CORE INDEX OF PERFECT MATCHING POLYTOPE FOR A 2-CONNECTED CUBIC GRAPH

XIUMEI WANG AND YIXUN LIN

School of Mathematics and Statistics
Zhengzhou University
Zhengzhou 450001, China

e-mail: wangxiumei@zzu.edu.cn.

Abstract

For a 2-connected cubic graph $G$, the perfect matching polytope $P(G)$ of $G$ contains a special point $x^c = (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$. The core index $\varphi(P(G))$ of the polytope $P(G)$ is the minimum number of vertices of $P(G)$ whose convex hull contains $x^c$. The Fulkerson’s conjecture asserts that every 2-connected cubic graph $G$ has six perfect matchings such that each edge appears in exactly two of them, namely, there are six vertices of $P(G)$ such that $x^c$ is the convex combination of them, which implies that $\varphi(P(G)) \leq 6$. It turns out that the latter assertion in turn implies the Fan-Raspaud conjecture: In every 2-connected cubic graph $G$, there are three perfect matchings $M_1$, $M_2$, and $M_3$ such that $M_1 \cap M_2 \cap M_3 = \emptyset$. In this paper we prove the Fan-Raspaud conjecture for $\varphi(P(G)) \leq 12$ with certain dimensional conditions.

Keywords: Fulkerson’s conjecture, Fan-Raspaud conjecture, cubic graph, perfect matching polytope, core index.

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1. Introduction

The celebrated Fulkerson’s conjecture in graph theory is the following (cf. [1, 5]).

**Conjecture A** (Fulkerson’s conjecture). Every 2-connected cubic graph has six perfect matchings such that each edge appears in exactly two of them.

We may state the polyhedral version of this conjecture as follows. Let $G$ be a 2-connected cubic graph. Thus each edge of $G$ is contained in a perfect matching of $G$. The characteristic vector of a perfect matching $M$ of $G$ is a vector $x \in \mathbb{R}^{E(G)}$ such that $x_e = 1$ if $e \in M$ and $x_e = 0$ otherwise. The perfect
matching polytope of $G$, denoted by $P(G)$, is the convex hull of the characteristic vectors of all perfect matchings in $G$. Now let $x^1, x^2, \ldots, x^6$ be the characteristic vectors of the six perfect matchings in the Fulkerson’s conjecture. Then $x^1 + x^2 + \cdots + x^6 = (2, 2, \ldots, 2)$ and thus $\frac{1}{6}(x^1 + x^2 + \cdots + x^6) = (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$. That is, $x^c := (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$ is the convex combination of $x^1, x^2, \ldots, x^6$.

We call this $x^c$ the core of the perfect matching polytope $P(G)$, which lies in $P(G)$ (see Proposition 3). Furthermore, a subset $Q \subseteq P(G)$ is called a core polytope of $P(G)$ if it is the convex hull of $k$ vertices of $P(G)$ such that $x^c \in Q$ and $k$ is minimum. Meanwhile, the above minimum value $k$ is called the core index of $P(G)$, denoted by $\varphi(P(G))$. In other words, the core index $\varphi(P(G))$ is the minimum number of vertices of $P(G)$ whose convex hull contains $x^c$. Therefore, the Fulkerson’s conjecture yields the following conjecture.

**Conjecture B.** For every 2-connected cubic graph $G$, $\varphi(P(G)) \leq 6$.

The study of Conjecture B would be meaningful to cope with the Fulkerson’s conjecture. In particular, the structure of the core polytope $Q$ inside a perfect matching polytope $P(G)$ is quite mysterious. Fan and Raspaud [5] proposed the following conjecture.

**Conjecture C (Fan-Raspaud conjecture).** In every 2-connected cubic graph there exist three perfect matchings $M_1$, $M_2$, and $M_3$ such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

Let us see the relation of these three conjectures.

**Proposition 1.** Conjecture A implies Conjecture B and Conjecture B implies Conjecture C.

**Proof.** The first assertion is clear, as $x^c := (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$ is contained in the convex hull of $\{x^1, x^2, \ldots, x^6\}$. We show the second assertion. Suppose Conjecture B holds. Let $x^1, x^2, \ldots, x^6$ be six vertices of $P(G)$ whose convex hull contains $x^c$. Then $x^c = c_1 x^1 + c_2 x^2 + \cdots + c_6 x^6$, where $\sum_{i=1}^{6} c_i = 1$ and $c_i \geq 0$. We may assume that $c_1, c_2, c_3$ are the three largest numbers among all $c_i$. Then $c_1 + c_2 + c_3 \geq \frac{1}{2} \sum_{i=1}^{6} c_i = \frac{1}{2}$. We claim that Conjecture C holds for the perfect matchings $M_1, M_2, M_3$ corresponding to $x^1, x^2, x^3$. Suppose not. Then there is an edge $e \in M_1 \cap M_2 \cap M_3$, namely, $x^e_i = 1$ for $i = 1, 2, 3$. Then $x^e_i = c_1 x^e_i + c_2 x^e_i + \cdots + c_6 x^e_i \geq c_1 x^1_i + c_2 x^2_i + c_3 x^3_i = c_1 + c_2 + c_3 \geq \frac{1}{2} > \frac{1}{3}$, which is a contradiction.

In brief, if the Fulkerson’s conjecture is true, then $\varphi(P(G)) \leq 6$, and thus the Fan-Raspaud conjecture holds. However, the Fulkerson’s conjecture is far from being proved at the moment. So we do not know the exact range of values $\varphi(P(G))$ for all 2-connected cubic graphs. In this circumstance we can use the parameter $\varphi(P(G))$ as a condition in proving the Fan-Raspaud conjecture.
The dimension of a polytope $P$, denoted by $d(P)$, is the dimension of its affine hull (the minimal affine subspace containing $P$). Let $G$ be a 2-connected cubic graph, and $P(G)$ the perfect matching polytope of $G$. If all vertices of $P(G)$ are affinely independent (namely, $P(G)$ is a simplex), then $d(P(G)) \leq 8$ (see [3, 9]). In our previous paper [9], we showed that Fan-Raspaud conjecture holds if $d(P(G)) \leq 9$, which implies $\varphi(P(G)) \leq 10$. In this paper, we obtain some improved results. The main results are the following:

1. The Fan-Raspaud conjecture is true if $d(P(G)) \leq 13$ and $\varphi(P(G)) \leq 11$.
2. The Fan-Raspaud conjecture is true if $G$ is a cubic brick, $d(P(G)) \leq 18$, and $\varphi(P(G)) \leq 12$.

