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DOMINATION PARAMETERS OF A GRAPH AND ITS COMPLEMENT

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Abstract

A dominating set in a graph G is a set S of vertices such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S, and the domination number of G is the minimum cardinality of a dominating set of G. Placing constraints on a dominating set yields different domination parameters, including total, connected, restrained, and clique domination numbers. In this paper, we study relationships among domination parameters of a graph and its complement.

Keywords: domination, complement, total domination, connected domination, clique domination, restrained domination.

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1. Introduction

The literature on the subject of domination parameters in graphs has been surveyed through 1997 and detailed in the two books [7, 8]. Our aim in this paper is to study graph relationships involving domination parameters in a graph G and its complement $\overline{G}$. We will also study relationships between the domination number of a graph and its total, restrained, clique and connected domination numbers.

For notation and graph theory terminology not defined herein, we refer the reader to [7]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. We denote the *complement* of a graph G by $\overline{G}$. For any vertex v, we call the subgraph of G induced by $N_G(v)$ the *link* of v and will denote it as $\mathcal{L}(v)$. We will denote the subgraph of $\overline{G}$ induced by $N_G(v)$ as $\overline{\mathcal{L}}(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. The degree of a vertex v in G is $d_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively. A vertex is *isolated* in G if its degree in G is zero. A graph is *isolate-free* if it has no isolated vertex. For any set $S \subset V(G)$, we denote the subgraph induced by S as $G[S]$. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $X \subseteq V$, the *degree of a vertex v in X*, denoted $d_X(v)$, is the number of vertices in X adjacent to v; that is, $d_X(v) = |N(v) \cap X|$. In particular, $d_G(v) = d_V(v)$.

For sets $A, B \subseteq V$, we let $G[A, B]$, or simply $[A, B]$ if the graph is clear from the context, denote the set of edges in G with one end in A and the other in B. A *nontrivial graph* is a graph with at least two vertices. We say that a graph is *F-free* if it does not contain F as an induced subgraph. In particular, if $F = K_{1,3}$, then we say that the graph is *claw-free*.

A *dominating set* in $G = (V, E)$ is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to at least one vertex in S, that is, $N[S] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a *$\gamma(G)$-set*. For subsets $X, Y \subseteq V$, the set X *dominates* the set Y in G if $Y \subseteq N[X]$. In particular, if X dominates V, then X is a dominating set of G. A vertex is called *$\gamma(G)$-good* if it is contained in some $\gamma(G)$-set, and *$\gamma(G)$-bad*, otherwise. In other words, a $\gamma(G)$-good vertex is contained in at least one $\gamma(G)$-set, while a $\gamma(G)$-bad vertex is not in any $\gamma(G)$-set. The minimum degree among the $\gamma(G)$-good (respectively, $\gamma(G)$-bad) vertices of G is denoted by $\delta_g(G)$ (respectively, $\delta_b(G)$).

A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a *$\gamma_t(G)$-set*. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book [10]. A survey of total domination in graphs can also be found in [9].

Another way of looking at total domination is that a dominating set S is a TD-set if the induced subgraph $G[S]$ has no isolated vertices. Placing the constraint that $G[S]$ is connected (respectively, a complete graph) yields *connected domination* (respectively, *clique domination*). More formally, a dominating set S is a *connected dominating set*, abbreviated CD-set, of a graph G if the induced subgraph $G[S]$ is connected. Every connected graph has a CD-set, since V is such a set. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a CD-set of G, and a CD-set of G of cardinality $\gamma_c(G)$ is called a *$\gamma_c(G)$-set*. Connected domination in graphs was first introduced by Sampathkumar *et al.* [14] and is now very well studied (see, for example, [4] and the recent papers [13, 15]). The study of connected domination has extensive application in the study of routing problems and virtual backbone based routing in wireless networks [6, 12, 17]. A subset $S \subset V$ of vertices in a graph $G = (V, E)$ is a *dominating clique* in G if S dominates V in G and $G[S]$ is complete. If a graph G has a dominating clique, then the minimum cardinality among all dominating cliques of G is the *clique domination number* of G, denoted by $\gamma_{\rm cl}(G)$.

A *restrained dominating set* of a graph G is a set S of vertices in G such that every vertex in $V \setminus S$ is adjacent to a vertex in S and to some other vertex in $V \setminus S$. Every connected graph has an RD-set, since V is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of an RD-set of G, and an RD-set of G of cardinality $\gamma_r(G)$ is called a *$\gamma_r(G)$-set*.

