

## CORE INDEX OF PERFECT MATCHING POLYTOPE FOR A 2-CONNECTED CUBIC GRAPH

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### Abstract

For a 2-connected cubic graph  $G$ , the perfect matching polytope  $P(G)$  of  $G$  contains a special point  $x^c = (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$ . The *core index*  $\varphi(P(G))$  of the polytope  $P(G)$  is the minimum number of vertices of  $P(G)$  whose convex hull contains  $x^c$ . The Fulkerson's conjecture asserts that every 2-connected cubic graph  $G$  has six perfect matchings such that each edge appears in exactly two of them, namely, there are six vertices of  $P(G)$  such that  $x^c$  is the convex combination of them, which implies that  $\varphi(P(G)) \leq 6$ . It turns out that the latter assertion in turn implies the Fan-Raspaud conjecture: In every 2-connected cubic graph  $G$ , there are three perfect matchings  $M_1$ ,  $M_2$ , and  $M_3$  such that  $M_1 \cap M_2 \cap M_3 = \emptyset$ . In this paper we prove the Fan-Raspaud conjecture for  $\varphi(P(G)) \leq 12$  with certain dimensional conditions.

**Keywords:** Fulkerson's conjecture, Fan-Raspaud conjecture, cubic graph, perfect matching polytope, core index.

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### 1. INTRODUCTION

The celebrated Fulkerson's conjecture in graph theory is the following (cf. [1, 5]).

**Conjecture A** (Fulkerson's conjecture). *Every 2-connected cubic graph has six perfect matchings such that each edge appears in exactly two of them.*

We may state the polyhedral version of this conjecture as follows. Let  $G$  be a 2-connected cubic graph. Thus each edge of  $G$  is contained in a perfect matching of  $G$ . The *characteristic vector* of a perfect matching  $M$  of  $G$  is a vector  $x \in \mathbb{R}^{E(G)}$  such that  $x_e = 1$  if  $e \in M$  and  $x_e = 0$  otherwise. The *perfect*

*matching polytope* of  $G$ , denoted by  $P(G)$ , is the convex hull of the characteristic vectors of all perfect matchings in  $G$ . Now let  $x^1, x^2, \dots, x^6$  be the characteristic vectors of the six perfect matchings in the Fulkerson's conjecture. Then  $x^1 + x^2 + \dots + x^6 = (2, 2, \dots, 2)$  and thus  $\frac{1}{6}(x^1 + x^2 + \dots + x^6) = (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$ . That is,  $x^c := (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$  is the convex combination of  $x^1, x^2, \dots, x^6$ .

We call this  $x^c$  the *core* of the perfect matching polytope  $P(G)$ , which lies in  $P(G)$  (see Proposition 3). Furthermore, a subset  $Q \subseteq P(G)$  is called a *core polytope* of  $P(G)$  if it is the convex hull of  $k$  vertices of  $P(G)$  such that  $x^c \in Q$  and  $k$  is minimum. Meanwhile, the above minimum value  $k$  is called the *core index* of  $P(G)$ , denoted by  $\varphi(P(G))$ . In other words, the core index  $\varphi(P(G))$  is the minimum number of vertices of  $P(G)$  whose convex hull contains  $x^c$ . Therefore, the Fulkerson's conjecture yields the following conjecture.

**Conjecture B.** *For every 2-connected cubic graph  $G$ ,  $\varphi(P(G)) \leq 6$ .*

The study of Conjecture B would be meaningful to cope with the Fulkerson's conjecture. In particular, the structure of the core polytope  $Q$  inside a perfect matching polytope  $P(G)$  is quite mysterious. Fan and Raspaud [5] proposed the following conjecture.

**Conjecture C** (Fan-Raspaud conjecture). *In every 2-connected cubic graph there exist three perfect matchings  $M_1, M_2$ , and  $M_3$  such that  $M_1 \cap M_2 \cap M_3 = \emptyset$ .*

Let us see the relation of these three conjectures.

**Proposition 1.** *Conjecture A implies Conjecture B and Conjecture B implies Conjecture C.*

**Proof.** The first assertion is clear, as  $x^c := (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$  is contained in the convex hull of  $\{x^1, x^2, \dots, x^6\}$ . We show the second assertion. Suppose Conjecture B holds. Let  $x^1, x^2, \dots, x^6$  be six vertices of  $P(G)$  whose convex hull contains  $x^c$ . Then  $x^c = c_1x^1 + c_2x^2 + \dots + c_6x^6$ , where  $\sum_{i=1}^6 c_i = 1$  and  $c_i \geq 0$ . We may assume that  $c_1, c_2, c_3$  are the three largest numbers among all  $c_i$ . Then  $c_1 + c_2 + c_3 \geq \frac{1}{2} \sum_{i=1}^6 c_i = \frac{1}{2}$ . We claim that Conjecture C holds for the perfect matchings  $M_1, M_2, M_3$  corresponding to  $x^1, x^2, x^3$ . Suppose not. Then there is an edge  $e \in M_1 \cap M_2 \cap M_3$ , namely,  $x_e^i = 1$  for  $i = 1, 2, 3$ . Then  $x_e^c = c_1x_e^1 + c_2x_e^2 + \dots + c_6x_e^6 \geq c_1x_e^1 + c_2x_e^2 + c_3x_e^3 = c_1 + c_2 + c_3 \geq \frac{1}{2} > \frac{1}{3}$ , which is a contradiction. ■

In brief, if the Fulkerson's conjecture is true, then  $\varphi(P(G)) \leq 6$ , and thus the Fan-Raspaud conjecture holds. However, the Fulkerson's conjecture is far from being proved at the moment. So we do not know the exact range of values  $\varphi(P(G))$  for all 2-connected cubic graphs. In this circumstance we can use the parameter  $\varphi(P(G))$  as a condition in proving the Fan-Raspaud conjecture.

