DOMINATION PARAMETERS OF A GRAPH
AND ITS COMPLEMENT

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Abstract

A dominating set in a graph $G$ is a set $S$ of vertices such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$, and the domination number of $G$ is the minimum cardinality of a dominating set of $G$. Placing constraints on a dominating set yields different domination parameters, including total, connected, restrained, and clique domination numbers. In this paper, we study relationships among domination parameters of a graph and its complement.

Keywords: domination, complement, total domination, connected domination, clique domination, restrained domination.

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1. Introduction

The literature on the subject of domination parameters in graphs has been surveyed through 1997 and detailed in the two books [7, 8]. Our aim in this paper
is to study graph relationships involving domination parameters in a graph $G$ and its complement $\overline{G}$. We will also study relationships between the domination number of a graph and its total, restrained, clique and connected domination numbers.

For notation and graph theory terminology not defined herein, we refer the reader to [7]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E\}$, and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. We denote the complement of a graph $G$ by $\overline{G}$. For any vertex $v$, we call the subgraph of $G$ induced by $N_G(v)$ the link of $v$ and will denote it as $L(v)$. We will denote the subgraph of $\overline{G}$ induced by $N_G(v)$ as $\overline{L}(v)$.

For a set $S \subseteq V$, its open neighborhood is the set $N_G(S) = \bigcup_{v \in S} N(v)$, and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. The degree of a vertex $v$ in $G$ is $d_G(v) = |N_G(v)|$. If the graph $G$ is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$, respectively. A vertex is isolated in $G$ if its degree in $G$ is zero. A graph is isolate-free if it has no isolated vertex. For any set $S \subseteq V(G)$, we denote the subgraph induced by $S$ as $G[S]$. The minimum and maximum degree among the vertices of $G$ is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $X \subseteq V$, the degree of a vertex $v$ in $X$, denoted $d_X(v)$, is the number of vertices in $X$ adjacent to $v$; that is, $d_X(v) = |N(v) \cap X|$. In particular, $d_G(v) = d_V(v)$.

For sets $A, B \subseteq V$, we let $G[A, B]$, or simply $[A, B]$ if the graph is clear from the context, denote the set of edges in $G$ with one end in $A$ and the other in $B$. A nontrivial graph is a graph with at least two vertices. We say that a graph is $F$-free if it does not contain $F$ as an induced subgraph. In particular, if $F = K_{1,3}$, then we say that the graph is claw-free.

A dominating set in $G = (V, E)$ is a set $S$ of vertices of $G$ such that every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$, that is, $N[S] = V$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. For subsets $X, Y \subseteq V$, the set $X$ dominates the set $Y$ in $G$ if $Y \subseteq N[X]$. In particular, if $X$ dominates $V$, then $X$ is a dominating set of $G$. A vertex is called $\gamma(G)$-good if it is contained in some $\gamma(G)$-set, and $\gamma(G)$-bad, otherwise. In other words, a $\gamma(G)$-good vertex is contained in at least one $\gamma(G)$-set, while a $\gamma(G)$-bad vertex is not in any $\gamma(G)$-set. The minimum degree among the $\gamma(G)$-good (respectively, $\gamma(G)$-bad) vertices of $G$ is denoted by $\delta_G(G)$ (respectively, $\delta_b(G)$).

A total dominating set, abbreviated TD-set, of $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of $G$. A TD-set of $G$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination is now well studied in graph theory. The literature on the subject of
total domination in graphs has been surveyed and detailed in the recent book [10]. A survey of total domination in graphs can also be found in [9].

Another way of looking at total domination is that a dominating set $S$ is a TD-set if the induced subgraph $G[S]$ has no isolated vertices. Placing the constraint that $G[S]$ is connected (respectively, a complete graph) yields connected domination (respectively, clique domination). More formally, a dominating set $S$ is a connected dominating set, abbreviated CD-set, of a graph $G$ if the induced subgraph $G[S]$ is connected. Every connected graph has a CD-set, since $V$ is such a set. The connected domination number of $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a CD-set of $G$, and a CD-set of $G$ of cardinality $\gamma_c(G)$ is called a $\gamma_c(G)$-set. Connected domination in graphs was first introduced by Sampathkumar et al. [14] and is now very well studied (see, for example, [4] and the recent papers [13, 15]). The study of connected domination has extensive application in the study of routing problems and virtual backbone based routing in wireless networks [6, 12, 17]. A subset $S \subset V$ of vertices in a graph $G = (V, E)$ is a dominating clique in $G$ if $S$ dominates $V$ in $G$ and $G[S]$ is complete. If a graph $G$ has a dominating clique, then the minimum cardinality among all dominating cliques of $G$ is the clique domination number of $G$, denoted by $\gamma_{cl}(G)$.