Since the dimension of a cubic brick $G$ is $d(P(G)) = m - n = n/2$ (see Lemma 4 with $b = 1$), $d(P(G)) \leq 18$ is equivalent to $n = |V(G)| \leq 36$. Hence the above result (2) means that the Fan-Raspaud conjecture is true for cubic bricks with up to 36 vertices (provided $\varphi(P(G)) \leq 12$). Recently, in [2], the computer search shows that the Fulkerson’s conjecture is true for snarks with up to 36 vertices, and so is the Fan-Raspaud conjecture. Here, a snark is a cyclically 4-edge connected cubic graph which cannot be 3-edge colored and has girth at least 5.

The organization of the paper is as follows. In Section 2, we present some basic properties. Section 3 is devoted to the results on 2-connected cubic graphs with $\varphi(P(G)) \leq 11$. Section 4 is concerned with cubic bricks with $\varphi(P(G)) \leq 12$.

2. Preliminary on Perfect Matching Polytopes

The basic notions on polyhedral combinatorics can be found in [6, 8]. The well-known characterization of perfect matching polytope, due to Edmonds (1965), is the following (cf. [7]).

**Lemma 2.** The perfect matching polytope of a graph $G$ is the set of vectors $x \in \mathbb{R}^{E(G)}$ satisfying

1. $x_e \geq 0$ (for all $e \in E(G)$),
2. $\sum_{e \in \delta(v)} x_e = 1$ (for all $v \in V(G)$),
3. $\sum_{e \in \delta(A)} x_e \geq 1$ (for all $A \subseteq V(G), |A|$ is odd),

where $\delta(v)$ stands for the set of edges incident with $v \in V(G)$, and $\delta(A)$ is the set of edges with exactly one end in $A$. 
For a 2-connected cubic graph $G$, every edge is contained in a perfect matching (see Corollary 3.4.3 of [7]). A graph is matching-covered if every edge of this graph is contained in a perfect matching of it. Hence $G$ is matching-covered and there are at least three different perfect matchings in $G$. Thus the perfect matching polytope $P(G)$ has at least three vertices. In particular, we have

**Proposition 3.** Let $G$ be a 2-connected cubic graph. Then the core $x_c = \left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ lies in $P(G)$.

**Proof.** Clearly, $x_c$ satisfies (1) and (2) of Lemma 2. It suffices to verify (3). For a subset $A \subseteq V(G)$ with odd cardinality, the degree sum of the vertices in $A$ is $3|A| = 2|E(G[A])| + |\delta(A)|$, where $G[A]$ is the subgraph of $G$ induced by $A$. So $|\delta(A)|$ is odd. If $|\delta(A)| = 1$, then the only edge of $\delta(A)$ is a cut edge of $G$, contradicting the assumption that $G$ is 2-connected. Therefore $|\delta(A)| \geq 3$ and thus $\sum_{e \in \delta(A)} x^e_c = \frac{1}{3} |\delta(A)| \geq 1$. That is, the point $x_c$ satisfies (3), as required. $lacksquare$

The following characterization of the dimension of perfect matching polytope found by Edmonds, Lovász, and Pulleyblank (see [4] or Theorem 7.6.6 of [7]) is used in the proof of the main results.

**Lemma 4.** For every matching-covered graph $G$, the dimension of perfect matching polytope $P(G)$ is $d(P(G)) = m - n + 1 - b$, where $m, n, b$ are the numbers of edges, vertices, and bricks of $G$, respectively.

Here, a brick is a 3-connected and bicritical graph, where a graph $G$ is bicritical if $G - u - v$ has a perfect matching for any two distinct vertices $u, v$ in $G$. Clearly, a brick is non-bipartite and matching-covered. The number of bricks of a matching-covered graph $G$ is the number of bricks produced in a procedure of ‘tight cut decomposition’, see [4, 7].

With respect to the dimension, the following Carathéodory theorem is classical (Theorem 5.1 of [8]).

**Lemma 5.** For any $V \subseteq \mathbb{R}^m$ and $x$ in the convex hull of $V$, there exist affinely independent vectors $x^1, \ldots, x^k$ in $V$ such that $x$ is contained in the convex hull of $\{x^1, \ldots, x^k\}$.

We obtain an upper bound of the core index as follows.

**Proposition 6.** For every 2-connected cubic graph $G$, $\varphi(P(G)) \leq d(P(G)) + 1$.

**Proof.** Let $V$ be the set of vertices in $P(G)$. Then $x^c$ is contained in the convex hull of $V$. By the Carathéodory theorem, $x^c$ is contained in the convex hull of $d(P(G)) + 1$ affinely independent vectors. The assertion follows. $lacksquare$
3. Results on Core Index and Dimension

For convenience, we refer to the property specified in the Fan-Raspaud conjecture as the 3PM-property. We start with some simple facts.