The *dimension* of a polytope  $P$ , denoted by  $d(P)$ , is the dimension of its affine hull (the minimal affine subspace containing  $P$ ). Let  $G$  be a 2-connected cubic graph, and  $P(G)$  the perfect matching polytope of  $G$ . If all vertices of  $P(G)$  are affinely independent (namely,  $P(G)$  is a simplex), then  $d(P(G)) \leq 8$  (see [3, 9]). In our previous paper [9], we showed that Fan-Raspaud conjecture holds if  $d(P(G)) \leq 9$ , which implies  $\varphi(P(G)) \leq 10$ . In this paper, we obtain some improved results. The main results are the following:

- (1) The Fan-Raspaud conjecture is true if  $d(P(G)) \leq 13$  and  $\varphi(P(G)) \leq 11$ .
- (2) The Fan-Raspaud conjecture is true if  $G$  is a cubic brick,  $d(P(G)) \leq 18$ , and  $\varphi(P(G)) \leq 12$ .

Since the dimension of a cubic brick  $G$  is  $d(P(G)) = m - n = n/2$  (see Lemma 4 with  $b = 1$ ),  $d(P(G)) \leq 18$  is equivalent to  $n = |V(G)| \leq 36$ . Hence the above result (2) means that the Fan-Raspaud conjecture is true for cubic bricks with up to 36 vertices (provided  $\varphi(P(G)) \leq 12$ ). Recently, in [2], the computer search shows that the Fulkerson's conjecture is true for snarks with up to 36 vertices, and so is the Fan-Raspaud conjecture. Here, a *snark* is a cyclically 4-edge connected cubic graph which cannot be 3-edge colored and has girth at least 5.

The organization of the paper is as follows. In Section 2, we present some basic properties. Section 3 is devoted to the results on 2-connected cubic graphs with  $\varphi(P(G)) \leq 11$ . Section 4 is concerned with cubic bricks with  $\varphi(P(G)) \leq 12$ .

## 2. PRELIMINARY ON PERFECT MATCHING POLYTOPES

The basic notions on polyhedral combinatorics can be found in [6, 8]. The well-known characterization of perfect matching polytope, due to Edmonds (1965), is the following (cf. [7]).

**Lemma 2.** *The perfect matching polytope of a graph  $G$  is the set of vectors  $x \in \mathbb{R}^{E(G)}$  satisfying*

- (1)  $x_e \geq 0$  (for all  $e \in E(G)$ ),
- (2)  $\sum_{e \in \delta(v)} x_e = 1$  (for all  $v \in V(G)$ ),
- (3)  $\sum_{e \in \delta(A)} x_e \geq 1$  (for all  $A \subseteq V(G)$ ,  $|A|$  is odd),

where  $\delta(v)$  stands for the set of edges incident with  $v \in V(G)$ , and  $\delta(A)$  is the set of edges with exactly one end in  $A$ .

For a 2-connected cubic graph  $G$ , every edge is contained in a perfect matching (see Corollary 3.4.3 of [7]). A graph is *matching-covered* if every edge of this graph is contained in a perfect matching of it. Hence  $G$  is matching-covered and there are at least three different perfect matchings in  $G$ . Thus the perfect matching polytope  $P(G)$  has at least three vertices. In particular, we have

**Proposition 3.** *Let  $G$  be a 2-connected cubic graph. Then the core  $x^c = (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$  lies in  $P(G)$ .*

**Proof.** Clearly,  $x^c$  satisfies (1) and (2) of Lemma 2. It suffices to verify (3). For a subset  $A \subseteq V(G)$  with odd cardinality, the degree sum of the vertices in  $A$  is  $3|A| = 2|E(G[A])| + |\delta(A)|$ , where  $G[A]$  is the subgraph of  $G$  induced by  $A$ . So  $|\delta(A)|$  is odd. If  $|\delta(A)| = 1$ , then the only edge of  $\delta(A)$  is a cut edge of  $G$ , contradicting the assumption that  $G$  is 2-connected. Therefore  $|\delta(A)| \geq 3$  and thus  $\sum_{e \in \delta(A)} x_e^c = \frac{1}{3}|\delta(A)| \geq 1$ . That is, the point  $x^c$  satisfies (3), as required. ■

The following characterization of the dimension of perfect matching polytope found by Edmonds, Lovász, and Pulleyblank (see [4] or Theorem 7.6.6 of [7]) is used in the proof of the main results.

**Lemma 4.** *For every matching-covered graph  $G$ , the dimension of perfect matching polytope  $P(G)$  is  $d(P(G)) = m - n + 1 - b$ , where  $m, n, b$  are the numbers of edges, vertices, and bricks of  $G$ , respectively.*

Here, a *brick* is a 3-connected and bicritical graph, where a graph  $G$  is *bicritical* if  $G - u - v$  has a perfect matching for any two distinct vertices  $u, v$  in  $G$ . Clearly, a brick is non-bipartite and matching-covered. The number of bricks of a matching-covered graph  $G$  is the number of bricks produced in a procedure of ‘tight cut decomposition’, see [4, 7].