A restrained dominating set of a graph $G$ is a set $S$ of vertices in $G$ such that every vertex in $V \setminus S$ is adjacent to a vertex in $S$ and to some other vertex in $V \setminus S$. Every connected graph has an RD-set, since $V$ is such a set. The restrained domination number of $G$, denoted by $\gamma_r(G)$, is the minimum cardinality of an RD-set of $G$, and an RD-set of $G$ of cardinality $\gamma_r(G)$ is called a $\gamma_r(G)$-set.

A proper vertex coloring of a graph $G$ is an assignment of colors (elements of some set) to the vertices of $G$, one color to each vertex, so that adjacent vertices are assigned distinct colors. If $k$ colors are used, then the coloring is referred to as a $k$-coloring. In a given coloring of $G$, a color class of the coloring is a set consisting of all those vertices assigned the same color. The vertex chromatic number $\chi(G)$ of $G$ is the minimum integer $k$ such that $G$ is $k$-colorable. A $\chi(G)$-coloring of $G$ is a coloring of $G$ with $\chi(G)$ colors.

Given a graph $G$, two edges are said to cross in the plane if in a drawing of the graph in the plane they intersect at a point that is not a vertex. The graph $G$ is planar if it can be drawn in the plane with no edges crossing. The crossing number of $G$, denoted $cr(G)$, is the minimum number of crossing edges amongst all drawings of $G$ in the plane. Note that if $G$ is planar, then necessarily $cr(G) = 0$.

2. Bounds on the Domination Number

In this section, we determine bounds on the domination number. If the graph $G$ is clear from the context, then we write $\delta, \overline{\delta}, \Delta, \overline{\Delta}, \gamma$ and $\overline{\gamma}$ rather than $\delta(G)$,
δ(Ḡ), ∆(Ḡ), ∆(Ḡ), γ(Ḡ) and γ(Ḡ), respectively.

2.1. Dominating the complement of a graph

We begin with results bounding the domination number of the complement of a graph. If \( v \) is an arbitrary vertex in a graph \( G \), then the closed neighborhood, \( N_G[v] \), of \( v \) is a dominating set of \( Ḡ \). In particular, choosing \( v \) to be a vertex of minimum degree in \( G \), we have that \( γ(Ḡ) \leq δ(G) + 1 \). Furthermore, a set formed by taking a vertex from each color class of an arbitrary \( χ(G) \)-coloring of \( G \) is a dominating set of \( Ḡ \), and so \( γ(Ḡ) \leq χ(G) \). We state these well known observations formally as follows.

**Observation 1.** Let \( G \) be a graph. Then the following hold.

(a) \( γ(Ḡ) \leq δ(G) + 1 \).
(b) \( γ(Ḡ) \leq χ(G) \).

By Observation 1, \( γ(Ḡ) \leq ∆(Ḡ) + 1 \). From Brook’s Coloring Theorem [2], \( χ(G) \leq ∆(Ḡ) + 1 \) with equality if and only if \( G \) is the complete graph or an odd cycle. Noting that the domination number of the complement of any odd cycle \( C_n \), where \( n \geq 5 \), is equal to 2, we observe that if \( G \) is a graph, then \( γ(Ḡ) \leq ∆(Ḡ) + 1 \) with equality if and only if \( G \) is a complete graph. Next we give an upper bound on \( γ(Ḡ) \) in terms of \( γ(G) \) and \( δ(G) \).

**Theorem 2.** If \( G \) is a graph with \( γ(G) \geq 2 \), then \( γ(Ḡ) \leq \left\lceil \frac{δ(G)}{γ(G)−1} \right\rceil + 1 \).

**Proof.** Let \( v \) be a vertex of \( G \) having degree \( δ \). Let \( A = N_G(v) \), and so \( |A| = δ \). Let \( k = \lceil δ/(γ − 1) \rceil \) and partition the set \( A \) into \( k \) sets \( A_1, \ldots, A_k \) each of cardinality at most \( γ − 1 \). Thus, \( A = \bigcup_{i=1}^{k} A_i \) and \( 1 \leq |A_i| \leq γ − 1 \) for each \( i \), \( 1 \leq i \leq k \). In particular, we note that no set \( A_i \) dominates \( V \) in \( G \). For each set \( A_i \), \( 1 \leq i \leq k \), select one vertex \( a_i \in V \setminus A_i \) that is not dominated by \( A_i \) in \( G \), and let \( A' = \bigcup_{i=1}^{k} \{a_i\} \). Then, \(|A'| \leq k \) and \( A' \) dominates \( A \) in \( Ḡ \). Therefore, the set \( A' \cup \{v\} \) is a dominating set of \( Ḡ \), and so \( γ \leq |A'| + 1 \leq k + 1 = 1 + \lceil δ/(γ − 1) \rceil \). ■

As an immediate consequence of Theorem 2, we have the following corollaries.

**Corollary 1.** If \( G \) is a graph with \( γ(Ḡ) > γ(G) \geq 2 \), then \( δ(G) \geq γ(G) \).

The next result shows that if \( G \) is a graph satisfying \( γ(G) \geq γ(Ḡ) − 1 \), then the bound of Observation 1(a) can be improved.