If \( \varphi(P(G)) = k \), then there are vertices \( x^1, x^2, \ldots, x^k \) of \( P(G) \) whose convex hull contains the core \( x^c \), i.e.,

\[
x^c = c_1 x^1 + c_2 x^2 + \cdots + c_k x^k, \quad \sum_{i=1}^{k} c_i = 1, \ c_i > 0.
\]

Let \( S^c = \{c_1, c_2, \ldots, c_k\} \), which stands for the convex combination representation of the core \( x^c \) in the convex hull of \( \{x^1, x^2, \ldots, x^k\} \). Meanwhile, each \( c_i \in S^c \) is called a \( c \)-element. For a set \( S \), by an \( h \)-combination \( X \) of \( S \) we mean a subset \( X \subseteq S \) with \( |X| = h \). Furthermore, for every edge \( e \in E(G) \), we have

\[
c_1 x^1_e + c_2 x^2_e + \cdots + c_k x^k_e = \frac{1}{3}.
\]

Let \( M_1, M_2, \ldots, M_k \) be the perfect matchings corresponding to \( x^1, x^2, \ldots, x^k \) respectively, and let \( \mathcal{M} = \{M_1, M_2, \ldots, M_k\} \). For an edge \( e \in E(G) \), let \( \mathcal{M}_e = \{M_i : x^i_e = 1, 1 \leq i \leq k\} \) (\( \subseteq \mathcal{M} \)), which is the set of perfect matchings containing the edge \( e \). Then (5) is equivalent to

\[
\sum_{M_i \in \mathcal{M}_e} c_i = \frac{1}{3}.
\]

Conversely, a subset \( S \) of \( S^c \) with \( \sum_{c_i \in S} c_i = \frac{1}{3} \) is not necessarily corresponding to an edge. We now give a useful definition as follows.

A subset \( S \) of \( S^c \) is called an edge-combination if (1) \( \sum_{c_i \in S} c_i = \frac{1}{3} \); (2) there exists a 3-combination \( X \) of \( S \) such that

\[
\bigcap_{c_i \in X} M_i \neq \emptyset \quad \text{implies} \quad \bigcap_{c_i \in S} M_i \neq \emptyset.
\]

For instance, if the 3PM-property does not hold, then any 3-combination \( S \) of \( S^c \) with \( \sum_{c_i \in S} c_i = \frac{1}{3} \) is an edge-combination. Moreover, for any edge-combination \( S \) of \( S^c \), we have \( |S| \geq 3 \) and \( \bigcap_{c_i \in S} M_i \neq \emptyset \), and thus \( S \) indeed corresponds to an edge in \( \bigcap_{c_i \in S} M_i \). This is the intention of the term “edge-combination”. We call the number of all the edge-combinations in \( S^c \) the edge-combination number of \( S^c \), denoted by \( \mathcal{E}(S^c) \).

**Lemma 7.** If \( |E(G)| < \mathcal{E}(S^c) \), then the 3PM property holds.

**Proof.** If the 3PM-property does not hold, then for each edge-combination \( S \), we have \( \bigcap_{c_i \in S} M_i \neq \emptyset \). Thus there exists an edge \( e \) contained in \( \bigcap_{c_i \in S} M_i \). So each
edge-combination $S$ corresponds to an edge $e \in \bigcap_{c_i \in S} M_i$. Furthermore, we claim that an edge $e \in E(G)$ cannot correspond to two different edge-combinations. In fact, if $e$ corresponds to two edge-combinations $S$ and $S'$ with $S \neq S'$, then $e \in \bigcap_{c_i \in S} M_i$ and $e \in \bigcap_{c_i \in S'} M_i$, whence $e \in \bigcap_{c_i \in S \cup S'} M_i$. By the definition of edge-combination, we have $\sum_{c_i \in S} c_i = \frac{1}{3}$ and $\sum_{c_i \in S'} c_i = \frac{1}{3}$. Hence $\sum_{M_i \in M_e} c_i \geq \sum_{c_i \in S \cup S'} c_i > \sum_{c_i \in S} c_i = \frac{1}{3}$, contradicting the equation (6). In this way, we define an injection (one-to-one mapping) from the set of edge-combinations to $E(G)$. Therefore $|\mathcal{E}(S^c)| \leq |E(G)|$, contradicting the condition of the lemma. ■

Lemma 8. If the 3PM-property does not hold, then $S^c = \{c_1, c_2, \ldots, c_k\}$ satisfies the following:

(i) For any 3-combination $X \subseteq S^c$, $\sum_{c_i \in X} c_i \leq \frac{1}{3}$;

(ii) For any 3-combination $X \subseteq S^c$, there exists an edge-combination $S \subseteq S^c$ such that $X \subseteq S$;

(iii) $S^c$ can be partitioned into three parts $\{S_1, S_2, S_3\}$ such that $\sum_{c_i \in S_k} c_i = \frac{1}{3}$ for $k = 1, 2, 3$.

Proof. Suppose that the 3PM-property does not hold. We show the three assertions as follows.

(i) Suppose that for $X = \{c_{i_1}, c_{i_2}, c_{i_3}\}$, $c_{i_1} + c_{i_2} + c_{i_3} > \frac{1}{3}$. Since $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$, there is an edge $e \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$ such that $x_e^c = c_{i_1}x_e^1 + c_{i_2}x_e^2 + \cdots + c_{i_3}x_e^3 \geq c_{i_1} + c_{i_2} + c_{i_3} > \frac{1}{3}$, a contradiction.

(ii) For any 3-combination $X \subseteq S^c$, since $\bigcap_{c_i \in X} M_i \neq \emptyset$, there exists an edge $e$ in $\bigcap_{c_i \in X} M_i$. Let $S = \{c_i : M_i \in \mathcal{M}_e\}$. Then $X \subseteq S$ and $\sum_{c_i \in S} c_i = \sum_{M_i \in \mathcal{M}_e} c_i = \frac{1}{3}$, which implies that $S$ is an edge-combination.

(iii) We take a vertex $v$ in $G$ and let $e_1, e_2, e_3$ be the three edges incident with this vertex $v$ in $G$. Since no perfect matching $M_i$ can contain two of $e_1, e_2, e_3$, all perfect matchings $M_1, M_2, \ldots, M_k$ are partitioned into three disjoint sets, each of which contains one of $e_1, e_2, e_3$. Therefore $S^c$ is partitioned into three parts $\{S_1, S_2, S_3\}$ such that $\sum_{c_i \in S_k} c_i = \sum_{M_i \in \mathcal{M}_k} c_i = \frac{1}{3}$ for $k = 1, 2, 3$. ■

Our previous paper [9] shows the following.