With respect to the dimension, the following Carathéodory theorem is classical (Theorem 5.1 of [8]).

**Lemma 5.** *For any  $V \subseteq \mathbb{R}^m$  and  $x$  in the convex hull of  $V$ , there exist affinely independent vectors  $x^1, \dots, x^k$  in  $V$  such that  $x$  is contained in the convex hull of  $\{x^1, \dots, x^k\}$ .*

We obtain an upper bound of the core index as follows.

**Proposition 6.** *For every 2-connected cubic graph  $G$ ,  $\varphi(P(G)) \leq d(P(G)) + 1$ .*

**Proof.** Let  $V$  be the set of vertices in  $P(G)$ . Then  $x^c$  is contained in the convex hull of  $V$ . By the Carathéodory theorem,  $x^c$  is contained in the convex hull of  $d(P(G)) + 1$  affinely independent vectors. The assertion follows. ■

3. RESULTS ON CORE INDEX AND DIMENSION

For convenience, we refer to the property specified in the Fan-Raspaud conjecture as the 3PM-property. We start with some simple facts.

If  $\varphi(P(G)) = k$ , then there are vertices  $x^1, x^2, \dots, x^k$  of  $P(G)$  whose convex hull contains the core  $x^c$ , i.e.,

$$(4) \quad x^c = c_1x^1 + c_2x^2 + \dots + c_kx^k, \quad \sum_{i=1}^k c_i = 1, c_i > 0.$$

Let  $S^c = \{c_1, c_2, \dots, c_k\}$ , which stands for the convex combination representation of the core  $x^c$  in the convex hull of  $\{x^1, x^2, \dots, x^k\}$ . Meanwhile, each  $c_i \in S^c$  is called a  $c$ -element. For a set  $S$ , by an  $h$ -combination  $X$  of  $S$  we mean a subset  $X \subseteq S$  with  $|X| = h$ . Furthermore, for every edge  $e \in E(G)$ , we have

$$(5) \quad c_1x_e^1 + c_2x_e^2 + \dots + c_kx_e^k = \frac{1}{3}.$$

Let  $M_1, M_2, \dots, M_k$  be the perfect matchings corresponding to  $x^1, x^2, \dots, x^k$  respectively, and let  $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ . For an edge  $e \in E(G)$ , let  $\mathcal{M}_e = \{M_i : x_e^i = 1, 1 \leq i \leq k\}$  ( $\subseteq \mathcal{M}$ ), which is the set of perfect matchings containing the edge  $e$ . Then (5) is equivalent to

$$(6) \quad \sum_{M_i \in \mathcal{M}_e} c_i = \frac{1}{3}.$$

Conversely, a subset  $S$  of  $S^c$  with  $\sum_{c_i \in S} c_i = \frac{1}{3}$  is not necessarily corresponding to an edge. We now give a useful definition as follows.

A subset  $S$  of  $S^c$  is called an *edge-combination* if (1)  $\sum_{c_i \in S} c_i = \frac{1}{3}$ ; (2) there exists a 3-combination  $X$  of  $S$  such that

$$\bigcap_{c_i \in X} M_i \neq \emptyset \text{ implies } \bigcap_{c_i \in S} M_i \neq \emptyset.$$

For instance, if the 3PM-property does not hold, then any 3-combination  $S$  of  $S^c$  with  $\sum_{c_i \in S} c_i = \frac{1}{3}$  is an edge-combination. Moreover, for any edge-combination  $S$  of  $S^c$ , we have  $|S| \geq 3$  and  $\bigcap_{c_i \in S} M_i \neq \emptyset$ , and thus  $S$  indeed corresponds to an edge in  $\bigcap_{c_i \in S} M_i$ . This is the intention of the term ‘‘edge-combination’’. We call the number of all the edge-combinations in  $S^c$  the *edge-combination number* of  $S^c$ , denoted by  $\mathcal{E}(S^c)$ .

**Lemma 7.** *If  $|E(G)| < \mathcal{E}(S^c)$ , then the 3PM property holds.*

**Proof.** If the 3PM-property does not hold, then for each edge-combination  $S$ , we have  $\bigcap_{c_i \in S} M_i \neq \emptyset$ . Thus there exists an edge  $e$  contained in  $\bigcap_{c_i \in S} M_i$ . So each

edge-combination  $S$  corresponds to an edge  $e \in \bigcap_{c_i \in S} M_i$ . Furthermore, we claim that an edge  $e \in E(G)$  cannot correspond to two different edge-combinations. In fact, if  $e$  corresponds to two edge-combinations  $S$  and  $S'$  with  $S \neq S'$ , then  $e \in \bigcap_{c_i \in S} M_i$  and  $e \in \bigcap_{c_i \in S'} M_i$ , whence  $e \in \bigcap_{c_i \in S \cup S'} M_i$ . By the definition of edge-combination, we have  $\sum_{c_i \in S} c_i = \frac{1}{3}$  and  $\sum_{c_i \in S'} c_i = \frac{1}{3}$ . Hence  $\sum_{M_i \in \mathcal{M}_e} c_i \geq \sum_{c_i \in S \cup S'} c_i > \sum_{c_i \in S} c_i = \frac{1}{3}$ , contradicting the equation (6). In this way, we define an injection (one-to-one mapping) from the set of edge-combinations to  $E(G)$ . Therefore  $\mathcal{E}(S^c) \leq |E(G)|$ , contradicting the condition of the lemma. ■