**Corollary 2.** If \( G \) is a graph satisfying \( γ(G) \geq γ(Ḡ) − 1 \), then \( γ(Ḡ) < 2 + √δ(G) \).

**Proof.** Let \( G \) be a graph satisfying \( γ \geq γ − 1 \). If \( γ = 1 \), then \( γ \leq 2 \), and the result follows. Accordingly, we may assume that \( γ \geq 2 \). By Theorem 2, \( γ \leq \lceil δ/(γ − 1) \rceil + 1 \). This simplifies to \((γ−2)(γ−1) < δ \). By assumption, \( γ \geq γ − 1 \). Hence, \((γ−2)(γ−2) < δ \), and the result follows. ■
From Corollary 2, we have the following Nordhaus-Gaddum type result for graphs $G$ with $\gamma(G) = \gamma(\overline{G})$.

**Corollary 3.** If $G$ is a graph with $\gamma(G) = \gamma(\overline{G})$, then $\gamma(G) + \gamma(\overline{G}) < 4 + \sqrt{\delta(G)} + \sqrt{\delta(\overline{G})}$.

### 2.2. Graphs $G$ with $\gamma(G) < \gamma(\overline{G})$

For a subset $S \subseteq V$ in a graph $G = (V,E)$, let $X_S(G)$ be the set of all vertices $x$ in $V \setminus S$ such that $x$ dominates $S$ in $G$; that is, $X_S(G) = \{x \in V \setminus S \mid S \subseteq N(x)\}$. We observe that if $X_S(G) = \emptyset$, then $S$ is a dominating set of $\overline{G}$. We state this formally as follows.

**Observation 3.** If $G$ is a graph and $S \subseteq V$ satisfies $|S| < \gamma(\overline{G})$, then $X_S(G) \neq \emptyset$.

The following result establishes properties about the set $X_S(G)$.

**Theorem 4.** Let $G$ be a graph with $\gamma(\overline{G}) = \gamma(G) + k$, where $k \geq 2$, and let $S$ be a $\gamma(G)$-set. It follows that $|X_S| \geq k$. Moreover, any subset $X' \subseteq X_S$ of size $|X_S| - k + 2$ is a dominating set of $G$.

**Proof.** By the definition of $X_S$, the set $S$ dominates $V \setminus (S \cup X_S)$ in $\overline{G}$. This gives that $S \cup X_S$ is a dominating set of $\overline{G}$, and so $\gamma(G) \leq |X_S| \geq |X_S| = |S| + |X_S| \geq \gamma(\overline{G}) = \gamma(G) + k$ which implies $|X_S| \geq k$.

Let $u$ be an arbitrary vertex in $V \setminus S$, and let $U = N_G(u) \cap X_S$. Since $S$ dominates $V \setminus (S \cup X_S)$ in $\overline{G}$, and $u$ dominates $X_S \setminus U$ in $\overline{G}$, the set $S \cup U \cup \{u\}$ is a dominating set of $\overline{G}$. Then, $\gamma(G) + k = \gamma(\overline{G}) \leq \gamma(G) + |U| + 1$. Consequently, $k - 1 \leq |U| = |N_G(u) \cap (X_S \setminus X'|) + |N_G(u) \cap X'| \leq k - 2 + |N_G(u) \cap X'|$ and so $N_G(u) \cap X' \neq \emptyset$. Hence, $X'$ dominates $V \setminus S$ in $G$. Since every vertex of $X'$ dominates $S$ in $G$, the set $X'$ is a dominating set of $G$. 

Let $G$ be a graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$. Further, let $S$ be a $\gamma(G)$-set, and let $X = X_S(G)$. By definition of the set $X$, we note that the edges, $G[X,S]$, in $G$ between $X$ and $S$ induce a complete bipartite graph $K_{|X|,|S|}$. By Theorem 4, $\gamma \leq |X|$. Thus, we have the following corollary of Theorem 4.

**Corollary 4.** If $G$ is a graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, then $G$ contains $K_{\gamma,\gamma}$ as a subgraph.

We observe from Corollary 4 that if $G$ is a graph that contains no 4-cycle (and thus does not contain $K_{r,r}$ for $r \geq 2$ as a subgraph), then $\gamma(G) = 1$ or $\gamma(G) \geq \gamma(\overline{G}) - 1$. We establish next a property of claw-free graphs $G$ with $\gamma(G) \leq \gamma(\overline{G}) - 2$. 


**Theorem 5.** Let $G$ be a graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, and let $S$ be a $\gamma(G)$-set. If $G$ is claw-free, then $\gamma(G) \leq 2$ or $S \cup X_S(G)$ is a clique in $G$.