Proposition 9. For a 2-connected cubic graph $G$ with $d(P(G)) \leq 9$, the 3PM-property holds.

Now we present several improved results. Proposition 1 says that if $\phi(P(G)) \leq 6$, then the 3PM-property holds. The following improvement is straightforward.

Proposition 10. For a 2-connected cubic graph $G$ with $\phi(P(G)) \leq 8$, the 3PM-property holds.
**Proof.** Suppose that (4) holds for \( k = 8 \) and \( c_1, c_2, c_3, c_4 \) are the four largest elements in \( S^c \) with \( c_1 \geq c_2 \geq c_3 \geq c_4 \). Then \( c_1 + c_2 + c_3 + c_4 \geq \frac{1}{2} \). If \( c_1 + c_2 + c_3 < \frac{3}{5} \), then \( c_3 < \frac{1}{5} \) and \( c_4 \geq \frac{1}{2} - (c_1 + c_2 + c_3) > \frac{1}{5} \), a contradiction to the assumption that \( c_3 \geq c_4 \). Therefore \( c_1 + c_2 + c_3 \geq \frac{3}{5} > \frac{1}{5} \), and thus the assertion follows from (i) of Lemma 8.

This result is independent of the dimension of the perfect matching polytope.

In the results below, we have to combine the dimensional condition.

**Lemma 11.** For a 2-connected cubic graph \( G \), if \( d(P(G)) = d \), then \( |E(G)| \leq 6(d - 1) \).

**Proof.** Since \( G \) is cubic, we have \( 2|E(G)| = 3|V(G)| \). Moreover, as a result of the brick decompositions of graphs, \( G \) has at most \( \frac{|E(G)| - |V(G)|}{2} = \frac{|E(G)|}{b} \) bricks (see Lemma 5.12 of [4]). Hence by the formula of dimension of perfect matching polytope, we have \( d = m - n + 1 - b \geq m - \frac{2m}{3} + 1 - \frac{m}{6} = \frac{m}{6} + 1 \), which implies that \( m \leq 6(d - 1) \).

**Theorem 12.** For a 2-connected cubic graph \( G \) with \( d(P(G)) \leq 14 \), if \( \varphi(P(G)) \leq 10 \), then the 3PM-property holds.

**Proof.** By Lemma 11 and \( d(P(G)) \leq 14 \), we have \( |E(G)| \leq 6(d - 1) \leq 78 \). Suppose, to the contrary, that the 3PM-property does not hold.

The case of \( \varphi(P(G)) \leq 8 \) has been settled in Proposition 10. We consider the case of \( \varphi(P(G)) = 9 \) now. In this case, the convex combination of (4) with \( k = 9 \) holds. Suppose, without loss of generality, that \( c_1 \geq c_2 \geq \cdots \geq c_9 \). By Lemma 8(i), the sum of any three \( c \)-elements is at most \( \frac{1}{4} \). On the other hand, \( c_1 + c_2 + c_3 \geq \frac{1}{3} \sum_{i=1}^{9} c_i = \frac{1}{3} \). It follows that \( c_1 + c_2 + c_3 = \frac{1}{3} \), and so \( c_4 + c_5 + c_6 = c_7 + c_8 + c_9 = \frac{1}{3} \). Therefore, \( c_7 \geq \frac{1}{9} \) and \( c_6 \leq \frac{1}{9} \). Since \( c_6 \geq c_7 \), we have \( c_6 = c_7 = \frac{1}{5} \), and so \( c_4 = c_5 = \frac{1}{5} \). By the same way, we have \( c_1 = c_2 = c_3 = \frac{1}{5} \), and \( c_7 = c_8 = c_9 = \frac{1}{5} \). Hence every 3-combination of \( S^c \) is an edge-combination, and so \( \mathcal{E}(S^c) = \binom{9}{3} = 84 > 78 \geq |E(G)| \). This is a contradiction to Lemma 7.

We next consider the case of \( \varphi(P(G)) = 10 \) with convex combination representation (4) with \( k = 10 \). By Lemma 8(i), the sum of any two \( c \)-elements is less than \( \frac{1}{4} \). We further observe that the sum of any five \( c \)-elements is greater than \( \frac{1}{4} \). This is because if there are five \( c \)-elements whose sum is less than or equal to \( \frac{1}{4} \), then the sum of the remaining five \( c \)-elements is at least \( \frac{2}{5} \), say \( c_{11} + c_{12} + \cdots + c_{15} \geq \frac{2}{5} \), and thus the sum of the three largest members of \( \{c_{i1}, c_{i2}, \ldots, c_{i5}\} \) is greater than \( \frac{1}{5} \), contradicting Lemma 8(i). Therefore, each edge of \( G \) is covered by precisely three or four of the ten perfect matchings \( M_1, M_2, \ldots, M_{10} \).

By Lemma 8(iii), \( S^c \) can be partitioned into three parts each of which has sum \( \frac{1}{3} \). Note that the only partition \( \{k_1, k_2, k_3\} \) of integer 10 is \( \{3, 3, 4\} \), where \( k_1 + k_2 + k_3 = 10 \), and \( 3 \leq k_i \leq 4 \). Hence the above partition of the ten \( c \)-elements
is a \( \{3,3,4\} \)-partition. Therefore, we assume, without loss of generality, that 
\[ c_1 + c_2 + c_3 = c_4 + c_5 + c_6 = c_7 + c_8 + c_9 + c_{10} = \frac{1}{9}. \]