A *proper vertex coloring* of a graph G is an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If k colors are used, then the coloring is referred to as a *k-coloring*. In a given coloring of G, a *color class* of the coloring is a set consisting of all those vertices assigned the same color. The *vertex chromatic number* $\chi(G)$ of G is the minimum integer k such that G is k-colorable. A *$\chi(G)$-coloring* of G is a coloring of G with $\chi(G)$ colors.

Given a graph G, two edges are said to *cross* in the plane if in a drawing of the graph in the plane they intersect at a point that is not a vertex. The graph G is *planar* if it can be drawn in the plane with no edges crossing. The *crossing number* of G, denoted $cr(G)$, is the minimum number of crossing edges amongst all drawings of G in the plane. Note that if G is planar, then necessarily $cr(G) = 0$.

2. Bounds on the Domination Number

In this section, we determine bounds on the domination number. If the graph G is clear from the context, then we write δ, $\overline{\delta}$, Δ, $\overline{\Delta}$, γ and $\overline{\gamma}$ rather than $\delta(G)$,

$\delta(\overline{G})$, $\Delta(G)$, $\Delta(\overline{G})$, $\gamma(G)$ and $\gamma(\overline{G})$, respectively.

2.1. Dominating the complement of a graph

We begin with results bounding the domination number of the complement of a graph. If v is an arbitrary vertex in a graph G, then the closed neighborhood, $N_G[v]$, of v is a dominating set of $\overline{G}$. In particular, choosing v to be a vertex of minimum degree in G, we have that $\gamma(\overline{G}) \le \delta(G) + 1$. Furthermore, a set formed by taking a vertex from each color class of an arbitrary $\chi(G)$-coloring of G is a dominating set of $\overline{G}$, and so $\gamma(\overline{G}) \le \chi(G)$. We state these well known observations formally as follows.

Observation 1. *Let G be a graph. Then the following hold.*
(a) $\gamma(\overline{G}) \le \delta(G) + 1$.
(b) $\gamma(\overline{G}) \le \chi(G)$.

By Observation 1, $\gamma(\overline{G}) \le \Delta(G) + 1$. From Brook's Coloring Theorem [2], $\chi(G) \le \Delta(G) + 1$ with equality if and only if G is the complete graph or an odd cycle. Noting that the domination number of the complement of any odd cycle C_n, where $n \ge 5$, is equal to 2, we observe that if G is a graph, then $\gamma(\overline{G}) \le \Delta(G) + 1$ with equality if and only if G is a complete graph. Next we give an upper bound on $\gamma(\overline{G})$ in terms of $\gamma(G)$ and $\delta(G)$.

Theorem 2. *If G is a graph with $\gamma(G) \ge 2$, then $\gamma(\overline{G}) \le \left\lceil \frac{\delta(G)}{\gamma(G)-1} \right\rceil + 1$.*

Proof. Let v be a vertex of G having degree δ. Let $A = N_G(v)$, and so $|A| = \delta$. Let $k = \lceil \delta/(\gamma - 1) \rceil$ and partition the set A into k sets $A_1, \ldots, A_k$ each of cardinality at most $\gamma - 1$. Thus, $A = \bigcup_{i=1}^{k} A_i$ and $1 \le |A_i| \le \gamma - 1$ for each i, $1 \le i \le k$. In particular, we note that no set A_i dominates V in G. For each set A_i, $1 \le i \le k$, select one vertex $a_i \in V \setminus A_i$ that is not dominated by A_i in G, and let $A' = \bigcup_{i=1}^{k} \{a_i\}$. Then, $|A'| \le k$ and A' dominates A in $\overline{G}$. Therefore, the set $A' \cup \{v\}$ is a dominating set of $\overline{G}$, and so $\overline{\gamma} \le |A'| + 1 \le k + 1 = 1 + \lceil \delta/(\gamma - 1) \rceil$. ■

As an immediate consequence of Theorem 2, we have the following corollaries.

Corollary 1. *If G is a graph with $\gamma(\overline{G}) > \gamma(G) \ge 2$, then $\delta(G) \ge \gamma(G)$.*

The next result shows that if G is a graph satisfying $\gamma(G) \ge \gamma(\overline{G}) - 1$, then the bound of Observation 1(a) can be improved.