**Proof.** Let $G = (V, E)$ be a claw-free graph with $\gamma \leq \gamma(G) - 2$, and let $S$ be a $\gamma(G)$-set. Following our earlier notation, let $X = X_S(G)$. By Theorem 4, the set $X$ is a dominating set of $G$, and so $\gamma \leq |X|$. Suppose that $G[S \cup X]$ is not a clique. Then there are two vertices, say $a$ and $b$, in $S \cup X$ that are not adjacent in $G$. Since every vertex in $X$ is by definition adjacent in $G$ to every vertex in $S$, we observe that both $a$ and $b$ are in $S$ or both $a$ and $b$ are in $X$. Let $c$ be an arbitrary vertex in $V \setminus \{a, b\}$.

We show that $c$ is dominated by $\{a, b\}$. Suppose to the contrary that $c$ is adjacent to neither $a$ nor $b$. On the one hand, suppose that $\{a, b\} \subseteq S$. Then, $c \notin X$. However since $X$ is a dominating set in $G$, there is a vertex $x \in X$ that is adjacent to $c$ in $G$. But then the set $\{a, b, c, x\}$ induces a claw in $G$, a contradiction. On the other hand, suppose that $\{a, b\} \subseteq X$. Then, $c \notin S$. However since $S$ is a dominating set in $G$, there is a vertex $x \in S$ that is adjacent to $c$ in $G$. But then the set $\{a, b, c, x\}$ induces a claw in $G$, a contradiction. In both cases, we have that $c$ is dominated by $\{a, b\}$, implying that $\{a, b\}$ is a dominating set in $G$, and therefore, that $\gamma \leq 2$.

Let $G$ be a claw-free graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, and let $S$ be a $\gamma(G)$-set and let $X = X_S(G)$. If $\gamma(G) \geq 3$, then by Theorem 5, the set $S \cup X$ is a clique in $G$, and therefore, an independent set in $\overline{G}$. Hence, as an immediate consequence of Theorem 5, we have the following result, where $\alpha(G)$ and $\omega(G)$ denote the vertex independence number and the clique number, respectively, of $G$.

**Corollary 5.** If $G$ is a claw-free graph with $\gamma(G) \leq \gamma(\overline{G}) - 2$, then $\gamma(G) \leq 2$ or $\gamma(G) \leq \omega(G)/2 = \alpha(\overline{G})/2$.

### 2.3. Graphs $G$ with a $\gamma(G)$-bad vertex

Recall that a vertex in a graph $G$ is a $\gamma(G)$-bad vertex if it is contained in no $\gamma(G)$-set. We establish next an upper bound on the sum of the domination numbers of a graph $G$ and its complement $\overline{G}$ in terms of the degree of a $\gamma(G)$-bad vertex.

**Theorem 6.** If a graph $G$ contains a vertex $v$ that is a $\gamma(\overline{G})$-bad vertex, then $\gamma(G) + \gamma(\overline{G}) \leq d_G(v) + 3$.

**Proof.** Let $G = (V, E)$ be a graph that contains a $\gamma(\overline{G})$-bad vertex $v$. Let $A = N_G(v)$, and so $|A| = d_G(v)$. Since the set $A \cup \{v\}$ is a dominating set in $\overline{G}$, we have that $\gamma(\overline{G}) \leq |A| + 1$. However if $\gamma(\overline{G}) = |A| + 1$, then $A \cup \{v\}$ is a $\gamma(\overline{G})$-set, contradicting the fact that $v$ is a $\gamma(\overline{G})$-bad vertex. Therefore, $\gamma(\overline{G}) < |A| + 1$, or, equivalently, $|A| \geq \gamma(\overline{G})$. 


Let \( B = V \setminus N_G[v] \). If \( B = \emptyset \), then \( v \) dominates \( V \) in the graph \( G \), implying that \( v \) is isolated in \( \overline{G} \) and therefore belongs to every \( \gamma(\overline{G}) \)-set, a contradiction. Hence, \( B \neq \emptyset \). We show next that each vertex in \( B \) has at least \( \tau - 1 \) neighbors in \( G \) that belong to the set \( A \). Let \( x \in B \), and let \( A_x = A \cap N_G(x) \). Then in the graph \( \overline{G} \), the vertex \( x \) dominates the set \( A \setminus A_x \). Thus since the vertex \( v \) dominates the set \( B \) in \( \overline{G} \), we have that the set \( A_x \cup \{v, x\} \) is a dominating set in \( \overline{G} \), implying that \( \gamma(\overline{G}) \leq |A_x| + 2 \). However if \( \gamma(\overline{G}) = |A_x| + 2 \), then \( A_x \cup \{v, x\} \) is a \( \gamma(\overline{G}) \)-set, contradicting the fact that \( v \) is a \( \gamma(\overline{G}) \)-bad vertex. Therefore, \( \gamma(\overline{G}) < |A_x| + 2 \), or, equivalently, \( \gamma(\overline{G}) \leq |A_x| + 1 \). Thus in the graph \( \overline{G} \), we have that \( d_A(x) = |A_x| \geq \gamma(\overline{G}) - 1 \). This is true for every vertex \( x \in B \).

