( N_G(v) \) the link of \( v \) and will denote it as \( L(v) \). We will denote the subgraph of \( \overline{G} \) induced by \( N_G(v) \) as \( \overline{L}(v) \).

For a set \( S \subseteq V \), its open neighborhood is the set \( N_G(S) = \bigcup_{v \in S} N(v) \), and its closed neighborhood is the set \( N_G[S] = N_G(S) \cup S \). The degree of a vertex \( v \) in \( G \) is \( d_G(v) = |N_G(v)| \). If the graph \( G \) is clear from the context, we simply write \( d(v), N(v), N[v], N(S) \) and \( N[S] \) rather than \( d_G(v), N_G(v), N_G[v], N_G(S) \) and \( N_G[S] \), respectively. A vertex is isolated in \( G \) if its degree in \( G \) is zero. A graph is isolate-free if it has no isolated vertex. For any set \( S \subset V(G) \), we denote the subgraph induced by \( S \) as \( G[S] \). The minimum and maximum degree among the vertices of \( G \) is denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. For a subset \( X \subseteq V \), the degree of a vertex \( v \) in \( X \), denoted \( d_X(v) \), is the number of vertices in \( X \) adjacent to \( v \); that is, \( d_X(v) = |N(v) \cap X| \). In particular, \( d_G(v) = d_v(v) \).

For sets \( A, B \subseteq V \), we let \( G[A, B] \), or simply \( [A, B] \) if the graph is clear from the context, denote the set of edges in \( G \) with one end in \( A \) and the other in \( B \). A nontrivial graph is a graph with at least two vertices. We say that a graph is \( F \)-free if it does not contain \( F \) as an induced subgraph. In particular, if \( F = K_{1,3} \), then we say that the graph is claw-free.

A dominating set in \( G = (V, E) \) is a set \( S \) of vertices of \( G \) such that every vertex in \( V \setminus S \) is adjacent to at least one vertex in \( S \), that is, \( N[S] = V \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). A dominating set of \( G \) of cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set. For subsets \( X, Y \subseteq V \), the set \( X \) dominates the set \( Y \) in \( G \) if \( Y \subseteq N[X] \). In particular, if \( X \) dominates \( V \), then \( X \) is a dominating set of \( G \). A vertex is called \( \gamma(G) \)-good if it is contained in some \( \gamma(G) \)-set, and \( \gamma(G) \)-bad, otherwise. In other words, a \( \gamma(G) \)-good vertex is contained in at least one \( \gamma(G) \)-set, while a \( \gamma(G) \)-bad vertex is not in any \( \gamma(G) \)-set. The minimum degree among the \( \gamma(G) \)-good (respectively, \( \gamma(G) \)-bad) vertices of \( G \) is denoted by \( \delta_{\gamma}(G) \) (respectively, \( \delta_{\gamma}(G) \)).

A total dominating set, abbreviated TD-set, of \( G \) is a set \( S \) of vertices of \( G \) such that every vertex in \( V(G) \) is adjacent to at least one vertex in \( S \). The total domination number of \( G \), denoted by \( \gamma_t(G) \), is the minimum cardinality of a TD-set of \( G \). A TD-set of \( G \) of cardinality \( \gamma_t(G) \) is called a \( \gamma_t(G) \)-set. Total domination is now well studied in graph theory. The literature on the subject of
total domination in graphs has been surveyed and detailed in the recent book [10]. A survey of total domination in graphs can also be found in [9].

Another way of looking at total domination is that a dominating set \( S \) is a TD-set if the induced subgraph \( G[S] \) has no isolated vertices. Placing the constraint that \( G[S] \) is connected (respectively, a complete graph) yields connected domination (respectively, clique domination). More formally, a dominating set \( S \) is a connected dominating set, abbreviated CD-set, of a graph \( G \) if the induced subgraph \( G[S] \) is connected. Every connected graph has a CD-set, since \( V \) is such a set. The connected domination number of \( G \), denoted by \( \gamma_c(G) \), is the minimum cardinality of a CD-set of \( G \), and a CD-set of \( G \) of cardinality \( \gamma_c(G) \) is called a \( \gamma_c(G) \)-set. Connected domination in graphs was first introduced by Sampathkumar et al. [14] and is now very well studied (see, for example, [4] and the recent papers [13, 15]). The study of connected domination has extensive application in the study of routing problems and virtual backbone based routing in wireless networks [6, 12, 17]. A subset \( S \subset V \) of vertices in a graph \( G = (V, E) \) is a dominating clique in \( G \) if \( S \) dominates \( V \) in \( G \) and \( G[S] \) is complete. If a graph \( G \) has a dominating clique, then the minimum cardinality among all dominating cliques of \( G \) is the clique domination number of \( G \), denoted by \( \gamma_{cl}(G) \).