Corollary 2. *If G is a graph satisfying $\gamma(G) \ge \gamma(\overline{G}) - 1$, then $\gamma(\overline{G}) < 2 + \sqrt{\delta(G)}$.*

Proof. Let G be a graph satisfying $\gamma \ge \overline{\gamma} - 1$. If $\gamma = 1$, then $\overline{\gamma} \le 2$, and the result follows. Accordingly, we may assume that $\gamma \ge 2$. By Theorem 2, $\overline{\gamma} \le \lceil \delta/(\gamma - 1) \rceil + 1$. This simplifies to $(\overline{\gamma} - 2)(\gamma - 1) < \delta$. By assumption, $\gamma \ge \overline{\gamma} - 1$. Hence, $(\overline{\gamma} - 2)(\overline{\gamma} - 2) < \delta$, and the result follows. ■

From Corollary 2, we have the following Nordhaus-Gaddum type result for graphs G with $\gamma(G) = \gamma(\overline{G})$.

Corollary 3. *If G is a graph with $\gamma(G) = \gamma(\overline{G})$, then $\gamma(G)+\gamma(\overline{G}) < 4+\sqrt{\delta(G)}+\sqrt{\delta(\overline{G})}$.*

2.2. Graphs G with $\gamma(G) < \gamma(\overline{G})$

For a subset $S \subset V$ in a graph $G = (V, E)$, let $X_S(G)$ be the set of all vertices x in $V \setminus S$ such that x dominates S in G; that is, $X_S(G) = \{x \in V \setminus S \mid S \subseteq N(x)\}$. We observe that if $X_S(G) = \emptyset$, then S is a dominating set of $\overline{G}$. We state this formally as follows.

Observation 3. *If G is a graph and $S \subset V$ satisfies $|S| < \gamma(\overline{G})$, then $X_S(G) \neq \emptyset$.*

The following result establishes properties about the set $X_S(G)$.

Theorem 4. *Let G be a graph with $\gamma(\overline{G}) = \gamma(G) + k$, where $k \geq 2$, and let S be a $\gamma(G)$-set. It follows that $|X_S| \geq k$. Moreover, any subset $X' \subseteq X_S$ of size $|X_S| - k + 2$ is a dominating set of G.*

Proof. By the definition of X_S, the set S dominates $V \setminus (S \cup X_S)$ in $\overline{G}$. This gives that $S \cup X_S$ is a dominating set of $\overline{G}$, and so $\gamma(G) + |X_S| = |S| + |X_S| \geq \gamma(\overline{G}) = \gamma(G) + k$ which implies $|X_S| \geq k$.

Let u be an arbitrary vertex in $V \setminus S$, and let $U = N_G(u) \cap X_S$. Since S dominates $V \setminus (S \cup X_S)$ in $\overline{G}$, and u dominates $X_S \setminus U$ in $\overline{G}$, the set $S \cup U \cup \{u\}$ is a dominating set of $\overline{G}$. Then, $\gamma(G) + k = \gamma(\overline{G}) \leq \gamma(G) + |U| + 1$. Consequently, $k - 1 \leq |U| = |N_G(u) \cap (X_S \setminus X')| + |N_G(u) \cap X'| \leq k - 2 + |N_G(u) \cap X'|$ and so $N_G(u) \cap X' \neq \emptyset$. Hence, X' dominates $V \setminus S$ in G. Since every vertex of X' dominates S in G, the set X' is a dominating set of G. ∎

Let G be a graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$. Further, let S be a $\gamma(G)$-set, and let $X = X_S(G)$. By definition of the set X, we note that the edges, $G[X, S]$, in G between X and S induce a complete bipartite graph $K_{|X|,|S|}$. By Theorem 4, $\gamma \leq |X|$. Thus, we have the following corollary of Theorem 4.

Corollary 4. *If G is a graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, then G contains $K_{\gamma,\gamma}$ as a subgraph.*

We observe from Corollary 4 that if G is a graph that contains no 4-cycle (and thus does not contain $K_{r,r}$ for $r \geq 2$ as a subgraph), then $\gamma(G) = 1$ or $\gamma(G) \geq \gamma(\overline{G}) - 1$. We establish next a property of claw-free graphs G with $\gamma(G) \leq \gamma(\overline{G}) - 2$.

Theorem 5. *Let G be a graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, and let S be a $\gamma(G)$-set. If G is claw-free, then $\gamma(G) \leq 2$ or $S \cup X_S(G)$ is a clique in G.*

Proof. Let $G = (V, E)$ be a claw-free graph with $\gamma \leq \overline{\gamma} - 2$, and let S be a $\gamma(G)$-set. Following our earlier notation, let $X = X_S(G)$. By Theorem 4, the set X is a dominating set of G, and so $\gamma \leq |X|$. Suppose that $G[S \cup X]$ is not a clique. Then there are two vertices, say a and b, in $S \cup X$ that are not adjacent in G. Since every vertex in X is by definition adjacent in G to every vertex in S, we observe that both a and b are in S or both a and b are in X. Let c be an arbitrary vertex in $V \setminus \{a, b\}$.