Recall that \( |A| \geq \gamma(\overline{G}) \). Let \( A' \) be an arbitrary subset of \( A \) of cardinality \( \gamma(\overline{G}) - 2 \), and let \( A^* = A \setminus A' \). Thus, \( |A'| = \gamma(\overline{G}) - 2 \) and \( |A^*| = |A| - |A'| = d_G(v) - \gamma(\overline{G}) + 2 \). Since \( d_A(x) \geq \gamma(\overline{G}) - 1 \) for every vertex \( x \in B \), the set \( A^* \) dominates the set \( B \) in \( G \). Thus, \( A^* \cup \{v\} \) is a dominating set in \( G \), implying that \( \gamma(G) \leq |A^*| + 1 = d_G(v) - \gamma(\overline{G}) + 3 \).

As a consequence of Theorem 6, we have the following result.

**Corollary 6.** If \( G \) is an \( r \)-regular graph that contains a \( \gamma(\overline{G}) \)-bad vertex, then \( \gamma(G) + \gamma(\overline{G}) \leq r + 3 \).

### 2.4. Domination and planarity

In this section, we study some relationships between planarity, the crossing number of \( G \) and the domination number of \( \overline{G} \). Fundamental to our results in this section is the famous Four Color Theorem.

**Theorem 7** [1]. If \( G \) is a planar graph, then \( \chi(G) \leq 4 \).

We first establish the following upper bound on the domination number of the complement of a graph. For this purpose, for a vertex \( v \) in a graph \( G \), we denote by \( G_v \) the subgraph of \( G \) induced by the neighbors of \( v \); that is, \( G_v = G[N(v)] \). If \( \mathcal{C} \) is a minimum coloring of the vertices of \( G_v \), and \( S \) is a set of vertices comprising of exactly one vertex from each color class of \( \mathcal{C} \), then the set \( S \cup \{v\} \) forms a dominating set of \( \overline{G} \), implying that \( \gamma(\overline{G}) \leq |\mathcal{C}| + 1 = \chi(G_v) + 1 \). We state this formally as follows.

**Observation 8.** If \( v \) is an arbitrary vertex in a graph \( G \), then \( \gamma(\overline{G}) \leq \chi(G_v) + 1 \).

As a consequence of Theorem 7 and Observation 8, we have the following results.

**Corollary 7.** If a graph \( G \) contains a vertex \( v \) with the property that \( G_v \) is a planar graph, then \( \gamma(\overline{G}) \leq 5 \).
Corollary 8. If a graph $G$ satisfies $\gamma(G) > 2cr(G)$, then $\gamma(G) \leq (1 + \sqrt{4n - 1})/2$.

Proof. Let $G^*$ be a drawing of $G$ in the plane with exactly $cr(G)$ crossing edges, and let $S$ be the set of vertices of $G$ incident with at least one crossing edge of $G^*$. Clearly, $|S| \leq 2cr(G)$. Since, by assumption, $\gamma(G) > 2cr(G)$, it follows there exists some vertex $v$ in $G$ that is not dominated by $S$. This implies that $G_v$ is a planar graph. Thus, by Corollary 7, $\gamma(G_v) \leq 5$.

3. Total, Connected, Restrained, and Clique Domination

In this section, we establish relationships involving the domination, total domination, restrained domination, connected domination and clique domination numbers of a graph. We begin with the following lemma.

Lemma 9. If there exists a $\gamma(G)$-set for a graph $G$ that is not a dominating set in $\overline{G}$, then $\gamma_t(G) \leq \gamma_c(G) \leq \gamma(G) + 1$.

Proof. Let $S$ be a $\gamma(G)$-set in a graph $G = (V, E)$ that is not a dominating set in $\overline{G}$. Then there exists a vertex $v \in V \setminus S$ that is not adjacent to any vertex of $S$ in $\overline{G}$. Hence in $G$, the vertex $v$ is adjacent to every vertex of $S$, implying that the graph $G[S \cup \{v\}]$ is connected. Since every superset of a dominating set is also a dominating set, the set $S \cup \{v\}$ is a CD-set, and so $\gamma_c(G) \leq |S \cup \{v\}| = \gamma(G) + 1$. Since the total domination of a graph is at most its connected domination number, the desired result follows from the observation that $\gamma_t(G) \leq \gamma_c(G)$.

By the contrapositive of Lemma 9, we note that if a graph $G$ satisfies $\gamma_t(G) \geq \gamma(G) + 2$, then every $\gamma(G)$-set is a dominating set in $\overline{G}$. Further as a consequence of Lemma 9 and the well-known result due to Jaeger and Payan [11] that if $G$ is a graph of order $n$, then $\gamma(G) \gamma(\overline{G}) \leq n$, we have the following result.

Corollary 10. Let $G$ be a graph of order $n$ satisfying $\gamma(G) < \gamma(\overline{G})$. Then the following holds.