**Lemma 8.** *If the 3PM-property does not hold, then  $S^c = \{c_1, c_2, \dots, c_k\}$  satisfies the following:*

- (i) *For any 3-combination  $X \subseteq S^c$ ,  $\sum_{c_i \in X} c_i \leq \frac{1}{3}$ ;*
- (ii) *For any 3-combination  $X \subseteq S^c$ , there exists an edge-combination  $S \subseteq S^c$  such that  $X \subseteq S$ ;*
- (iii)  *$S^c$  can be partitioned into three parts  $\{S_1, S_2, S_3\}$  such that  $\sum_{c_i \in S_k} c_i = \frac{1}{3}$  for  $k = 1, 2, 3$ .*

**Proof.** Suppose that the 3PM-property does not hold. We show the three assertions as follows.

(i) Suppose that for  $X = \{c_{i_1}, c_{i_2}, c_{i_3}\}$ ,  $c_{i_1} + c_{i_2} + c_{i_3} > \frac{1}{3}$ . Since  $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$ , there is an edge  $e \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$  such that  $x_e^c = c_1 x_e^1 + c_2 x_e^2 + \dots + c_k x_e^k \geq c_{i_1} x_e^{i_1} + c_{i_2} x_e^{i_2} + c_{i_3} x_e^{i_3} = c_{i_1} + c_{i_2} + c_{i_3} > \frac{1}{3}$ , a contradiction.

(ii) For any 3-combination  $X \subseteq S^c$ , since  $\bigcap_{c_i \in X} M_i \neq \emptyset$ , there exists an edge  $e$  in  $\bigcap_{c_i \in X} M_i$ . Let  $S = \{c_i : M_i \in \mathcal{M}_e\}$ . Then  $X \subseteq S$  and  $\sum_{c_i \in S} c_i = \sum_{M_i \in \mathcal{M}_e} c_i = \frac{1}{3}$ , which implies that  $S$  is an edge-combination.

(iii) We take a vertex  $v$  in  $G$  and let  $e_1, e_2, e_3$  be the three edges incident with this vertex  $v$  in  $G$ . Since no perfect matching  $M_i$  can contain two of  $e_1, e_2, e_3$ , all perfect matchings  $M_1, M_2, \dots, M_k$  are partitioned into three disjoint sets, each of which contains one of  $e_1, e_2, e_3$ . Therefore  $S^c$  is partitioned into three parts  $\{S_1, S_2, S_3\}$  such that  $\sum_{c_i \in S_k} c_i = \sum_{M_i \in \mathcal{M}_{e_k}} c_i = \frac{1}{3}$  for  $k = 1, 2, 3$ . ■

Our previous paper [9] shows the following.

**Proposition 9.** *For a 2-connected cubic graph  $G$  with  $d(P(G)) \leq 9$ , the 3PM-property holds.*

Now we present several improved results. Proposition 1 says that if  $\varphi(P(G)) \leq 6$ , then the 3PM-property holds. The following improvement is straightforward.

**Proposition 10.** *For a 2-connected cubic graph  $G$  with  $\varphi(P(G)) \leq 8$ , the 3PM-property holds.*

**Proof.** Suppose that (4) holds for  $k = 8$  and  $c_1, c_2, c_3, c_4$  are the four largest elements in  $S^c$  with  $c_1 \geq c_2 \geq c_3 \geq c_4$ . Then  $c_1 + c_2 + c_3 + c_4 \geq \frac{1}{2}$ . If  $c_1 + c_2 + c_3 < \frac{3}{8}$ , then  $c_3 < \frac{1}{8}$  and  $c_4 \geq \frac{1}{2} - (c_1 + c_2 + c_3) > \frac{1}{8}$ , a contradiction to the assumption that  $c_3 \geq c_4$ . Therefore  $c_1 + c_2 + c_3 \geq \frac{3}{8} > \frac{1}{3}$ , and thus the assertion follows from (i) of Lemma 8. ■

This result is independent of the dimension of the perfect matching polytope. In the results below, we have to combine the dimensional condition.

**Lemma 11.** *For a 2-connected cubic graph  $G$ , if  $d(P(G)) = d$ , then  $|E(G)| \leq 6(d - 1)$ .*

**Proof.** Since  $G$  is cubic, we have  $2|E(G)| = 3|V(G)|$ . Moreover, as a result of the brick decompositions of graphs,  $G$  has at most  $\frac{|E(G)| - |V(G)|}{2} = \frac{|E(G)|}{6}$  bricks (see Lemma 5.12 of [4]). Hence by the formula of dimension of perfect matching polytope, we have  $d = m - n + 1 - b \geq m - \frac{2m}{3} + 1 - \frac{m}{6} = \frac{m}{6} + 1$ , which implies that  $m \leq 6(d - 1)$ . ■

**Theorem 12.** *For a 2-connected cubic graph  $G$  with  $d(P(G)) \leq 14$ , if  $\varphi(P(G)) \leq 10$ , then the 3PM-property holds.*

**Proof.** By Lemma 11 and  $d(P(G)) \leq 14$ , we have  $|E(G)| \leq 6(d - 1) \leq 78$ . Suppose, to the contrary, that the 3PM-property does not hold.