By Lemma 8(i), we have 
\[ c_1 = c_2 = \cdots = c_6 = \frac{1}{9} \text{ and } c_7, \ldots, c_{10} \leq \frac{1}{9}. \]
In this context, we choose a 3-combination \( X = \{c_1, c_2, c_3\} \) where \( 7 \leq i \leq 10 \). If \( X \) is an edge-combination, then \( c_i = \frac{1}{9} \). Otherwise, by Lemma 8(ii), \( X \) can be extended to an edge-combination \( S \) with \( |S| = 4 \). Then there exists another \( c_j \) (\( 7 \leq j \leq 10 \)) such that \( c_i + c_j = \frac{1}{9} \). We further claim that among \( c_7, \ldots, c_{10} \), there is at most one such pair with \( c_i + c_j = \frac{1}{9} \). Suppose without loss of generality that \( c_7 + c_8 = c_9 + c_{10} = \frac{1}{9} \) or \( c_7 + c_8 = c_7 + c_9 = \frac{1}{9} \). The former contradicts the assumption \( c_7 + c_8 + c_9 + c_{10} = \frac{1}{3} \), and the latter implies that \( c_{10} > \frac{1}{9} \), also a contradiction. Therefore, we obtain that 
\[ c_1 = c_2 = \cdots = c_8 = \frac{1}{9} \text{ and } c_9 + c_{10} = \frac{1}{9}. \]

We proceed to compute the edge-combination number \( E(S^c) \) as follows.

1. There are \( \binom{3}{3} \) 3-combinations \( \{c_1, c_2, c_3\} \) chosen from \( \{c_1, c_2, \ldots, c_8\} \), each of which is an edge-combination.
2. There are \( \binom{5}{3} \) 4-combinations \( S = \{c_2, c_3, c_9, c_{10}\} \) such that \( \{c_1, c_j\} \) are chosen from \( \{c_1, c_2, \ldots, c_8\} \), each of which is an edge-combination.

To sum up, \( E(S^c) \geq \binom{3}{3} + \binom{5}{3} = 78 \geq |E(G)| \), contradicting Lemma 7. \( \blacksquare \)

**Theorem 13.** For a 2-connected cubic graph \( G \) with \( d(P(G)) \leq 13 \), if \( \varphi(P(G)) \leq 11 \), then the 3PM-property holds.

**Proof.** We consider the case \( \varphi(P(G)) = 11 \) and representation (4) with \( k = 11 \). By Lemma 11, \( d(P(G)) \leq 13 \) implies \( |E(G)| \leq 6(d - 1) \leq 72 \). Suppose, to the contrary, that the 3PM-property does not hold.

By Lemma 8(i), no two \( c \)-elements have sum \( \frac{1}{9} \). Also, no six \( c \)-elements have sum \( \frac{1}{9} \), for otherwise the remaining five \( c \)-elements would have \( \sum c_i = \frac{7}{9} \), and thus there are three of them with \( \sum c_i > \frac{1}{3} \), contradicting Lemma 8(i). Moreover, by Lemma 8(iii), the 11 \( c \)-elements are divided into three sets, each of which has sum \( \frac{1}{9} \). Note that the only partitions \( \{k_1, k_2, k_3\} \) of integer 11 are \( \{3,3,5\} \) and \( \{3,4,4\} \), where \( k_1 + k_2 + k_3 = 11, 3 \leq k_i \leq 5 \). We distinguish two cases as follows.

**Case 1.** There is a \( \{3,3,5\} \)-partition of \( S^c \). By Lemma 8(i), the first six \( c \)-elements are \( \frac{1}{9} \) and no other \( c \) is greater than \( \frac{1}{9} \). Therefore, we can sort all \( c \)-elements in the form that 
\[ c_1 = c_2 = \cdots = c_6 = \frac{1}{9} \geq c_7 \geq c_8 \geq \cdots \geq c_{11} \] and 
\[ c_7 + c_8 + \cdots + c_{11} = \frac{1}{9}. \] Note that there are at most two of \( \{c_7, c_8, \ldots, c_{11}\} \) being \( \frac{1}{9} \) (for otherwise \( c_{10} = 11 = 0 \)). We have the following subcases.

**Subcase 1.1.** \( c_7 = c_8 = \frac{1}{9} \) and \( c_9 + c_{10} + c_{11} = \frac{1}{9} \). Let us see the edge-combination number.
1. There are \( \binom{3}{3} \) 3-combinations \( \{c_1, c_2, c_3\} \) chosen from \( \{c_1, c_2, \ldots, c_8\} \), each of which is an edge-combination.
2. There are \( \binom{5}{3} \) 4-combinations \( S = \{c_2, c_3, c_9, c_{10}, c_{11}\} \) such that \( \{c_1, c_j\} \) are chosen from \( \{c_1, c_2, \ldots, c_8\} \), each of which is an edge-combination.
Therefore, \( \mathcal{E}(S^c) \geq \binom{9}{3} + \binom{9}{2} > 72 \geq |E(G)| \), a contradiction to Lemma 7.

**Subcase 1.2.** \( c_7 = \frac{1}{5} \), \( c_8, c_9, c_{10}, c_{11} < \frac{1}{5} \), and \( c_8 + c_9 + c_{10} + c_{11} = \frac{2}{5} \). We choose a \( 3 \)-combination \( X = \{c_i, c_j, c_k\} \), where \( 1 \leq i, j \leq 7 \) and \( 8 \leq k \leq 11 \). By Lemma 8(ii), \( X \) can be extended to an edge-combination \( S \). If \( |S| = 5 \), then \( S \setminus X \subset \{c_8, c_9, c_{10}, c_{11}\} \) and the sum of \( c_k \) and the two elements in \( S \setminus X \) is \( \frac{1}{5} \). So the remaining one of \( c_8, c_9, c_{10}, c_{11} \) equals \( \frac{1}{5} \), a contradiction. Therefore, \( |S| = 4 \) and there exists \( c_r(k) (8 \leq r(k) \leq 11) \) such that \( S = \{c_i, c_j, c_k, c_r(k)\} \).

Without loss of generality, we may assume that \( 9 \neq r(8) \). We obtain the following computation.