(a) $\gamma_t(G) \leq \gamma_c(G) \leq \gamma(G) + 1$.
(b) $\gamma_c(G) \leq (1 + \sqrt{4n + 1})/2$.

Proof. Part (a) is an immediate consequence of Lemma 9. To prove part (b), let $G$ be a graph of order $n$ satisfying $\gamma(G) < \gamma(\overline{G})$. By part (a) and our assumption that $\gamma(G) \leq \gamma(\overline{G}) - 1$, we have that $\gamma_c(G) \leq \gamma(G) + 1 \leq \gamma(\overline{G})$. Applying the result due to Jaeger and Payan, we therefore have that $(\gamma_c(G) - 1)\gamma_c(G) \leq \gamma(G)\gamma(\overline{G}) \leq n$. Solving for $\gamma_c(G)$, we have that $\gamma_c(G) \leq (1 + \sqrt{4n + 1})/2$.

In the following result, we consider the case when $\gamma(G) \leq \gamma(\overline{G}) + 1$. 

Theorem 9. Let $G$ be a graph satisfying $\gamma(G) \leq \gamma(G) + 1$. Then the following holds.

(a) If both $G$ and $\overline{G}$ are connected, then $\gamma_c(G) \leq \gamma(G) + 1$ or $\gamma_c(\overline{G}) \leq \gamma(\overline{G}) + 1$.
(b) If both $G$ and $\overline{G}$ are isolate-free, then $\gamma_t(G) \leq \gamma(G) + 1$ or $\gamma_t(\overline{G}) \leq \gamma(\overline{G}) + 1$.

Proof. Let $G = (V, E)$, and let $S$ be a $\gamma(G)$-set in the graph. We first establish part (a). Suppose that both $G$ and $\overline{G}$ are connected. If $G[S]$ is connected, then $S$ is a CD-set in $G$, implying that $\gamma_c(G) \leq |S| = \gamma(G)$. Hence we may assume that $G[S]$ is not connected, for otherwise part (a) is immediate. This implies that $\overline{G}[S]$ is connected. If the set $S$ is not a dominating set in $\overline{G}$, then by Lemma 9, we have that $\gamma_c(\overline{G}) \leq \gamma(\overline{G}) + 1$. If the set $S$ is a dominating set in $\overline{G}$, then $S$ is a CD-set in $\overline{G}$, implying that $\gamma_c(\overline{G}) \leq |S| = \gamma(G) \leq \gamma(\overline{G}) + 1$. This proves part (a).

Next we prove part (b). Suppose that both $G$ and $\overline{G}$ are isolate-free. If $G[S]$ is isolate-free, then $S$ is a TD-set in $G$, implying that $\gamma_t(G) \leq |S| = \gamma(G)$. Hence we may assume that $G[S]$ contains an isolated vertex, for otherwise part (b) is immediate. This implies that $\overline{G}[S]$ is connected. If the set $S$ is not a dominating set in $\overline{G}$, then by Lemma 9 we have that $\gamma_t(G) \leq \gamma(G) + 1$. If the set $S$ is a dominating set in $\overline{G}$, then $S$ is a TD-set in $\overline{G}$, implying that $\gamma_t(\overline{G}) \leq |S| = \gamma(\overline{G}) \leq \gamma(G) + 1$. This proves part (b).

We establish next an upper bound on the total domination number of a graph in terms of its domination number and the domination number of its complement.

Theorem 10. Let $G$ be an isolate-free graph, and let $S$ be a $\gamma(G)$-set. If $s$ is the number of isolated vertices in $G[S]$, then $\gamma_t(G) \leq \gamma(G) + \lceil s/(\gamma(\overline{G}) - 1) \rceil$.

Proof. Let $G = (V, E)$. Since $G$ is isolate-free, we note that $\gamma(\overline{G}) \geq 2$. Let $I$ be the set of isolated vertices in $G[S]$, and so $s = |I|$. Let $k = \lceil s/(\gamma - 1) \rceil$, and partition the set $I$ into $k$ sets $I_1, \ldots, I_k$ each of cardinality at most $\gamma - 1$. Thus, $I = \bigcup_{i=1}^{k} I_i$ and $1 \leq |I_i| \leq \gamma - 1$ for each $i$, $1 \leq i \leq k$. In particular, we note that no set $I_i$ dominates $V$ in $\overline{G}$. For each set $I_i$, $1 \leq i \leq k$, select one vertex $w_i \in V \setminus I_i$ that is not dominated by $I_i$ in $\overline{G}$, and let $W = \bigcup_{i=1}^{k} \{w_i\}$. Then, $|W| \leq k$. We note that in the graph $G$, the vertex $w_i$ is adjacent to every vertex of $I_i$, and so $S \cup W$ is a TD-set in $G$. Hence, $\gamma_t(G) \leq |S \cup W| \leq |S| + |W| \leq \gamma(G) + k = \gamma(G) + \lceil s/(\gamma - 1) \rceil$.