We show that c is dominated by $\{a, b\}$. Suppose to the contrary that c is adjacent to neither a nor b. On the one hand, suppose that $\{a, b\} \subseteq S$. Then, $c \notin X$. However since X is a dominating set in G, there is a vertex $x \in X$ that is adjacent to c in G. But then the set $\{a, b, c, x\}$ induces a claw in G, a contradiction. On the other hand, suppose that $\{a, b\} \subseteq X$. Then, $c \notin S$. However since S is a dominating set in G, there is a vertex $x \in S$ that is adjacent to c in G. But then the set $\{a, b, c, x\}$ induces a claw in G, a contradiction. In both cases, we have that c is dominated by $\{a, b\}$, implying that $\{a, b\}$ is a dominating set in G, and therefore, that $\gamma \leq 2$. ■

Let G be a claw-free graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, and let S be a $\gamma(G)$-set and let $X = X_S(G)$. If $\gamma(G) \geq 3$, then by Theorem 5, the set $S \cup X$ is a clique in G, and therefore, an independent set in $\overline{G}$. Hence, as an immediate consequence of Theorem 5, we have the following result, where $\alpha(G)$ and $\omega(G)$ denote the vertex independence number and the clique number, respectively, of G.

Corollary 5. *If G is a claw-free graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, then $\gamma(G) \leq 2$ or $\gamma(G) \leq \omega(G)/2 = \alpha(\overline{G})/2$.*

2.3. Graphs G with a $\gamma(G)$-bad vertex

Recall that a vertex in a graph G is a $\gamma(G)$-*bad* vertex if it is contained in no $\gamma(G)$-set. We establish next an upper bound on the sum of the domination numbers of a graph G and its complement $\overline{G}$ in terms of the degree of a $\gamma(G)$-bad vertex.

Theorem 6. *If a graph G contains a vertex v that is a $\gamma(\overline{G})$-bad vertex, then $\gamma(G) + \gamma(\overline{G}) \leq d_G(v) + 3$.*

Proof. Let $G = (V, E)$ be a graph that contains a $\gamma(\overline{G})$-bad vertex v. Let $A = N_G(v)$, and so $|A| = d_G(v)$. Since the set $A \cup \{v\}$ is a dominating set in $\overline{G}$, we have that $\gamma(\overline{G}) \leq |A| + 1$. However if $\gamma(\overline{G}) = |A| + 1$, then $A \cup \{v\}$ is a $\gamma(\overline{G})$-set, contradicting the fact that v is a $\gamma(\overline{G})$-bad vertex. Therefore, $\gamma(\overline{G}) < |A| + 1$, or, equivalently, $|A| \geq \gamma(\overline{G})$.

Let $B = V \setminus N_G[v]$. If $B = \emptyset$, then v dominates V in the graph G, implying that v is isolated in $\overline{G}$ and therefore belongs to every $\gamma(\overline{G})$-set, a contradiction. Hence, $B \neq \emptyset$. We show next that each vertex in B has at least $\overline{\gamma} - 1$ neighbors in G that belong to the set A. Let $x \in B$, and let $A_x = A \cap N_G(x)$. Then in the graph $\overline{G}$, the vertex x dominates the set $A \setminus A_x$. Thus since the vertex v dominates the set B in $\overline{G}$, we have that the set $A_x \cup \{v, x\}$ is a dominating set in $\overline{G}$, implying that $\gamma(\overline{G}) \leq |A_x| + 2$. However if $\gamma(\overline{G}) = |A_x| + 2$, then $A_x \cup \{v, x\}$ is a $\gamma(\overline{G})$-set, contradicting the fact that v is a $\gamma(\overline{G})$-bad vertex. Therefore, $\gamma(\overline{G}) < |A_x| + 2$, or, equivalently, $\gamma(\overline{G}) \leq |A_x| + 1$. Thus in the graph $\overline{G}$, we have that $d_A(x) = |A_x| \geq \gamma(\overline{G}) - 1$. This is true for every vertex $x \in B$.