A restrained dominating set of a graph \( G \) is a set \( S \) of vertices in \( G \) such that every vertex in \( V \setminus S \) is adjacent to a vertex in \( S \) and to some other vertex in \( V \setminus S \). Every connected graph has an RD-set, since \( V \) is such a set. The restrained domination number of \( G \), denoted by \( \gamma_r(G) \), is the minimum cardinality of an RD-set of \( G \), and an RD-set of \( G \) of cardinality \( \gamma_r(G) \) is called a \( \gamma_r(G) \)-set.

A proper vertex coloring of a graph \( G \) is an assignment of colors (elements of some set) to the vertices of \( G \), one color to each vertex, so that adjacent vertices are assigned distinct colors. If \( k \) colors are used, then the coloring is referred to as a \( k \)-coloring. In a given coloring of \( G \), a color class of the coloring is a set consisting of all those vertices assigned the same color. The vertex chromatic number \( \chi(G) \) of \( G \) is the minimum integer \( k \) such that \( G \) is \( k \)-colorable. A \( \chi(G) \)-coloring of \( G \) is a coloring of \( G \) with \( \chi(G) \) colors.

Given a graph \( G \), two edges are said to cross in the plane if in a drawing of the graph in the plane they intersect at a point that is not a vertex. The graph \( G \) is planar if it can be drawn in the plane with no edges crossing. The crossing number of \( G \), denoted \( cr(G) \), is the minimum number of crossing edges amongst all drawings of \( G \) in the plane. Note that if \( G \) is planar, then necessarily \( cr(G) = 0 \).

## 2. Bounds on the Domination Number

In this section, we determine bounds on the domination number. If the graph \( G \) is clear from the context, then we write \( \delta, \overline{\delta}, \Delta, \overline{\Delta}, \gamma \) and \( \overline{\gamma} \) rather than \( \delta(G) \), \( \overline{\delta}(G) \), \( \Delta(G) \), \( \overline{\Delta}(G) \), \( \gamma(G) \) and \( \overline{\gamma}(G) \).
\[ \delta(\overline{G}), \Delta(G), \Delta(\overline{G}), \gamma(G) \text{ and } \gamma(\overline{G}), \text{ respectively.} \]

### 2.1. Dominating the complement of a graph

We begin with results bounding the domination number of the complement of a graph. If \( v \) is an arbitrary vertex in a graph \( G \), then the closed neighborhood, \( N_G[v] \), of \( v \) is a dominating set of \( \overline{G} \). In particular, choosing \( v \) to be a vertex of minimum degree in \( G \), we have that \( \gamma(\overline{G}) \leq \delta(G) + 1 \). Furthermore, a set formed by taking a vertex from each color class of an arbitrary \( \chi(G) \)-coloring of \( G \) is a dominating set of \( \overline{G} \), and so \( \gamma(\overline{G}) \leq \chi(G) \). We state these well known observations formally as follows.

**Observation 1.** Let \( G \) be a graph. Then the following hold.

(a) \( \gamma(\overline{G}) \leq \delta(G) + 1 \).
(b) \( \gamma(\overline{G}) \leq \chi(G) \).

By Observation 1, \( \gamma(\overline{G}) \leq \Delta(G) + 1 \). From Brook’s Coloring Theorem [2], \( \chi(G) \leq \Delta(G) + 1 \) with equality if and only if \( G \) is the complete graph or an odd cycle. Noting that the domination number of the complement of any odd cycle \( C_n \), where \( n \geq 5 \), is equal to 2, we observe that if \( G \) is a graph, then \( \gamma(\overline{G}) \leq \Delta(G) + 1 \) with equality if and only if \( G \) is a complete graph. Next we give an upper bound on \( \gamma(\overline{G}) \) in terms of \( \gamma(G) \) and \( \delta(G) \).

**Theorem 2.** If \( G \) is a graph with \( \gamma(G) \geq 2 \), then \( \gamma(\overline{G}) \leq \left\lceil \frac{\delta(G)}{\gamma(G) - 1} \right\rceil + 1 \).

**Proof.** Let \( v \) be a vertex of \( G \) having degree \( \delta \). Let \( A = N_G(v) \), and so \( |A| = \delta \). Let \( k = \lceil \delta / (\gamma - 1) \rceil \) and partition the set \( A \) into \( k \) sets \( A_1, \ldots, A_k \) each of cardinality at most \( \gamma - 1 \). Thus, \( A = \bigcup_{i=1}^{k} A_i \), and \( 1 \leq |A_i| \leq \gamma - 1 \) for each \( i \), \( 1 \leq i \leq k \). In particular, we note that no set \( A_i \) dominates \( V \) in \( G \). For each set \( A_i \), \( 1 \leq i \leq k \), select one vertex \( a_i \in V \setminus A_i \) that is not dominated by \( A_i \) in \( G \), and let \( A' = \bigcup_{i=1}^{k} \{a_i\} \). Then, \( |A'| \leq k \) and \( A' \) dominates \( A \) in \( \overline{G} \). Therefore, the set \( A' \cup \{v\} \) is a dominating set of \( \overline{G} \), and so \( \gamma \leq |A'| + 1 \leq k + 1 = 1 + \lceil \delta / (\gamma - 1) \rceil \).