- There are \( \binom{7}{3} \) \( 3 \)-combinations \( \{c_i, c_j, c_k\} \) chosen from \( \{c_1, c_2, \ldots, c_7\} \), each of which is an edge-combination.
- There are \( \binom{7}{4} \) \( 4 \)-combinations \( \{c_i, c_j, c_8, c_r(8)\} \) and \( \binom{7}{2} \) \( 2 \)-combinations \( \{c_i, c_j, c_9, c_r(9)\} \) with \( \{c_i, c_j\} \) chosen from \( \{c_1, c_2, \ldots, c_7\} \), each of which is an edge-combination.

Therefore, \( \mathcal{E}(S^c) \geq \binom{7}{3} + 2 \binom{7}{2} > 72 \geq |E(G)| \), a contradiction to Lemma 7.

**Subcase 1.3.** \( c_i < \frac{1}{5} \) for \( 7 \leq i \leq 11 \). As in the previous subcase, we choose a \( 3 \)-combination \( X = \{c_1, c_2, c_7\} \), where \( 7 \leq i \leq 11 \). By Lemma 8(ii), \( X \) can be extended to an edge-combination \( S \). We can assert that \( |S| = 4 \). In fact, if \( |S| = 5 \), say \( S = \{c_1, c_2, c_7, c_8, c_9\} \), then \( c_7 + c_8 + c_9 = \frac{1}{5} \) and so \( c_{10} + c_{11} = \frac{2}{5} \). Thus \( \max \{c_{10}, c_{11}\} \geq \frac{1}{5} \), a contradiction. Hence there exists another \( c_j (7 \leq j \leq 11) \) such that \( S = \{c_1, c_2, c_i, c_j\} \) and \( c_i + c_j = \frac{1}{5} \).

Now we consider a graph \( F \) with vertex set \( \{c_7, c_8, c_9, c_{10}, c_{11}\} \), two vertices \( c_i \) and \( c_j \) being adjacent if and only if \( c_i + c_j = \frac{1}{5} \). Then no two edges in \( F \) are nonadjacent, for otherwise the vertex not incident to these edges would have \( c_i = \frac{1}{5} \), a contradiction. This implies that \( F \) is a star on 5 vertices. Suppose \( c_{11} \) is the center of the star. We calculate the edge-combination number as follows.

- There are \( \binom{5}{3} \) \( 3 \)-combinations \( \{c_i, c_j, c_k\} \) chosen from \( \{c_1, c_2, \ldots, c_6\} \), each of which is an edge-combination.
- There are \( \binom{5}{2}\binom{1}{1} \) \( 4 \)-combinations \( \{c_i, c_j, c_k, c_{11}\} \) such that \( \{c_i, c_j\} \) are chosen from \( \{c_1, c_2, \ldots, c_6\} \) and \( c_k \) is chosen from \( \{c_7, c_8, c_9, c_{10}\} \), each of which is an edge-combination.

To sum up, \( \mathcal{E}(S^c) \geq \binom{5}{3} + \binom{5}{2}\binom{1}{1} > 72 \geq |E(G)| \), a contradiction to Lemma 7. This completes the proof of Case 1.

**Case 2.** There is a \( \{3, 4, 4\} \)-partition of \( S^c \). So we may assume that

\[
c_1 + c_2 + c_3 = c_4 + c_5 + c_6 + c_7 = c_8 + c_9 + c_{10} + c_{11} = \frac{1}{3},
\]

We first observe that \( \min \{c_1, c_2, c_3\} \geq \max \{c_4, c_5, \ldots, c_{11}\} \). If not, say \( c_4 > c_3 \), then \( c_1 + c_2 + c_4 > \frac{1}{3} \), contradicting Lemma 8(i). So \( \max \{c_4, c_5, \ldots, c_{11}\} \leq \frac{1}{5}. \)

Moreover, if \( \max \{c_4, c_5, \ldots, c_{11}\} = \frac{1}{9} \), then in addition to \( c_1 = c_2 = c_3 = \frac{1}{9} \),
each of \{c_4, c_5, c_6, c_7\} and \{c_8, c_9, c_{10}, c_{11}\} has at most two elements being \(1/3\). We distinguish the following subcases.

**Subcase 2.1.** Each of \(\{c_4, c_5, c_6, c_7\}\) and \(\{c_8, c_9, c_{10}, c_{11}\}\) has exactly two elements being \(1/3\). We may assume that \(c_4 = c_5 = 1/3\) and \(c_6 + c_7 = 2/3\), and \(c_8 = c_9 = 1/3\) and \(c_{10} + c_{11} = 2/3\). Similar to Subcase 1.2, there are \(\binom{7}{3}\) edge-combinations \(\{c_i, c_j, c_k\}\) chosen from \(\{c_1, c_2, c_3, c_4, c_5, c_8, c_9\}\). There are \(2\binom{3}{3}\) edge-combinations \(\{c_i, c_j, c_k\}\) and \(\{c_i, c_j, c_{10}, c_{11}\}\), where \(10 \neq r(6)\), and \(\{c_i, c_j\}\) are chosen from \(\{c_1, c_2, c_3, c_4, c_5, c_8, c_9\}\). Therefore, \(E(S^c) \geq \binom{7}{3} + 2\binom{3}{3} > 72 \geq |E(G)|\), as required.