Recall that $|A| \geq \gamma(\overline{G})$. Let A' be an arbitrary subset of A of cardinality $\gamma(\overline{G}) - 2$, and let $A^* = A \setminus A'$. Thus, $|A'| = \gamma(\overline{G}) - 2$ and $|A^*| = |A| - |A'| = d_G(v) - \gamma(\overline{G}) + 2$. Since $d_A(x) \geq \gamma(\overline{G}) - 1$ for every vertex $x \in B$, the set A^* dominates the set B in G. Thus, $A^* \cup \{v\}$ is a dominating set in G, implying that $\gamma(G) \leq |A^*| + 1 = d_G(v) - \gamma(\overline{G}) + 3$. ∎

As a consequence of Theorem 6, we have the following result.

Corollary 6. *If G is an r-regular graph that contains a $\gamma(\overline{G})$-bad vertex, then $\gamma(G) + \gamma(\overline{G}) \leq r + 3$.*

2.4. Domination and planarity

In this section, we study some relationships between planarity, the crossing number of G and the domination number of $\overline{G}$. Fundamental to our results in this section is the famous Four Color Theorem.

Theorem 7 [1]**.** *If G is a planar graph, then $\chi(G) \leq 4$.*

We first establish the following upper bound on the domination number of the complement of a graph. For this purpose, for a vertex v in a graph G, we denote by G_v the subgraph of G induced by the neighbors of v; that is, $G_v = G[N(v)]$. If $\mathcal{C}$ is a minimum coloring of the vertices of G_v, and S is a set of vertices comprising of exactly one vertex from each color class of $\mathcal{C}$, then the set $S \cup \{v\}$ forms a dominating set of $\overline{G}$, implying that $\gamma(\overline{G}) \leq |\mathcal{C}| + 1 = \chi(G_v) + 1$. We state this formally as follows.

Observation 8. *If v is an arbitrary vertex in a graph G, then $\gamma(\overline{G}) \leq \chi(G_v) + 1$.*

As a consequence of Theorem 7 and Observation 8, we have the following results.

Corollary 7. *If a graph G contains a vertex v with the property that G_v is a planar graph, then $\gamma(\overline{G}) \leq 5$.*

Corollary 8. *If a graph G satisfies $\gamma(G) > 2cr(G)$, then $\gamma(\overline{G}) \leq 5$.*

Proof. Let G^* be a drawing of G in the plane with exactly $cr(G)$ crossing edges, and let S be the set of vertices of G incident with at least one crossing edge of G^*. Clearly, $|S| \leq 2cr(G)$. Since, by assumption, $\gamma(G) > 2cr(G)$, it follows there exists some vertex v in G that is not dominated by S. This implies that G_v is a planar graph. Thus, by Corollary 7, $\gamma(\overline{G}) \leq 5$. ∎

3. Total, Connected, Restrained, and Clique Domination

In this section, we establish relationships involving the domination, total domination, restrained domination, connected domination and clique domination numbers of a graph. We begin with the following lemma.

Lemma 9. *If there exists a $\gamma(G)$-set for a graph G that is not a dominating set in $\overline{G}$, then $\gamma_t(G) \leq \gamma_c(G) \leq \gamma(G) + 1$.*

Proof. Let S be a $\gamma(G)$-set in a graph $G = (V, E)$ that is not a dominating set in $\overline{G}$. Then there exists a vertex $v \in V \setminus S$ that is not adjacent to any vertex of S in $\overline{G}$. Hence in G, the vertex v is adjacent to every vertex of S, implying that the graph $G[S \cup \{v\}]$ is connected. Since every superset of a dominating set is also a dominating set, the set $S \cup \{v\}$ is a CD-set, and so $\gamma_c(G) \leq |S \cup \{v\}| = \gamma(G) + 1$. Since the total domination of a graph is at most its connected domination number, the desired result follows from the observation that $\gamma_t(G) \leq \gamma_c(G)$. ∎

By the contrapositive of Lemma 9, we note that if a graph G satisfies $\gamma_t(G) \geq \gamma(G) + 2$, then every $\gamma(G)$-set is a dominating set in $\overline{G}$. Further as a consequence of Lemma 9 and the well-known result due to Jaeger and Payan [11] that if G is a graph of order n, then $\gamma(G)\gamma(\overline{G}) \leq n$, we have the following result.

Corollary 10. *Let G be a graph of order n satisfying $\gamma(G) < \gamma(\overline{G})$. Then the following holds.*

(a) $\gamma_t(G) \leq \gamma_c(G) \leq \gamma(G) + 1$.
(b) $\gamma_c(G) \leq (1 + \sqrt{4n+1})/2$.