The case of  $\varphi(P(G)) \leq 8$  has been settled in Proposition 10. We consider the case of  $\varphi(P(G)) = 9$  now. In this case, the convex combination of (4) with  $k = 9$  holds. Suppose, without loss of generality, that  $c_1 \geq c_2 \geq \dots \geq c_9$ . By Lemma 8(i), the sum of any three  $c$ -elements is at most  $\frac{1}{3}$ . On the other hand,  $c_1 + c_2 + c_3 \geq \frac{1}{3} \sum_{i=1}^9 c_i = \frac{1}{3}$ . It follows that  $c_1 + c_2 + c_3 = \frac{1}{3}$ , and so  $c_4 + c_5 + c_6 = c_7 + c_8 + c_9 = \frac{1}{3}$ . Therefore,  $c_7 \geq \frac{1}{9}$  and  $c_6 \leq \frac{1}{9}$ . Since  $c_6 \geq c_7$ , we have  $c_6 = \frac{1}{9}$ , and so  $c_4 = c_5 = \frac{1}{9}$ . By the same way, we have  $c_1 = c_2 = c_3 = \frac{1}{9}$ , and  $c_7 = c_8 = c_9 = \frac{1}{9}$ . Hence every 3-combination of  $S^c$  is an edge-combination, and so  $\mathcal{E}(S^c) = \binom{9}{3} > 78 \geq |E(G)|$ . This is a contradiction to Lemma 7.

We next consider the case of  $\varphi(P(G)) = 10$  with convex combination representation (4) with  $k = 10$ . By Lemma 8(i), the sum of any two  $c$ -elements is less than  $\frac{1}{3}$ . We further observe that the sum of any five  $c$ -elements is greater than  $\frac{1}{3}$ . This is because if there are five  $c$ -elements whose sum is less than or equal to  $\frac{1}{3}$ , then the sum of the remaining five  $c$ -elements is at least  $\frac{2}{3}$ , say  $c_{i_1} + c_{i_2} + \dots + c_{i_5} \geq \frac{2}{3}$ , and thus the sum of the three largest members of  $\{c_{i_1}, c_{i_2}, \dots, c_{i_5}\}$  is greater than  $\frac{1}{3}$ , contradicting Lemma 8(i). Therefore, each edge of  $G$  is covered by precisely three or four of the ten perfect matchings  $M_1, M_2, \dots, M_{10}$ .

By Lemma 8(iii),  $S^c$  can be partitioned into three parts each of which has sum  $\frac{1}{3}$ . Note that the only partition  $\{k_1, k_2, k_3\}$  of integer 10 is  $\{3, 3, 4\}$ , where  $k_1 + k_2 + k_3 = 10$ , and  $3 \leq k_i \leq 4$ . Hence the above partition of the ten  $c$ -elements

is a  $\{3, 3, 4\}$ -partition. Therefore, we assume, without loss of generality, that  $c_1 + c_2 + c_3 = c_4 + c_5 + c_6 = c_7 + c_8 + c_9 + c_{10} = \frac{1}{3}$ .

By Lemma 8(i), we have  $c_1 = c_2 = \dots = c_6 = \frac{1}{9}$  and  $c_7, \dots, c_{10} \leq \frac{1}{9}$ . In this context, we choose a 3-combination  $X = \{c_1, c_2, c_i\}$  where  $7 \leq i \leq 10$ . If  $X$  is an edge-combination, then  $c_i = \frac{1}{9}$ . Otherwise, by Lemma 8(ii),  $X$  can be extended to an edge-combination  $S$  with  $|S| = 4$ . Then there exists another  $c_j$  ( $7 \leq j \leq 10$ ) such that  $c_i + c_j = \frac{1}{9}$ . We further claim that among  $c_7, \dots, c_{10}$ , there is at most one such pair with  $c_i + c_j = \frac{1}{9}$ . Suppose without loss of generality that  $c_7 + c_8 = c_9 + c_{10} = \frac{1}{9}$  or  $c_7 + c_8 = c_7 + c_9 = \frac{1}{9}$ . The former contradicts the assumption  $c_7 + c_8 + c_9 + c_{10} = \frac{1}{3}$ , and the latter implies that  $c_{10} > \frac{1}{9}$ , also a contradiction. Therefore, we obtain that  $c_1 = c_2 = \dots = c_8 = \frac{1}{9}$  and  $c_9 + c_{10} = \frac{1}{9}$ .

We proceed to compute the edge-combination number  $\mathcal{E}(S^c)$  as follows.

- There are  $\binom{8}{3}$  3-combinations  $\{c_i, c_j, c_k\}$  chosen from  $\{c_1, c_2, \dots, c_8\}$ , each of which is an edge-combination.

- There are  $\binom{8}{2}$  4-combinations  $S = \{c_i, c_j, c_9, c_{10}\}$  such that  $\{c_i, c_j\}$  are chosen from  $\{c_1, c_2, \dots, c_8\}$ , each of which is an edge-combination.

To sum up,  $\mathcal{E}(S^c) \geq \binom{8}{3} + \binom{8}{2} > 78 \geq |E(G)|$ , contradicting Lemma 7. ■

**Theorem 13.** *For a 2-connected cubic graph  $G$  with  $d(P(G)) \leq 13$ , if  $\varphi(P(G)) \leq 11$ , then the 3PM-property holds.*

**Proof.** We consider the case  $\varphi(P(G)) = 11$  and representation (4) with  $k = 11$ . By Lemma 11,  $d(P(G)) \leq 13$  implies  $|E(G)| \leq 6(d - 1) \leq 72$ . Suppose, to the contrary, that the 3PM-property does not hold.