As an immediate consequence of Theorem 2, we have the following corollaries.

**Corollary 1.** If \( G \) is a graph with \( \gamma(\overline{G}) > \gamma(G) \geq 2 \), then \( \delta(G) \geq \gamma(G) \).

The next result shows that if \( G \) is a graph satisfying \( \gamma(G) \geq \gamma(\overline{G}) - 1 \), then the bound of Observation 1(a) can be improved.

**Corollary 2.** If \( G \) is a graph satisfying \( \gamma(G) \geq \gamma(\overline{G}) - 1 \), then \( \gamma(\overline{G}) < 2 + \sqrt{\delta(G)} \).

**Proof.** Let \( G \) be a graph satisfying \( \gamma \geq \gamma - 1 \). If \( \gamma = 1 \), then \( \gamma \leq 2 \), and the result follows. Accordingly, we may assume that \( \gamma \geq 2 \). By Theorem 2, \( \gamma \leq \lceil \delta / (\gamma - 1) \rceil + 1 \). This simplifies to \( (\gamma - 2)(\gamma - 1) < \delta \). By assumption, \( \gamma \geq \gamma - 1 \). Hence, \( (\gamma - 2)(\gamma - 2) < \delta \), and the result follows.
From Corollary 2, we have the following Nordhaus-Gaddum type result for graphs $G$ with $\gamma(G) = \gamma(G)$.

**Corollary 3.** If $G$ is a graph with $\gamma(G) = \gamma(G)$, then $\gamma(G) + \gamma(G) < 4 + \sqrt{\delta(G)}$.

### 2.2. Graphs $G$ with $\gamma(G) < \gamma(G)$

For a subset $S \subseteq V$ in a graph $G = (V, E)$, let $X_S(G)$ be the set of all vertices $x$ in $V \setminus S$ such that $x$ dominates $S$ in $G$; that is, $X_S(G) = \{x \in V \setminus S \mid S \subseteq N(x)\}$. We observe that if $X_S(G) = \emptyset$, then $S$ is a dominating set of $G$. We state this formally as follows.

**Observation 3.** If $G$ is a graph and $S \subseteq V$ satisfies $|S| < \gamma(G)$, then $X_S(G) \neq \emptyset$.

The following result establishes properties about the set $X_S(G)$.

**Theorem 4.** Let $G$ be a graph with $\gamma(G) = \gamma(G) + k$, where $k \geq 2$, and let $S$ be a $\gamma(G)$-set. It follows that $|X_S| \geq k$. Moreover, any subset $X' \subseteq X_S$ of size $|X_S| - k + 2$ is a dominating set of $G$.

**Proof.** By the definition of $X_S$, the set $S$ dominates $V \setminus (S \cup X_S)$ in $G$. This gives that $S \cup X_S$ is a dominating set of $G$, and so $\gamma(G) + |X_S| = |S| + |X_S| \geq \gamma(G) = \gamma(G) + k$ which implies $|X_S| \geq k$.

Let $u$ be an arbitrary vertex in $V \setminus S$, and let $U = N_G(u) \cap X_S$. Since $S$ dominates $V \setminus (S \cup X_S)$ in $G$, and $u$ dominates $X_S \setminus U$ in $G$, the set $S \cup U \cup \{u\}$ is a dominating set of $G$. Then, $\gamma(G) + k = \gamma(G) \leq \gamma(G) + |U| + 1$. Consequently, $k - 1 \leq |U| = |N_G(u) \cap (X_S \setminus X')| + |N_G(u) \cap X'| \leq k - 2 + |N_G(u) \cap X'|$ and so $|N_G(u) \cap X'| \neq \emptyset$. Hence, $X'$ dominates $V \setminus S$ in $G$. Since every vertex of $X'$ dominates $S$ in $G$, the set $X'$ is a dominating set of $G$. \[\square\]

Let $G$ be a graph with $\gamma(G) \leq \gamma(G) - 2$. Further, let $S$ be a $\gamma(G)$-set, and let $X = X_S(G)$. By definition of the set $X$, we note that the edges, $G[X, S]$, in $G$ between $X$ and $S$ induce a complete bipartite graph $K_{|X|, |S|}$. By Theorem 4, $\gamma \leq |X|$. Thus, we have the following corollary of Theorem 4.

**Corollary 4.** If $G$ is a graph with $\gamma(G) \leq \gamma(G) - 2$, then $G$ contains $K_{\gamma, \gamma}$ as a subgraph.

We observe from Corollary 4 that if $G$ is a graph that contains no 4-cycle (and thus does not contain $K_{r,r}$ for $r \geq 2$ as a subgraph), then $\gamma(G) = 1$ or $\gamma(G) \geq \gamma(G) - 1$. We establish next a property of claw-free graphs $G$ with $\gamma(G) \leq \gamma(G) - 2$. 