**Subcase 2.2.** \(\{c_4, c_5, c_6, c_7\}\) has only one element being \(1/3\), say \(c_4 = 1/3\). Then \(c_5 + c_6 + c_7 = 2/3\). Assume that \(1/3 > c_5 \geq c_6 \geq c_7\). Then \(c_1 + c_2 + c_4 < 1/3\) for \(5 \leq i \leq 7\). By Lemma 8(ii), \(\{c_1, c_2, c_4\}\) can be extended to an edge-combination \(S\) with \(|S| = 4\) (if \(|S| = 5\), then we can get a \(\{3, 3, 5\}\)-partition of \(S^c\), which reduces to Case 1). Note that \(c_1 + c_k > 1/3\) for \(5 \leq i \leq 7\). There is a \(c_j\) with \(j > 7\) such that \(S = \{c_1, c_2, c_1, c_j\}\) and \(c_1 + c_j = 1/3\). Thus we can define a mapping from \(\{c_1 : 5 \leq i \leq 7\}\) to \(\{c_j : 8 \leq j \leq 11\}\) with \(c_i + c_j = 1/3\). In this respect, we claim that it is impossible that \(\{c_8, c_9, c_{10}, c_{11}\}\) has only one element being \(1/3\). To see this, assume that \(c_8 = 1/3\) and \(c_9 + c_{10} + c_{11} = 2/3\). Then we can also define a mapping from \(\{c_j : 9 \leq j \leq 11\}\) to \(\{c_4 : 5 \leq i \leq 7\}\) as above. Hence we obtain a bijection between \(\{c_5, c_6, c_7\}\) and \(\{c_9, c_{10}, c_{11}\}\). Consequently, \(c_5 + c_6 + c_7 + c_9 + c_{10} + c_{11} = 1/3 + 1/3 + 1/3 = 1\) and so \(c_4 + c_8 = 1/3\), which is impossible. Therefore, \(\{c_8, c_9, c_{10}, c_{11}\}\) has exactly two elements being \(1/3\), say \(c_8 = c_9 = 1/3\), and so \(c_{10} + c_{11} = 1/3\).

As stated in Subcase 1.3, we may define a graph \(F\) with vertex set \(\{c_5, c_6, c_7, c_{10}, c_{11}\}\), two vertices \(c_i\) and \(c_j\) being adjacent if and only if \(c_i + c_j = 1/3\). Then \(F\) is a star on 5 vertices. By the same calculation as in Subcase 1.3, we obtain \(E(S^c) \geq \binom{5}{3} + \binom{3}{3} + \binom{4}{1} > 72 \geq |E(G)|\), a contradiction to Lemma 7.

**Subcase 2.3.** \(\max \{c_4, c_5, \ldots, c_{11}\} < 1/3\). For this, we cannot assure \(\{c_1, c_2, c_3\}\) has some element being \(1/3\). We may assume that \(c_1 \geq c_2 \geq c_3 \geq c_4 \geq c_5 \geq c_6 \geq c_7\), \(c_8 \geq c_9 \geq c_{10} \geq c_{11}\), and \(c_4 \geq c_8\). We show that \(c_3 = c_4\). In fact, if \(c_3 > c_4\), then \(c_1 + c_2 + c_4 < 1/3\), and so there is an \(i\) with \(5 \leq i \leq 11\) such that \(c_1 + c_2 + c_4 + c_i = 1/3\) (by Lemma 8(ii)). Noting \(c_3 \leq 1/3\), we have \(c_1 + c_2 \geq 2/3\). This implies that \(c_4 + c_i \leq 1/3\), and thus \(c_8 + c_i \leq 1/3\). Consequently, there would be \(c_j (4 \leq j, k \leq 11)\) such that \(c_j + c_k \geq 2/3\). Then one of them is at least \(1/3\), contradicting our assumption. By the same argument, we can show that \(c_4 = c_5\), and further \(c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = 1/12\).

Furthermore, we claim that \(c_2 = c_3\). In fact, if \(c_1 \geq c_2 > c_3\), then \(c_2 \leq 1/2 (1 - 1/12) = 5/8\), and so \(c_2 + c_3 + c_4 \leq 1/3 + 1/6 + 1/5 = 13/15 < 1/3\). By Lemma 8(ii), there is an \(i (i \geq 5)\) such that \(c_2 + c_3 + c_4 + c_i = 1/3\). But this contradicts the fact that \(c_2 + c_3 + c_4 + c_i < 1/3\). Therefore, \(c_2 = c_3 = \cdots = c_{11} = 1/12\) which implies \(c_1 = 1/6\).
From this, we consider the edge-combination number $E(S^c)$ as follows.

• There are $\binom{10}{3}$ 3-combinations $\{c_i, c_j, c_k\}$ with $c_i = c_j = c_k = \frac{1}{12}$, apart from $c_1 = \frac{1}{6}$, at most $\binom{4}{3}$ of which correspond to a common edge-combination $\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\}$.

• There are $\binom{10}{2}$ 3-combinations $\{c_i, c_j\}$ with $c_1 = \frac{1}{6}$ and $c_i = c_j = \frac{1}{12}$, each of which corresponds to an edge-combination $\{\frac{1}{6}, \frac{1}{12}, \frac{1}{12}\}$.

To summarize, $E(S^c) \geq \binom{10}{3}/4 + \binom{10}{2} > 72 \geq |E(G)|$, a contradiction to Lemma 7. This completes the proof.

As a consequence, if $d(P(G)) \leq 10$ (whence $\varphi(P(G)) \leq 11$), then the 3PM-property holds. This includes our previous result (Proposition 9).

4. Bricks

Recall that a brick is 3-connected and bicritical. Relative to Theorem 12 and 13, we can get better results for cubic bricks.

**Lemma 14.** For a cubic brick $G$, if $d(P(G)) = d$, then $|E(G)| = 3d$.

**Proof.** Since $G$ is cubic, $2|E(G)| = 3|V(G)|$. Since $G$ is a brick, $b = 1$. By Lemma 4, $d = m - n + 1 - b = m - \frac{2m}{3} = \frac{m}{3}$, which implies $|E(G)| = 3d$.

**Theorem 15.** For a cubic brick $G$ with $d(P(G)) \leq 24$, if $\varphi(P(G)) \leq 11$, then the 3PM-property holds.

**Proof.** Recalling the proofs of Theorems 12 and 13, we note that when $\varphi(P(G)) \leq 11$, as long as $|E(G)| \leq 72$, all these proofs are valid. Now for a brick $G$, if $d(P(G)) \leq 24$, then $|E(G)| = 3d \leq 72$, as we needed in the proofs.