By Lemma 8(i), no two  $c$ -elements have sum  $\frac{1}{3}$ . Also, no six  $c$ -elements have sum  $\frac{1}{3}$ , for otherwise the remaining five  $c$ -elements would have  $\sum c_i = \frac{2}{3}$ , and thus there are three of them with  $\sum c_i > \frac{1}{3}$ , contradicting Lemma 8(i). Moreover, by Lemma 8(iii), the 11  $c$ -elements are divided into three sets, each of which has sum  $\frac{1}{3}$ . Note that the only partitions  $\{k_1, k_2, k_3\}$  of integer 11 are  $\{3, 3, 5\}$  and  $\{3, 4, 4\}$ , where  $k_1 + k_2 + k_3 = 11, 3 \leq k_i \leq 5$ . We distinguish two cases as follows.

*Case 1.* There is a  $\{3, 3, 5\}$ -partition of  $S^c$ . By Lemma 8(i), the first six  $c$ -elements are  $\frac{1}{9}$  and no other  $c_i$  is greater than  $\frac{1}{9}$ . Therefore, we can sort all  $c$ -elements in the form that  $c_1 = c_2 = \dots = c_6 = \frac{1}{9} \geq c_7 \geq c_8 \geq \dots \geq c_{11}$  and  $c_7 + c_8 + \dots + c_{11} = \frac{1}{3}$ . Note that there are at most two of  $\{c_7, c_8, \dots, c_{11}\}$  being  $\frac{1}{9}$  (for otherwise  $c_{10} = c_{11} = 0$ ). We have the following subcases.

*Subcase 1.1.*  $c_7 = c_8 = \frac{1}{9}$  and  $c_9 + c_{10} + c_{11} = \frac{1}{9}$ . Let us see the edge-combination number.

- There are  $\binom{8}{3}$  3-combinations  $\{c_i, c_j, c_k\}$  chosen from  $\{c_1, c_2, \dots, c_8\}$ , each of which is an edge-combination.

- There are  $\binom{8}{2}$  5-combinations  $\{c_i, c_j, c_9, c_{10}, c_{11}\}$  such that  $\{c_i, c_j\}$  are chosen from  $\{c_1, c_2, \dots, c_8\}$ , each of which is an edge-combination.



Therefore,  $\mathcal{E}(S^c) \geq \binom{8}{3} + \binom{8}{2} > 72 \geq |E(G)|$ , a contradiction to Lemma 7.

*Subcase 1.2.*  $c_7 = \frac{1}{9}$ ,  $c_8, c_9, c_{10}, c_{11} < \frac{1}{9}$ , and  $c_8 + c_9 + c_{10} + c_{11} = \frac{2}{9}$ . We choose a 3-combination  $X = \{c_i, c_j, c_k\}$ , where  $1 \leq i, j \leq 7$  and  $8 \leq k \leq 11$ . By Lemma 8(ii),  $X$  can be extended to an edge-combination  $S$ . If  $|S| = 5$ , then  $S \setminus X \subset \{c_8, c_9, c_{10}, c_{11}\}$  and the sum of  $c_k$  and the two elements in  $S \setminus X$  is  $\frac{1}{9}$ . So the remaining one of  $c_8, c_9, c_{10}, c_{11}$  equals  $\frac{1}{9}$ , a contradiction. Therefore,  $|S| = 4$  and there exists  $c_{r(k)}$  ( $8 \leq r(k) \leq 11$ ) such that  $S = \{c_i, c_j, c_k, c_{r(k)}\}$ . Without loss of generality, we may assume that  $9 \neq r(8)$ . We obtain the following computation.

- There are  $\binom{7}{3}$  3-combinations  $\{c_i, c_j, c_k\}$  chosen from  $\{c_1, c_2, \dots, c_7\}$ , each of which is an edge-combination.
- There are  $\binom{7}{2}$  4-combinations  $\{c_i, c_j, c_8, c_{r(8)}\}$  and  $\binom{7}{2}$  4-combinations  $\{c_i, c_j, c_9, c_{r(9)}\}$  with  $\{c_i, c_j\}$  chosen from  $\{c_1, c_2, \dots, c_7\}$ , each of which is an edge-combination.

Therefore,  $\mathcal{E}(S^c) \geq \binom{7}{3} + 2\binom{7}{2} > 72 \geq |E(G)|$ , a contradiction to Lemma 7.

*Subcase 1.3.*  $c_i < \frac{1}{9}$  for  $7 \leq i \leq 11$ . As in the previous subcase, we choose a 3-combination  $X = \{c_1, c_2, c_i\}$ , where  $7 \leq i \leq 11$ . By Lemma 8(ii),  $X$  can be extended to an edge-combination  $S$ . We can assert that  $|S| = 4$ . In fact, if  $|S| = 5$ , say  $S = \{c_1, c_2, c_7, c_8, c_9\}$ , then  $c_7 + c_8 + c_9 = \frac{1}{9}$  and so  $c_{10} + c_{11} = \frac{2}{9}$ . Thus  $\max\{c_{10}, c_{11}\} \geq \frac{1}{9}$ , a contradiction. Hence there exists another  $c_j$  ( $7 \leq j \leq 11$ ) such that  $S = \{c_1, c_2, c_i, c_j\}$  and  $c_i + c_j = \frac{1}{9}$ .