As an immediate consequence of Theorem 10, we have the following upper bound on the total domination number of a graph.

Corollary 11. If $G$ is an isolate-free graph, then $\gamma_t(G) \leq \gamma(G) + \left\lceil \frac{\gamma(G)}{\gamma(\overline{G}) - 1} \right\rceil$.

Theorem 11. If $G$ is a graph with $\gamma_t(G) \geq \gamma(G) + 2$, then $\gamma_t(\overline{G}) \leq 1 + \left\lceil \frac{\delta(G)}{\gamma(\overline{G})} \right\rceil$. 
Proof. Let $G = (V, E)$ be a graph with $\gamma_t(G) \geq \gamma(G) + 2$, and let $v$ be a vertex of $G$ having degree $\delta(G)$. Let $A = N_G(v)$, and so $|A| = \delta(G)$. Let $k = \lceil \delta(G)/\gamma_t(G) \rceil$ and partition the set $A$ into $k$ sets $A_1, \ldots, A_k$ each of cardinality at most $\gamma_t(G)$. Thus, $A = \bigcup_{i=1}^k A_i$ and $1 \leq |A_i| \leq \gamma_t(G)$ for each $i$, $1 \leq i \leq k$. If the set $A_i$ dominates $V \setminus N_G[v]$ in $G$ for some $i$, $1 \leq i \leq k$, then the set $A_i \cup \{v\}$ is a TD-set in $G$, implying that $\gamma_t(G) \leq |A_i| + 1 \leq \gamma_t(G) + 1$, a contradiction. Therefore, no set $A_i$ dominates $V \setminus N_G[v]$ in $G$. For each set $A_i$, $1 \leq i \leq k$, select one vertex $a_i \in V \setminus N_G[v]$ that is not dominated by $A_i$ in $G$, and let $A' = \bigcup_{i=1}^k \{a_i\}$. Then, $|A'| \leq k$ and $A'$ dominates $A$ in $\overline{G}$. Therefore, the set $A' \cup \{v\}$ is a TD-set in $\overline{G}$, and so $\gamma_t(\overline{G}) \leq |A'| + 1 \leq k + 1 = 1 + \lceil \delta(G)/\gamma_t(G) \rceil$. □

Next we consider the restrained domination number. We first prove a general lemma.

Lemma 12. If a graph $G$ has a $\gamma(G)$-set $S$ such that the induced subgraph $G[V \setminus S]$ has an isolated vertex, then $\gamma_t(G) \leq 3$.

Proof. Let $S$ be a $\gamma(G)$-set such that $G[V \setminus S]$ has an isolated vertex, say $w$. If $G[S]$ has an isolated vertex $v$, then $\{v, w\}$ is dominating set of $\overline{G}$, and so $\gamma_t(\overline{G}) \leq 2$. If $G[S]$ contains no isolated vertices, then by the minimality of $S$, for each $v \in S$, there exists a vertex, say $v' \in V \setminus S$, such that $N(v') \cap S = \{v\}$. In this case, the set $\{v, w, v'\}$ is a dominating set of $\overline{G}$, implying that $\gamma_t(\overline{G}) \leq 3$. □

As an immediate consequence of Lemma 12, we have the following result.

Corollary 13. If a graph $G$ has $\gamma_t(G) \geq 4$, then every $\gamma_t(G)$-set is a $\gamma_t(G)$-set. In particular, $\gamma_t(G) = \gamma_t(G)$.

We close this section with two results about the clique domination number of a graph.

Theorem 12. If $G$ is a graph with $\gamma_t(G) \geq \gamma_t(G) + 2$, then $\gamma_t(\overline{G}) \leq \gamma(G)$. Moreover, if $G$ is claw-free, then $\gamma_t(\overline{G}) \leq 3$.

Proof. Let $G$ be a graph with $\gamma_t(G) \geq \gamma_t(G) + 2$, and let $S$ be a $\gamma(G)$-set. Further, let $I(S)$ be the set of isolated vertices in $G[S]$. If $I(S) = \emptyset$, then $S$ is a TD-set of $G$, implying that $\gamma_t(G) \leq |S| = \gamma_t(G)$, a contradiction. Hence, $I(S) \neq \emptyset$. We show that $I(S)$ dominates $\overline{G}$. Suppose to the contrary that there exists a vertex $v$ that is not adjacent to any vertex of $I(S)$ in $\overline{G}$. Then in the graph $G$, the vertex $v$ is adjacent to every vertex of $I(S)$, implying that $S \cup \{v\}$ is a TD-set for $G$, and so $\gamma_t(G) \leq |S| + 1 = \gamma_t(G) + 1$, a contradiction. Hence, the set $I(S)$ dominates $\overline{G}$. Since $I(S)$ is an independent set in $G$, it forms a clique in $\overline{G}$. Therefore, $I(S)$ is a dominating clique in $\overline{G}$, implying that $\gamma_t(\overline{G}) \leq |I(S)| \leq \gamma_t(G)$.