Theorem 5. Let $G$ be a graph with $\gamma(G) \leq \gamma(G) - 2$, and let $S$ be a $\gamma(G)$-set. If $G$ is claw-free, then $\gamma(G) \leq 2$ or $S \cup X_S(G)$ is a clique in $G$.

Proof. Let $G = (V, E)$ be a claw-free graph with $\gamma \leq \gamma - 2$, and let $S$ be a $\gamma(G)$-set. Following our earlier notation, let $X = X_S(G)$. By Theorem 4, the set $X$ is a dominating set of $G$, and so $\gamma \leq |X|$. Suppose that $G[S \cup X]$ is not a clique. Then there are two vertices, say $a$ and $b$, in $S \cup X$ that are not adjacent in $G$. Since every vertex in $X$ is by definition adjacent in $G$ to every vertex in $S$, we observe that both $a$ and $b$ are in $S$ or both $a$ and $b$ are in $X$. Let $c$ be an arbitrary vertex in $V \setminus \{a, b\}$.

We show that $c$ is dominated by $\{a, b\}$. Suppose to the contrary that $c$ is adjacent to neither $a$ nor $b$. On the one hand, suppose that $\{a, b\} \subseteq S$. Then, $c \notin X$. However since $X$ is a dominating set in $G$, there is a vertex $x \in X$ that is adjacent to $c$ in $G$. But then the set $\{a, b, c, x\}$ induces a claw in $G$, a contradiction. On the other hand, suppose that $\{a, b\} \subseteq X$. Then, $c \notin S$. However since $S$ is a dominating set in $G$, there is a vertex $x \in S$ that is adjacent to $c$ in $G$. But then the set $\{a, b, c, x\}$ induces a claw in $G$, a contradiction. In both cases, we have that $c$ is dominated by $\{a, b\}$, implying that $\{a, b\}$ is a dominating set in $G$, and therefore, that $\gamma \leq 2$.

Let $G$ be a claw-free graph with $\gamma(G) \leq \gamma(G) - 2$, and let $S$ be a $\gamma(G)$-set and let $X = X_S(G)$. If $\gamma(G) \geq 3$, then by Theorem 5, the set $S \cup X$ is a clique in $G$, and therefore, an independent set in $\overline{G}$. Hence, as an immediate consequence of Theorem 5, we have the following result, where $\alpha(G)$ and $\omega(G)$ denote the vertex independence number and the clique number, respectively, of $G$.

Corollary 5. If $G$ is a claw-free graph with $\gamma(G) \leq \gamma(G) - 2$, then $\gamma(G) \leq 2$ or $\gamma(G) \leq \omega(G)/2 = \alpha(G)/2$.

2.3. Graphs $G$ with a $\gamma(G)$-bad vertex

Recall that a vertex in a graph $G$ is a $\gamma(G)$-bad vertex if it is contained in no $\gamma(G)$-set. We establish next an upper bound on the sum of the domination numbers of a graph $G$ and its complement $\overline{G}$ in terms of the degree of a $\gamma(G)$-bad vertex.

Theorem 6. If a graph $G$ contains a vertex $v$ that is a $\gamma(\overline{G})$-bad vertex, then $\gamma(G) + \gamma(\overline{G}) \leq d_G(v) + 3$.

Proof. Let $G = (V, E)$ be a graph that contains a $\gamma(\overline{G})$-bad vertex $v$. Let $A = N_G(v)$, and so $|A| = d_G(v)$. Since the set $A \cup \{v\}$ is a dominating set in $\overline{G}$, we have that $\gamma(\overline{G}) \leq |A| + 1$. However if $\gamma(\overline{G}) = |A| + 1$, then $A \cup \{v\}$ is a $\gamma(\overline{G})$-set, contradicting the fact that $v$ is a $\gamma(\overline{G})$-bad vertex. Therefore, $\gamma(\overline{G}) < |A| + 1$, or, equivalently, $|A| \geq \gamma(\overline{G})$. 

Let $B = V \setminus N_G[v]$. If $B = \emptyset$, then $v$ dominates $V$ in the graph $G$, implying that $v$ is isolated in $\overline{G}$ and therefore belongs to every $\gamma(\overline{G})$-set, a contradiction. Hence, $B \neq \emptyset$. We show next that each vertex in $B$ has at least $\gamma - 1$ neighbors in $G$ that belong to the set $A$. Let $x \in B$, and let $A_x = A \cap N_G(x)$. Then in the graph $\overline{G}$, the vertex $x$ dominates the set $A \setminus A_x$. Thus since the vertex $v$ dominates the set $B$ in $\overline{G}$, we have that the set $A_x \cup \{v, x\}$ is a dominating set in $\overline{G}$, implying that $\gamma(\overline{G}) \leq |A_x| + 2$. However if $\gamma(\overline{G}) = |A_x| + 2$, then $A_x \cup \{v, x\}$ is a $\gamma(\overline{G})$-set, contradicting the fact that $v$ is a $\gamma(\overline{G})$-bad vertex. Therefore, $\gamma(\overline{G}) < |A_x| + 2$, or, equivalently, $\gamma(\overline{G}) \leq |A_x| + 1$. Thus in the graph $\overline{G}$, we have that $d_A(x) = |A_x| \geq \gamma(\overline{G}) - 1$. This is true for every vertex $x \in B$.