**Theorem 16.** For a cubic brick $G$ with $d(P(G)) \leq 18$, if $\varphi(P(G)) \leq 12$, then the 3PM-property holds.

**Proof.** By Lemma 14, $|E(G)| = 3d \leq 54$. By Theorem 15, we need only consider the case $\varphi(P(G)) = 12$. Then the equations (4) and (5) hold for $k = 12$. Suppose, to the contrary, that the 3PM-property does not hold.

By Lemma 8(i), each edge-combination contains from three to six $c$-elements of $c_1, c_2, \ldots, c_{12}$. Moreover, by Lemma 8(iii), the 12 $c$-elements are divided into three sets, each of which has sum $\frac{1}{3}$. Note that the only partitions $\{k_1, k_2, k_3\}$ of integer 12, where $k_1 + k_2 + k_3 = 12$, $3 \leq k_1 \leq 6$, are $\{3, 3, 6\}, \{3, 4, 5\}$, and $\{4, 4, 4\}$. So we consider three cases as follows.

**Case 1.** There is a $\{3, 3, 6\}$-partition of $S^c$. By Lemma 8(i), there are six $c$-elements being $\frac{1}{5}$ and no other $c_i$ is greater than $\frac{1}{5}$. So we may assume that
It is impossible that be extended to an edge-combination to Case 1 in the proof of Theorem 13, we have the following subcases.

Subcase 1.1. \( c_7 = c_8 = \frac{1}{9} \) and \( c_9 + c_{10} + c_{11} + c_{12} = \frac{2}{9} \). Since each 3-combination \( \{c_i, c_j, c_k\} \) taken from \( \{c_1, c_2, \ldots, c_8\} \) is an edge-combination, we have \( E(S^c) \geq \binom{5}{3} > 54 \geq |E(G)| \), contradicting Lemma 7.

Subcase 1.2. \( c_7 = \frac{1}{9} \) and \( c_8 + c_9 + c_{10} + c_{11} + c_{12} = \frac{2}{9} \). Then there are \( \binom{3}{2} \) 3-combinations \( \{c_i, c_j, c_k\} \) taken from \( \{c_1, c_2, \ldots, c_7\} \), each of which is an edge-combination. For each \( \{c_i, c_j\} \) chosen from \( \{c_1, c_2, \ldots, c_7\} \), \( c_i + c_j + c_8 < \frac{1}{9} \). By Lemma 8(ii), \( \{c_i, c_j, c_8\} \) can be extended to an edge-combination. Therefore, \( E(S^c) \geq \binom{3}{2} + \binom{2}{2} > 54 \geq |E(G)| \), as required.

Subcase 1.3. \( c_i < \frac{1}{9} \) for \( 7 \leq i \leq 12 \) and \( c_7 \geq c_8 \geq \cdots \geq c_{12} \). For a pair of given \( i, j \) (\( 1 \leq i, j \leq 6 \)), \( c_i + c_j + c_k < \frac{1}{9} \) for any \( k \) with \( 7 \leq k \leq 12 \), and so \( \{c_i, c_j, c_k\} \) can be extended to an edge-combination \( S_k \). Note that \( S_k \setminus \{c_i, c_j\} \subseteq \{c_7, c_8, \ldots, c_{12}\} \). It is impossible that \( |S_k| = 6 \), for otherwise there would be a \( c_l \geq \frac{1}{9} \) for \( 7 \leq l \leq 12 \). So \( 4 \leq |S_k| \leq 5 \).

If there is an \( S_k \) with \( |S_k| = 5 \), then the sum of the three \( c \)-elements of \( S_k \setminus \{c_i, c_j\} \) is \( \frac{1}{9} \) and the sum of the remaining three \( c \)-elements in \( \{c_7, c_8, \ldots, c_{12}\} \) is \( \frac{2}{9} \), say \( c_{k1} + c_{k2} + c_{k3} = \frac{2}{9} \). Since the sum of any two \( c \)-elements of \( c_{k1}, c_{k2}, c_{k3} \) is more than \( \frac{1}{9} \), \( S_{k1}, S_{k2}, \) and \( S_{k3} \) are different. If each \( S_k \) has exactly four \( c \)-elements, say \( S_k = \{c_i, c_j, c_{k1}, c_{k2}\} \), then there are at least three different \( S_k \)'s.

Now we evaluate the number of edge-combinations. There are \( \binom{6}{3} \) 3-combinations taken from \( \{c_1, c_2, \ldots, c_6\} \), each of which is an edge-combination. Moreover, each 2-combination \( \{c_i, c_j\} \) taken from \( \{c_1, c_2, \ldots, c_6\} \) corresponds to at least three edge-combinations. Therefore, we have \( E(S^c) \geq \binom{6}{3} + 3(\binom{6}{2}) > 54 \geq |E(G)| \), as needed.

Case 2. There is a \( \{4, 4, 4\} \)-partition of \( S^c \). Then every edge-combination has cardinality 4. In this context, there are \( \binom{12}{3} \) 3-combinations \( \{c_i, c_j, c_k\} \), at most \( \binom{3}{3} \) of which correspond to a common edge-combination. Therefore, \( E(S^c) \geq \binom{12}{3}/4 > 54 \geq |E(G)| \), as needed.

Case 3. Every 3-partition of \( S^c \) is a \( \{3, 4, 5\} \)-partition. Then for any vertex \( v \) of \( G \), the three edges of \( G \) incident with \( v \) lie in exactly three, four, and five perfect matchings of \( M_1, M_2, \ldots, M_{12} \), respectively. Let \( N_1, N_2, N_3 \) be the sets of edges which lie in exactly three, four, and five perfect matchings of \( M_1, M_2, \ldots, M_{12} \), respectively. Then we see that \( N_1, N_2, \) and \( N_3 \) are three pairwise disjoint perfect matchings of \( G \), a contradiction to the assumption that 3PM-property fails. This completes the proof. ■
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