Proof. Part (a) is an immediate consequence of Lemma 9. To prove part (b), let G be a graph of order n satisfying $\gamma(G) < \gamma(\overline{G})$. By part (a) and our assumption that $\gamma(G) \leq \gamma(\overline{G}) - 1$, we have that $\gamma_c(G) \leq \gamma(G) + 1 \leq \gamma(\overline{G})$. Applying the result due to Jaeger and Payan, we therefore have that $(\gamma_c(G) - 1)\gamma_c(G) \leq \gamma(G)\gamma(\overline{G}) \leq n$. Solving for $\gamma_c(G)$, we have that $\gamma_c(G) \leq (1 + \sqrt{4n+1})/2$. ∎

In the following result, we consider the case when $\gamma(G) \leq \gamma(\overline{G}) + 1$.

Theorem 9. *Let G be a graph satisfying $\gamma(G) \le \gamma(\overline{G}) + 1$. Then the following holds.*

(a) *If both G and $\overline{G}$ are connected, then $\gamma_c(G) \le \gamma(G) + 1$ or $\gamma_c(\overline{G}) \le \gamma(\overline{G}) + 1$.*
(b) *If both G and $\overline{G}$ are isolate-free, then $\gamma_t(G) \le \gamma(G) + 1$ or $\gamma_t(\overline{G}) \le \gamma(\overline{G}) + 1$.*

Proof. Let $G = (V, E)$, and let S be a $\gamma(G)$-set in the graph. We first establish part (a). Suppose that both G and $\overline{G}$ are connected. If $G[S]$ is connected, then S is a CD-set in G, implying that $\gamma_c(G) \le |S| = \gamma(G)$. Hence we may assume that $G[S]$ is not connected, for otherwise part (a) is immediate. This implies that $\overline{G}[S]$ is connected. If the set S is not a dominating set in $\overline{G}$, then by Lemma 9, we have that $\gamma_c(G) \le \gamma(G) + 1$. If the set S is a dominating set in $\overline{G}$, then S is a CD-set in $\overline{G}$, implying that $\gamma_c(\overline{G}) \le |S| = \gamma(G) \le \gamma(\overline{G}) + 1$. This proves part (a).

Next we prove part (b). Suppose that both G and $\overline{G}$ are isolate-free. If $G[S]$ is isolate-free, then S is a TD-set in G, implying that $\gamma_t(G) \le |S| = \gamma(G)$. Hence we may assume that $G[S]$ contains an isolated vertex, for otherwise part (b) is immediate. This implies that $\overline{G}[S]$ is connected. If the set S is not a dominating set in $\overline{G}$, then by Lemma 9 we have that $\gamma_t(G) \le \gamma(G) + 1$. If the set S is a dominating set in $\overline{G}$, then S is a TD-set in $\overline{G}$, implying that $\gamma_t(\overline{G}) \le |S| = \gamma(G) \le \gamma(\overline{G}) + 1$. This proves part (b). ■

We establish next an upper bound on the total domination number of a graph in terms of its domination number and the domination number of its complement.

Theorem 10. *Let G be an isolate-free graph, and let S be a $\gamma(G)$-set. If s is the number of isolated vertices in $G[S]$, then $\gamma_t(G) \le \gamma(G) + \lceil s/(\gamma(\overline{G}) - 1)\rceil$.*

Proof. Let $G = (V, E)$. Since G is isolate-free, we note that $\gamma(\overline{G}) \ge 2$. Let I be the set of isolated vertices in $G[S]$, and so $s = |I|$. Let $k = \lceil s/(\overline{\gamma} - 1)\rceil$, and partition the set I into k sets $I_1, \ldots, I_k$ each of cardinality at most $\overline{\gamma} - 1$. Thus, $I = \bigcup_{i=1}^{k} I_i$ and $1 \le |I_i| \le \overline{\gamma} - 1$ for each i, $1 \le i \le k$. In particular, we note that no set I_i dominates V in $\overline{G}$. For each set I_i, $1 \le i \le k$, select one vertex $w_i \in V \setminus I_i$ that is not dominated by I_i in $\overline{G}$, and let $W = \bigcup_{i=1}^{k}\{w_i\}$. Then, $|W| \le k$. We note that in the graph G, the vertex w_i is adjacent to every vertex of I_i, and so $S \cup W$ is a TD-set in G. Hence, $\gamma_t(G) \le |S \cup W| \le |S| + |W| \le \gamma(G) + k = \gamma(G) + \lceil s/(\overline{\gamma} - 1)\rceil$. ■

As an immediate consequence of Theorem 10, we have the following upper bound on the total domination number of a graph.