Recall that $|A| \geq \gamma(\overline{G})$. Let $A'$ be an arbitrary subset of $A$ of cardinality $\gamma(\overline{G}) - 2$, and let $A'' = A \setminus A'$. Thus, $|A'| = \gamma(\overline{G}) - 2$ and $|A''| = |A| - |A'| = d_G(v) - \gamma(\overline{G}) + 2$. Since $d_A(x) \geq \gamma(\overline{G}) - 1$ for every vertex $x \in B$, the set $A''$ dominates the set $B$ in $G$. Thus, $A'' \cup \{v\}$ is a dominating set in $G$, implying that $\gamma(G) \leq |A''| + 1 = d_G(v) - \gamma(\overline{G}) + 3$.

As a consequence of Theorem 6, we have the following result.

**Corollary 6.** If $G$ is an $r$-regular graph that contains a $\gamma(\overline{G})$-bad vertex, then $\gamma(G) + \gamma(\overline{G}) \leq r + 3$.

### 2.4. Domination and planarity

In this section, we study some relationships between planarity, the crossing number of $G$ and the domination number of $\overline{G}$. Fundamental to our results in this section is the famous Four Color Theorem.

**Theorem 7** [1]. If $G$ is a planar graph, then $\chi(G) \leq 4$.

We first establish the following upper bound on the domination number of the complement of a graph. For this purpose, for a vertex $v$ in a graph $G$, we denote by $G_v$ the subgraph of $G$ induced by the neighbors of $v$; that is, $G_v = G[N(v)]$. If $\mathcal{C}$ is a minimum coloring of the vertices of $G_v$, and $S$ is a set of vertices comprising of exactly one vertex from each color class of $\mathcal{C}$, then the set $S \cup \{v\}$ forms a dominating set of $\overline{G}$, implying that $\gamma(\overline{G}) \leq |\mathcal{C}| + 1 = \chi(G_v) + 1$. We state this formally as follows.

**Observation 8.** If $v$ is an arbitrary vertex in a graph $G$, then $\gamma(\overline{G}) \leq \chi(G_v) + 1$.

As a consequence of Theorem 7 and Observation 8, we have the following results.

**Corollary 7.** If a graph $G$ contains a vertex $v$ with the property that $G_v$ is a planar graph, then $\gamma(\overline{G}) \leq 5$. 

Corollary 8. If a graph $G$ satisfies $\gamma(G) > 2\text{cr}(G)$, then $\gamma(G) \leq 5$.

\textbf{Proof.} Let $G^*$ be a drawing of $G$ in the plane with exactly $\text{cr}(G)$ crossing edges, and let $S$ be the set of vertices of $G$ incident with at least one crossing edge of $G^*$. Clearly, $|S| \leq 2\text{cr}(G)$. Since, by assumption, $\gamma(G) > 2\text{cr}(G)$, it follows there exists some vertex $v$ in $G$ that is not dominated by $S$. This implies that $G_v$ is a planar graph. Thus, by Corollary 7, $\gamma(G) \leq 5$. 

3. Total, Connected, Restrained, and Clique Domination

In this section, we establish relationships involving the domination, total domination, restrained domination, connected domination and clique domination numbers of a graph. We begin with the following lemma.

Lemma 9. If there exists a $\gamma(G)$-set for a graph $G$ that is not a dominating set in $\overline{G}$, then $\gamma_t(G) \leq \gamma_c(G) \leq \gamma(G) + 1$.

\textbf{Proof.} Let $S$ be a $\gamma(G)$-set in a graph $G = (V, E)$ that is not a dominating set in $\overline{G}$. Then there exists a vertex $v \in V \setminus S$ that is not adjacent to any vertex of $S$ in $\overline{G}$. Hence in $G$, the vertex $v$ is adjacent to every vertex of $S$, implying that the graph $G[S \cup \{v\}]$ is connected. Since every superset of a dominating set is also a dominating set, the set $S \cup \{v\}$ is a CD-set, and so $\gamma_c(G) \leq |S \cup \{v\}| = \gamma(G) + 1$. Since the total domination of a graph is at most its connected domination number, the desired result follows from the observation that $\gamma_t(G) \leq \gamma_c(G)$. 

By the contrapositive of Lemma 9, we note that if a graph $G$ satisfies $\gamma_t(G) \geq \gamma(G) + 2$, then every $\gamma(G)$-set is a dominating set in $\overline{G}$. Further as a consequence of Lemma 9 and the well-known result due to Jaeger and Payan [11] that if $G$ is a graph of order $n$, then $\gamma(G) \gamma(\overline{G}) \leq n$, we have the following result.