Now we consider a graph  $F$  with vertex set  $\{c_7, c_8, c_9, c_{10}, c_{11}\}$ , two vertices  $c_i$  and  $c_j$  being adjacent if and only if  $c_i + c_j = \frac{1}{9}$ . Then no two edges in  $F$  are nonadjacent, for otherwise the vertex not incident to these edges would have  $c_i = \frac{1}{9}$ , a contradiction. This implies that  $F$  is a star on 5 vertices. Suppose  $c_{11}$  is the center of the star. We calculate the edge-combination number as follows.

- There are  $\binom{6}{3}$  3-combinations  $\{c_i, c_j, c_k\}$  chosen from  $\{c_1, c_2, \dots, c_6\}$ , each of which is an edge-combination.
- There are  $\binom{6}{2}\binom{4}{1}$  4-combinations  $\{c_i, c_j, c_k, c_{11}\}$  such that  $\{c_i, c_j\}$  are chosen from  $\{c_1, c_2, \dots, c_6\}$  and  $c_k$  is chosen from  $\{c_7, c_8, c_9, c_{10}\}$ , each of which is an edge-combination.

To sum up,  $\mathcal{E}(S^c) \geq \binom{6}{3} + \binom{6}{2}\binom{4}{1} > 72 \geq |E(G)|$ , a contradiction to Lemma 7. This completes the proof of Case 1.

*Case 2.* There is a  $\{3, 4, 4\}$ -partition of  $S^c$ . So we may assume that

$$c_1 + c_2 + c_3 = c_4 + c_5 + c_6 + c_7 = c_8 + c_9 + c_{10} + c_{11} = \frac{1}{3}.$$

We first observe that  $\min\{c_1, c_2, c_3\} \geq \max\{c_4, c_5, \dots, c_{11}\}$ . If not, say  $c_4 > c_3$ , then  $c_1 + c_2 + c_4 > \frac{1}{3}$ , contradicting Lemma 8(i). So  $\max\{c_4, c_5, \dots, c_{11}\} \leq \frac{1}{9}$ . Moreover, if  $\max\{c_4, c_5, \dots, c_{11}\} = \frac{1}{9}$ , then in addition to  $c_1 = c_2 = c_3 = \frac{1}{9}$ ,

each of  $\{c_4, c_5, c_6, c_7\}$  and  $\{c_8, c_9, c_{10}, c_{11}\}$  has at most two elements being  $\frac{1}{9}$ . We distinguish the following subcases.

*Subcase 2.1.* Each of  $\{c_4, c_5, c_6, c_7\}$  and  $\{c_8, c_9, c_{10}, c_{11}\}$  has exactly two elements being  $\frac{1}{9}$ . We may assume that  $c_4 = c_5 = \frac{1}{9}$  and  $c_6 + c_7 = \frac{1}{9}$ , and  $c_8 = c_9 = \frac{1}{9}$  and  $c_{10} + c_{11} = \frac{1}{9}$ . Similar to Subcase 1.2, there are  $\binom{7}{3}$  edge-combinations  $\{c_i, c_j, c_k\}$  chosen from  $\{c_1, c_2, c_3, c_4, c_5, c_8, c_9\}$ . There are  $2\binom{7}{2}$  edge-combinations  $\{c_i, c_j, c_6, c_{r(6)}\}$  and  $\{c_i, c_j, c_{10}, c_{r(10)}\}$ , where  $10 \neq r(6)$ , and  $\{c_i, c_j\}$  are chosen from  $\{c_1, c_2, c_3, c_4, c_5, c_8, c_9\}$ . Therefore,  $\mathcal{E}(S^c) \geq \binom{7}{3} + 2\binom{7}{2} > 72 \geq |E(G)|$ , as required.

*Subcase 2.2.*  $\{c_4, c_5, c_6, c_7\}$  has only one element being  $\frac{1}{9}$ , say  $c_4 = \frac{1}{9}$ . Then  $c_5 + c_6 + c_7 = \frac{2}{9}$ . Assume that  $\frac{1}{9} > c_5 \geq c_6 \geq c_7$ . Then  $c_1 + c_2 + c_i < \frac{1}{3}$  for  $5 \leq i \leq 7$ . By Lemma 8(ii),  $\{c_1, c_2, c_i\}$  can be extended to an edge-combination  $S$  with  $|S| = 4$  (if  $|S| = 5$ , then we can get a  $\{3, 3, 5\}$ -partition of  $S^c$ , which reduces to Case 1). Note that  $c_i + c_k > \frac{1}{9}$  for  $5 \leq i, k \leq 7$ . There is a  $c_j$  with  $j > 7$  such that  $S = \{c_1, c_2, c_i, c_j\}$  and  $c_i + c_j = \frac{1}{9}$ . Thus we can define a mapping from  $\{c_i : 5 \leq i \leq 7\}$  to  $\{c_j : 8 \leq j \leq 11\}$  with  $c_i + c_j = \frac{1}{9}$ . In this respect, we claim that it is impossible that  $\{c_8, c_9, c_{10}, c_{11}\}$  has only one element being  $\frac{1}{9}$ . To see this, assume that  $c_8 = \frac{1}{9}$  and  $c_9 + c_{10} + c_{11} = \frac{2}{9}$ . Then we can also define a mapping from  $\{c_j : 9 \leq j \leq 11\}$  to  $\{c_i : 5 \leq i \leq 7\}$  as above. Hence we obtain a bijection between  $\{c_5, c_6, c_7\}$  and  $\{c_9, c_{10}, c_{11}\}$ . Consequently,  $c_5 + c_6 + c_7 + c_9 + c_{10} + c_{11} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$  and so  $c_4 + c_8 = \frac{1}{3}$ , which is impossible. Therefore,  $\{c_8, c_9, c_{10}, c_{11}\}$  has exactly two elements being  $\frac{1}{9}$ , say  $c_8 = c_9 = \frac{1}{9}$ , and so  $c_{10} + c_{11} = \frac{1}{9}$ .