Now, suppose that $G$ is claw-free. If $|I(S)| \leq 3$, then the result follows. Hence, we may assume that $|I(S)| \geq 4$ and there exists a subset $\{a, b, c\} \subseteq I(S)$
that is not a dominating set in $G$. Then there exists a vertex $v$ that is not adjacent to $a$, $b$, or $c$ in $G$. But then in the graph $G$, we have that $\{a, b, c, v\}$ induces a claw, a contradiction. Therefore, every subset of $I(S)$ of cardinality 3 is a dominating set in $G$, implying that $\gamma_d(G) \leq 3$.

4. Bounds on the Domination Number of a Graph in Terms of the Adjacency Matrix of its Complement

We begin this section by stating two well-known theorems. The first result counts the number of walks of length $k$ for an arbitrary positive integer $k$ in a graph (see [3]; see also Theorem 1.17 in [5]). The second result is a consequence of a result due to Vizing [16] and provides an upper bound for the domination number of a graph in terms of its order and size.

**Theorem 13** [3]. Let $G$ be a graph of order $n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and with adjacency matrix $A$. For each positive integer $k$, the number of different walks of length $k$ from the vertex $v_i$ to the vertex $v_j$ is the $(i,j)$-entry in the matrix $A^k$.

**Theorem 14** [16]. If $G$ is graph of order $n$ and size $m$, then $\gamma(G) \leq n + 1 - \sqrt{1 + 2m}$.

Let $G$ be a graph of order $n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and with adjacency matrix $A$, and let $a_{ij}^{(k)}$ denote the $(i,j)$-entry in $A^k$. Recall that if $v$ is a vertex in $G$, then the subgraph of $G$ induced by $N_G(v)$ is called the link of $v$ and is denoted by $L(v)$, while the subgraph of $\overline{G}$ induced by $N_G(v)$ is denoted $\overline{L}(v)$.

Theorem 13 implies that the $(i, i)$-entry of $A^2$, $1 \leq i \leq n$, is the degree $d_G(v_i)$ of $v_i$, and the $(i, i)$-entry of $A^3$, $1 \leq i \leq n$, is equal to twice the number of edges in $L(v_i)$. Suppose that $a_{ii}^{(3)} < a_{ii}^{(2)}$ for some $i$, $1 \leq i \leq n$. Since $a_{ii}^{(2)} = d_G(v_i)$ and $\frac{1}{2} a_{ii}^{(3)}$ is the number of edges in $L(v_i)$, this implies that $L(v_i)$ contains an isolated vertex, $v$ say. Thus the set $\{v, v_i\}$ is a dominating set in the graph $G$, implying that $\gamma(\overline{G}) \leq 2$. We state this formally as follows.

**Observation 15**. Let $G$ be an isolate-free graph of order $n$ with adjacency matrix $A$. If the $(i, i)$-entry of $A^3$ is less than the $(i, i)$-entry of $A^2$ for some $i$, $1 \leq i \leq n$, then $\gamma(\overline{G}) \leq 2$.

Using Observation 15, we obtain the following bound on the domination number of the complement of a graph.

**Theorem 16**. Let $G$ be a graph of order $n$ with adjacency matrix $A$, and let $a_{ij}^{(k)}$ denote the $(i,j)$-entry in $A^k$. For every $i$, $1 \leq i \leq n$, we have that

$$\gamma(\overline{G}) \leq a_{ii}^{(2)} + 2 - \sqrt{1 + a_{ii}^{(2)}(a_{ii}^{(2)} - 1) - a_{ii}^{(3)}}.$$
Proof. Let \( i \) be an arbitrary integer with \( 1 \leq i \leq n \). Since \( a^{(2)}_{ii} = d_G(v_i) \) and \( \frac{1}{2}a^{(3)}_{ii} \) is the number of edges in \( L(v_i) \), this implies that \( \overline{L}(v_i) \) has order \( a^{(2)}_{ii} \) and size
\[
\left( \frac{a^{(2)}_{ii}}{2} \right) - \frac{1}{2}a^{(3)}_{ii} = \frac{1}{2} \left( a^{(2)}_{ii} (a^{(2)}_{ii} - 1) - a^{(3)}_{ii} \right).
\]
Thus, by Theorem 14, we have that
\[
\gamma(\overline{L}(v_i)) \leq a^{(2)}_{ii} + 1 - \sqrt{1 + a^{(2)}_{ii} (a^{(2)}_{ii} - 1) - a^{(3)}_{ii}}.
\]
The desired bound now follows from the observation that every \( \gamma(\overline{L}(v_i)) \)-set can be extended to a dominating set in \( G \) by adding to it the vertex \( v_i \), and so
\[
\gamma(G) \leq \gamma(\overline{L}(v_i)) + 1.
\]

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