Corollary 10. Let $G$ be a graph of order $n$ satisfying $\gamma(G) < \gamma(\overline{G})$. Then the following holds.

(a) $\gamma_t(G) \leq \gamma_c(G) \leq \gamma(G) + 1$.
(b) $\gamma_c(G) \leq (1 + \sqrt{4n + 1})/2$.

\textbf{Proof.} Part (a) is an immediate consequence of Lemma 9. To prove part (b), let $G$ be a graph of order $n$ satisfying $\gamma(G) < \gamma(\overline{G})$. By part (a) and our assumption that $\gamma(G) \leq \gamma(\overline{G}) - 1$, we have that $\gamma_c(G) \leq \gamma(G) + 1 \leq \gamma(\overline{G})$. Applying the result due to Jaeger and Payan, we therefore have that $(\gamma_c(G) - 1)\gamma_c(G) \leq \gamma(G)\gamma(\overline{G}) \leq n$. Solving for $\gamma_c(G)$, we have that $\gamma_c(G) \leq (1 + \sqrt{4n + 1})/2$. 

In the following result, we consider the case when $\gamma(G) \leq \gamma(\overline{G}) + 1$. 


Theorem 9. Let $G$ be a graph satisfying $\gamma(G) \leq \gamma(\overline{G}) + 1$. Then the following holds.

(a) If both $G$ and $\overline{G}$ are connected, then $\gamma_c(G) \leq \gamma(G) + 1$ or $\gamma_c(\overline{G}) \leq \gamma(\overline{G}) + 1$.

(b) If both $G$ and $\overline{G}$ are isolate-free, then $\gamma_t(G) \leq \gamma(G) + 1$ or $\gamma_t(\overline{G}) \leq \gamma(\overline{G}) + 1$.

Proof. Let $G = (V, E)$, and let $S$ be a $\gamma(G)$-set in the graph. We first establish part (a). Suppose that both $G$ and $\overline{G}$ are connected. If $G[S]$ is connected, then $S$ is a CD-set in $G$, implying that $\gamma_c(G) \leq |S| = \gamma(G)$. Hence we may assume that $G[S]$ is not connected, for otherwise part (a) is immediate. This implies that $\overline{G}[S]$ is connected. If the set $S$ is not a dominating set in $\overline{G}$, then by Lemma 9, we have that $\gamma_c(\overline{G}) \leq \gamma(\overline{G}) + 1$. If the set $S$ is a dominating set in $\overline{G}$, then $S$ is a CD-set in $\overline{G}$, implying that $\gamma_c(\overline{G}) \leq |S| = \gamma(G) \leq \gamma(\overline{G}) + 1$. This proves part (a).

Next we prove part (b). Suppose that both $G$ and $\overline{G}$ are isolate-free. If $G[S]$ is isolate-free, then $S$ is a TD-set in $G$, implying that $\gamma_t(G) \leq |S| = \gamma(G)$. Hence we may assume that $G[S]$ contains an isolated vertex, for otherwise part (b) is immediate. This implies that $\overline{G}[S]$ is connected. If the set $S$ is not a dominating set in $\overline{G}$, then by Lemma 9 we have that $\gamma_t(G) \leq \gamma(G) + 1$. If the set $S$ is a dominating set in $\overline{G}$, then $S$ is a TD-set in $\overline{G}$, implying that $\gamma_t(\overline{G}) \leq |S| = \gamma(G) \leq \gamma(\overline{G}) + 1$. This proves part (b).

We establish next an upper bound on the total domination number of a graph in terms of its domination number and the domination number of its complement.

Theorem 10. Let $G$ be an isolate-free graph, and let $S$ be a $\gamma(G)$-set. If $s$ is the number of isolated vertices in $G[S]$, then $\gamma_t(G) \leq \gamma(G) + \lceil s/(\gamma(\overline{G}) - 1) \rceil$.

Proof. Let $G = (V, E)$. Since $G$ is isolate-free, we note that $\gamma(\overline{G}) \geq 2$. Let $I$ be the set of isolated vertices in $G[S]$, and so $s = |I|$. Let $k = \lceil s/(\gamma - 1) \rceil$, and partition the set $I$ into $k$ sets $I_1, \ldots, I_k$ each of cardinality at most $\gamma - 1$. Thus, $I = \bigcup_{i=1}^{k} I_i$ and $1 \leq |I_i| \leq \gamma - 1$ for each $i$, $1 \leq i \leq k$. In particular, we note that no set $I_i$ dominates $V$ in $\overline{G}$. For each set $I_i$, $1 \leq i \leq k$, select one vertex $w_i \in V \setminus I_i$ that is not dominated by $I_i$ in $\overline{G}$, and let $W = \bigcup_{i=1}^{k} \{w_i\}$. Then, $|W| \leq k$. We note that in the graph $G$, the vertex $w_i$ is adjacent to every vertex of $I_i$, and so $S \cup W$ is a TD-set in $G$. Hence, $\gamma_t(G) \leq |S \cup W| \leq |S| + |W| \leq \gamma(G) + k = \gamma(G) + \lceil s/(\gamma - 1) \rceil$.