Corollary 11. *If G is an isolate-free graph, then $\gamma_t(G) \le \gamma(G) + \left\lceil \frac{\gamma(G)}{\gamma(\overline{G})-1} \right\rceil$.*

Theorem 11. *If G is a graph with $\gamma_t(G) \ge \gamma(G) + 2$, then $\gamma_t(\overline{G}) \le 1 + \left\lceil \frac{\delta(G)}{\gamma(G)} \right\rceil$.*

Proof. Let $G = (V, E)$ be a graph with $\gamma_t(G) \geq \gamma(G)+2$, and let v be a vertex of G having degree $\delta(G)$. Let $A = N_G(v)$, and so $|A| = \delta(G)$. Let $k = \lceil \delta(G)/\gamma(G) \rceil$ and partition the set A into k sets $A_1, \ldots, A_k$ each of cardinality at most $\gamma(G)$. Thus, $A = \bigcup_{i=1}^{k} A_i$ and $1 \leq |A_i| \leq \gamma(G)$ for each i, $1 \leq i \leq k$. If the set A_i dominates $V \setminus N_G[v]$ in G for some i, $1 \leq i \leq k$, then the set $A_i \cup \{v\}$ is a TD-set in G, implying that $\gamma_t(G) \leq |A_i| + 1 \leq \gamma(G) + 1$, a contradiction. Therefore, no set A_i dominates $V \setminus N_G[v]$ in G. For each set A_i, $1 \leq i \leq k$, select one vertex $a_i \in V \setminus N_G[v]$ that is not dominated by A_i in G, and let $A' = \bigcup_{i=1}^{k}\{a_i\}$. Then, $|A'| \leq k$ and A' dominates A in $\overline{G}$. Therefore, the set $A' \cup \{v\}$ is a TD-set in $\overline{G}$, and so $\gamma_t(\overline{G}) \leq |A'| + 1 \leq k + 1 = 1 + \lceil \delta(G)/\gamma(G) \rceil$. ■

Next we consider the restrained domination number. We first prove a general lemma.

Lemma 12. *If a graph G has a $\gamma(G)$-set S such that the induced subgraph $G[V \setminus S]$ has an isolated vertex, then $\gamma(\overline{G}) \leq 3$.*

Proof. Let S be a $\gamma(G)$-set such that $G[V \setminus S]$ has an isolated vertex, say w. If $G[S]$ has an isolated vertex v, then $\{v, w\}$ is dominating set of $\overline{G}$, and so $\gamma(\overline{G}) \leq 2$. If $G[S]$ contains no isolated vertices, then by the minimality of S, for each $v \in S$, there exists a vertex, say $v' \in V \setminus S$, such that $N(v') \cap S = \{v\}$. In this case, the set $\{v, w, v'\}$ is a dominating set of $\overline{G}$, implying that $\gamma(\overline{G}) \leq 3$. ■

As an immediate consequence of Lemma 12, we have the following result.

Corollary 13. *If a graph G has $\gamma(\overline{G}) \geq 4$, then every $\gamma(G)$-set is a $\gamma_r(G)$-set. In particular, $\gamma(G) = \gamma_r(G)$.*

We close this section with two results about the clique domination number of a graph.

Theorem 12. *If G is a graph with $\gamma_t(G) \geq \gamma(G) + 2$, then $\gamma_{\rm cl}(\overline{G}) \leq \gamma(G)$. Moreover, if G is claw-free, then $\gamma_{\rm cl}(\overline{G}) \leq 3$.*

Proof. Let G be a graph with $\gamma_t(G) \geq \gamma(G)+2$, and let S be a $\gamma(G)$-set. Further, let $I(S)$ be the set of isolated vertices in $G[S]$. If $I(S) = \emptyset$, then S is a TD-set of G, implying that $\gamma_t(G) \leq |S| = \gamma(G)$, a contradiction. Hence, $I(S) \neq \emptyset$. We show that $I(S)$ dominates $\overline{G}$. Suppose to the contrary that there exists a vertex v that is not adjacent to any vertex of $I(S)$ in $\overline{G}$. Then in the graph G, the vertex v is adjacent to every vertex of $I(S)$, implying that $S \cup \{v\}$ is a TD-set for G, and so $\gamma_t(G) \leq |S| + 1 = \gamma(G) + 1$, a contradiction. Hence, the set $I(S)$ dominates $\overline{G}$. Since $I(S)$ is an independent set in G, it forms a clique in $\overline{G}$. Therefore, $I(S)$ is a dominating clique in $\overline{G}$, implying that $\gamma_{\rm cl}(\overline{G}) \leq |I(S)| \leq \gamma(G)$.