As an immediate consequence of Theorem 10, we have the following upper bound on the total domination number of a graph.

Corollary 11. If $G$ is an isolate-free graph, then $\gamma_t(G) \leq \gamma(G) + \left\lceil \frac{\gamma(G)}{\gamma(\overline{G}) - 1} \right\rceil$.

Theorem 11. If $G$ is a graph with $\gamma_t(G) \geq \gamma(G) + 2$, then $\gamma_t(\overline{G}) \leq 1 + \left\lceil \frac{\delta(G)}{\gamma(G)} \right\rceil$. 
Proof. Let $G = (V, E)$ be a graph with $\gamma_t(G) \geq \gamma(G)+2$, and let $v$ be a vertex of $G$ having degree $\delta(G)$. Let $A = N_G(v)$, and so $|A| = \delta(G)$. Let $k = \lceil \delta(G)/\gamma(G) \rceil$ and partition the set $A$ into $k$ sets $A_1, \ldots, A_k$ each of cardinality at most $\gamma(G)$. Thus, $A = \bigcup_{i=1}^{k} A_i$ and $1 \leq |A_i| \leq \gamma(G)$ for each $i$, $1 \leq i \leq k$. If the set $A_i$ dominates $V \setminus N_G[v]$ in $G$ for some $i$, $1 \leq i \leq k$, then the set $A_i \cup \{v\}$ is a TD-set in $G$, implying that $\gamma_t(G) \leq |A_i| + 1 \leq \gamma(G) + 1$, a contradiction. Therefore, no set $A_i$ dominates $V \setminus N_G[v]$ in $G$. For each set $A_i$, $1 \leq i \leq k$, select one vertex $a_i \in V \setminus N_G[v]$ that is not dominated by $A_i$ in $G$, and let $A' = \bigcup_{i=1}^{k} \{a_i\}$. Then, $|A'| \leq k$ and $A'$ dominates $A$ in $\overline{G}$. Therefore, the set $A' \cup \{v\}$ is a TD-set in $\overline{G}$, and so $\gamma_t(\overline{G}) \leq |A'| + 1 \leq k + 1 = 1 + \lceil \delta(G)/\gamma(G) \rceil$. 

Next we consider the restrained domination number. We first prove a general lemma.

Lemma 12. If a graph $G$ has a $\gamma(G)$-set $S$ such that the induced subgraph $G[V \setminus S]$ has an isolated vertex, then $\gamma(\overline{G}) \leq 3$.

Proof. Let $S$ be a $\gamma(G)$-set such that $G[V \setminus S]$ has an isolated vertex, say $w$. If $G[S]$ has an isolated vertex $v$, then $\{v, w\}$ is dominating set of $\overline{G}$, and so $\gamma(\overline{G}) \leq 2$. If $G[S]$ contains no isolated vertices, then by the minimality of $S$, for each $v \in S$, there exists a vertex, say $v' \in V \setminus S$, such that $N(v') \cap S = \{v\}$. In this case, the set $\{v, w, v'\}$ is a dominating set of $\overline{G}$, implying that $\gamma(\overline{G}) \leq 3$.

As an immediate consequence of Lemma 12, we have the following result.

Corollary 13. If a graph $G$ has $\gamma(\overline{G}) \geq 4$, then every $\gamma(G)$-set is a $\gamma_r(G)$-set. In particular, $\gamma(G) = \gamma_r(G)$.

We close this section with two results about the clique domination number of a graph.

Theorem 12. If $G$ is a graph with $\gamma_t(G) \geq \gamma(G) + 2$, then $\gamma(\overline{G}) \leq \gamma(G)$. Moreover, if $G$ is claw-free, then $\gamma(\overline{G}) \leq 3$.

Proof. Let $G$ be a graph with $\gamma_t(G) \geq \gamma(G) + 2$, and let $S$ be a $\gamma(G)$-set. Further, let $I(S)$ be the set of isolated vertices in $G[S]$. If $I(S) = \emptyset$, then $S$ is a TD-set of $G$, implying that $\gamma_t(G) \leq |S| = \gamma(G)$, a contradiction. Hence, $I(S) \neq \emptyset$. We show that $I(S)$ dominates $\overline{G}$. Suppose to the contrary that there exists a vertex $v$ that is not adjacent to any vertex of $I(S)$ in $\overline{G}$. Then in the graph $G$, the vertex $v$ is adjacent to every vertex of $I(S)$, implying that $S \cup \{v\}$ is a TD-set for $G$, and so $\gamma_t(G) \leq |S| + 1 = \gamma(G) + 1$, a contradiction. Hence, the set $I(S)$ dominates $\overline{G}$. Since $I(S)$ is an independent set in $G$, it forms a clique in $\overline{G}$. Therefore, $I(S)$ is a dominating clique in $\overline{G}$, implying that $\gamma(\overline{G}) \leq |I(S)| \leq \gamma(G)$.

Now, suppose that $G$ is claw free. If $|I(S)| \leq 3$, then the result follows. Hence, we may assume that $|I(S)| \geq 4$ and there exists a subset $\{a, b, c\} \subseteq I(S)$
that is not a dominating set in \( \overline{G} \). Then there exists a vertex \( v \) that is not adjacent to \( a, b, \) or \( c \) in \( \overline{G} \). But then in the graph \( G \), we have that \( \{a, b, c, v\} \) induces a claw, a contradiction. Therefore, every subset of \( I(S) \) of cardinality 3 is a dominating set in \( \overline{G} \), implying that \( \gamma_{cl}(\overline{G}) \leq 3 \). \( \blacksquare \)

4. Bounds on the Domination Number of a Graph in Terms of the Adjacency Matrix of its Complement

We begin this section by stating two well-known theorems. The first result counts the number of walks of length \( k \) for an arbitrary positive integer \( k \) in a graph (see [3]; see also Theorem 1.17 in [5]). The second result is a consequence of a result due to Vizing [16] and provides an upper bound for the domination number of a graph in terms of its order and size.

**Theorem 13** [3]. Let \( G \) be a graph of order \( n \) with \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and with adjacency matrix \( A \). For each positive integer \( k \), the number of different walks of length \( k \) from the vertex \( v_i \) to the vertex \( v_j \) is the \((i, j)-entry in the matrix \( A^k \).\)

**Theorem 14** [16]. If \( G \) is a graph of order \( n \) and size \( m \), then \( \gamma(G) \leq n + 1 - \sqrt{1 + 2m} \).

Let \( G \) be a graph of order \( n \) with \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and with adjacency matrix \( A \), and let \( a_{ij}^{(k)} \) denote the \((i, j)\)-entry in \( A^k \). Recall that if \( v \) is a vertex in \( G \), then the subgraph of \( G \) induced by \( N_G(v) \) is called the link of \( v \) and is denoted by \( \mathcal{L}(v) \), while the subgraph of \( \overline{G} \) induced by \( N_{\overline{G}}(v) \) is denoted \( \overline{\mathcal{L}}(v) \). Theorem 13 implies that the \((i, i)\)-entry of \( A^2 \), \( 1 \leq i \leq n \), is the degree \( d_G(v_i) \) of \( v_i \), and the \((i, i)\)-entry of \( A^3 \), \( 1 \leq i \leq n \), is equal to twice the number of edges in \( \mathcal{L}(v_i) \). Suppose that \( a_{ii}^{(3)} < a_{ii}^{(2)} \) for some \( i, 1 \leq i \leq n \). Since \( a_{ii}^{(2)} = d_G(v_i) \) and \( \frac{1}{2}a_{ii}^{(3)} \) is the number of edges in \( \mathcal{L}(v_i) \), this implies that \( \mathcal{L}(v_i) \) contains an isolated vertex, \( v \) say. Thus the set \( \{v, v_i\} \) is a dominating set in the graph \( G \), implying that \( \gamma(G) \leq 2 \). We state this formally as follows.

**Observation 15.** Let \( G \) be an isolate-free graph of order \( n \) with adjacency matrix \( A \). If the \((i, i)\)-entry of \( A^3 \) is less than the \((i, i)\)-entry of \( A^2 \) for some \( i, 1 \leq i \leq n \), then \( \gamma(G) \leq 2 \).

Using Observation 15, we obtain the following bound on the domination number of the complement of a graph.

**Theorem 16.** Let \( G \) be a graph of order \( n \) with adjacency matrix \( A \), and let \( a_{ij}^{(k)} \) denote the \((i, j)\)-entry in \( A^k \). For every \( i, 1 \leq i \leq n \), we have that

\[
\gamma(\overline{G}) \leq a_{ii}^{(2)} + 2 - \sqrt{1 + a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)}}.
\]
Proof. Let \( i \) be an arbitrary integer with \( 1 \leq i \leq n \). Since \( a_{ii}^{(2)} = d_G(v_i) \) and \( \frac{1}{2}a_{ii}^{(3)} \) is the number of edges in \( L(v_i) \), this implies that \( \overline{L}(v_i) \) has order \( a_{ii}^{(2)} \) and size

\[
\left( \frac{a_{ii}^{(2)}}{2} \right) - \frac{1}{2}a_{ii}^{(3)} = \frac{1}{2} \left( a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)} \right).
\]

Thus, by Theorem 14, we have that

\[
\gamma(\overline{L}(v_i)) \leq a_{ii}^{(2)} + 1 - \sqrt{1 + a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)}}.
\]

The desired bound now follows from the observation that every \( \gamma(\overline{L}(v_i)) \)-set can be extended to a dominating set in \( \overline{G} \) by adding to it the vertex \( v_i \), and so \( \gamma(\overline{G}) \leq \gamma(\overline{L}(v_i)) + 1. \)

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