Now, suppose that G is claw free. If $|I(S)| \leq 3$, then the result follows. Hence, we may assume that $|I(S)| \geq 4$ and there exists a subset $\{a, b, c\} \subseteq I(S)$

that is not a dominating set in $\overline{G}$. Then there exists a vertex v that is not adjacent to a, b, or c in $\overline{G}$. But then in the graph G, we have that $\{a, b, c, v\}$ induces a claw, a contradiction. Therefore, every subset of $I(S)$ of cardinality 3 is a dominating set in $\overline{G}$, implying that $\gamma_{\rm cl}(\overline{G}) \le 3$. ∎

4. Bounds on the Domination Number of a Graph in Terms of the Adjacency Matrix of its Complement

We begin this section by stating two well-known theorems. The first result counts the number of walks of length k for an arbitrary positive integer k in a graph (see [3]; see also Theorem 1.17 in [5]). The second result is a consequence of a result due to Vizing [16] and provides an upper bound for the domination number of a graph in terms of its order and size.

Theorem 13 [3]. *Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and with adjacency matrix A. For each positive integer k, the number of different walks of length k from the vertex v_i to the vertex v_j is the (i, j)-entry in the matrix A^k.*

Theorem 14 [16]. *If G is graph of order n and size m, then $\gamma(G) \le n + 1 - \sqrt{1 + 2m}$.*

Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and with adjacency matrix A, and let $a_{ij}^{(k)}$ denote the (i, j)-entry in A^k. Recall that if v is a vertex in G, then the subgraph of G induced by $N_G(v)$ is called the link of v and is denoted by $\mathcal{L}(v)$, while the subgraph of $\overline{G}$ induced by $N_G(v)$ is denoted $\overline{\mathcal{L}}(v)$. Theorem 13 implies that the (i, i)-entry of A^2, $1 \le i \le n$, is the degree $d_G(v_i)$ of v_i, and the (i, i)-entry of A^3, $1 \le i \le n$, is equal to twice the number of edges in $\mathcal{L}(v_i)$. Suppose that $a_{ii}^{(3)} < a_{ii}^{(2)}$ for some i, $1 \le i \le n$. Since $a_{ii}^{(2)} = d_G(v_i)$ and $\frac{1}{2}a_{ii}^{(3)}$ is the number of edges in $\mathcal{L}(v_i)$, this implies that $\mathcal{L}(v_i)$ contains an isolated vertex, v say. Thus the set $\{v, v_i\}$ is a dominating set in the graph $\overline{G}$, implying that $\gamma(\overline{G}) \le 2$. We state this formally as follows.

Observation 15. *Let G be an isolate-free graph of order n with adjacency matrix A. If the (i, i)-entry of A^3 is less than the (i, i)-entry of A^2 for some i, $1 \le i \le n$, then $\gamma(\overline{G}) \le 2$.*

Using Observation 15, we obtain the following bound on the domination number of the complement of a graph.

Theorem 16. *Let G be a graph of order n with adjacency matrix A, and let $a_{ij}^{(k)}$ denote the (i, j)-entry in A^k. For every i, $1 \le i \le n$, we have that*

$$\gamma(\overline{G}) \le a_{ii}^{(2)} + 2 - \sqrt{1 + a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)}}.$$

Proof. Let i be an arbitrary integer with $1 \le i \le n$. Since $a_{ii}^{(2)} = d_G(v_i)$ and $\frac{1}{2}a_{ii}^{(3)}$ is the number of edges in $\mathcal{L}(v_i)$, this implies that $\overline{\mathcal{L}}(v_i)$ has order $a_{ii}^{(2)}$ and size

$$\binom{a_{ii}^{(2)}}{2} - \frac{1}{2}a_{ii}^{(3)} = \frac{1}{2}\left(a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)}\right).$$

Thus, by Theorem 14, we have that

$$\gamma(\overline{\mathcal{L}}(v_i)) \le a_{ii}^{(2)} + 1 - \sqrt{1 + a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)}}.$$

The desired bound now follows from the observation that every $\gamma\left(\overline{\mathcal{L}}(v_i)\right)$-set can be extended to a dominating set in $\overline{G}$ by adding to it the vertex v_i, and so $\gamma\left(\overline{G}\right) \le \gamma\left(\overline{\mathcal{L}}(v_i)\right) + 1$. ■

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