As stated in Subcase 1.3, we may define a graph  $F$  with vertex set  $\{c_5, c_6, c_7, c_{10}, c_{11}\}$ , two vertices  $c_i$  and  $c_j$  being adjacent if and only if  $c_i + c_j = \frac{1}{9}$ . Then  $F$  is a star on 5 vertices. By the same calculation as in Subcase 1.3, we obtain  $\mathcal{E}(S^c) \geq \binom{6}{3} + \binom{6}{2} \binom{4}{1} > 72 \geq |E(G)|$ , a contradiction to Lemma 7.

*Subcase 2.3.*  $\max\{c_4, c_5, \dots, c_{11}\} < \frac{1}{9}$ . For this, we cannot assure  $\{c_1, c_2, c_3\}$  has some element being  $\frac{1}{9}$ . We may assume that  $c_1 \geq c_2 \geq c_3 \geq c_4 \geq c_5 \geq c_6 \geq c_7$ ,  $c_8 \geq c_9 \geq c_{10} \geq c_{11}$ , and  $c_4 \geq c_8$ . We show that  $c_3 = c_4$ . In fact, if  $c_3 > c_4$ , then  $c_1 + c_2 + c_4 < \frac{1}{3}$ , and so there is an  $i$  with  $5 \leq i \leq 11$  such that  $c_1 + c_2 + c_4 + c_i = \frac{1}{3}$  (by Lemma 8(ii)). Noting  $c_3 \leq \frac{1}{9}$ , we have  $c_1 + c_2 \geq \frac{2}{9}$ . This implies that  $c_4 + c_i \leq \frac{1}{9}$ , and thus  $c_8 + c_i \leq \frac{1}{9}$ . Consequently, there would be  $c_j$  and  $c_k$  ( $4 \leq j, k \leq 11$ ) such that  $c_j + c_k \geq \frac{2}{9}$ . Then one of them is at least  $\frac{1}{9}$ , contradicting our assumption. By the same argument, we can show that  $c_4 = c_5$ , and further  $c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = \frac{1}{12}$ .

Furthermore, we claim that  $c_2 = c_3$ . In fact, if  $c_1 \geq c_2 > c_3$ , then  $c_2 \leq \frac{1}{2} \left( \frac{1}{3} - \frac{1}{12} \right) = \frac{1}{8}$ , and so  $c_2 + c_3 + c_4 \leq \frac{1}{8} + \frac{1}{6} = \frac{7}{24} < \frac{1}{3}$ . By Lemma 8(ii), there is an  $i$  ( $i \geq 5$ ) such that  $c_2 + c_3 + c_4 + c_i = \frac{1}{3}$ . But this contradicts the fact that  $c_2 + c_3 + c_4 + c_i > \frac{4}{12} = \frac{1}{3}$ . Therefore,  $c_2 = c_3 = \dots = c_{11} = \frac{1}{12}$  which implies  $c_1 = \frac{1}{6}$ .

From this, we consider the edge-combination number  $\mathcal{E}(S^c)$  as follows.

- There are  $\binom{10}{3}$  3-combinations  $\{c_i, c_j, c_k\}$  with  $c_i = c_j = c_k = \frac{1}{12}$ , apart from  $c_1 = \frac{1}{6}$ , at most  $\binom{4}{3}$  of which correspond to a common edge-combination  $\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\}$ .

- There are  $\binom{10}{2}$  3-combinations  $\{c_1, c_i, c_j\}$  with  $c_1 = \frac{1}{6}$  and  $c_i = c_j = \frac{1}{12}$ , each of which corresponds to an edge-combination  $\{\frac{1}{6}, \frac{1}{12}, \frac{1}{12}\}$ .

To summarize,  $\mathcal{E}(S^c) \geq \binom{10}{3}/4 + \binom{10}{2} > 72 \geq |E(G)|$ , a contradiction to Lemma 7. This completes the proof. ■

As a consequence, if  $d(P(G)) \leq 10$  (whence  $\varphi(P(G)) \leq 11$ ), then the 3PM-property holds. This includes our previous result (Proposition 9).

#### 4. BRICKS

Recall that a brick is 3-connected and bicritical. Relative to Theorem 12 and 13, we can get better results for cubic bricks.

**Lemma 14.** *For a cubic brick  $G$ , if  $d(P(G)) = d$ , then  $|E(G)| = 3d$ .*

**Proof.** Since  $G$  is cubic,  $2|E(G)| = 3|V(G)|$ . Since  $G$  is a brick,  $b = 1$ . By Lemma 4,  $d = m - n + 1 - b = m - \frac{2m}{3} = \frac{m}{3}$ , which implies  $|E(G)| = 3d$ . ■

**Theorem 15.** *For a cubic brick  $G$  with  $d(P(G)) \leq 24$ , if  $\varphi(P(G)) \leq 11$ , then the 3PM-property holds.*

**Proof.** Recalling the proofs of Theorems 12 and 13, we note that when  $\varphi(P(G)) \leq 11$ , as long as  $|E(G)| \leq 72$ , all these proofs are valid. Now for a brick  $G$ , if  $d(P(G)) \leq 24$ , then  $|E(G)| = 3d \leq 72$ , as we needed in the proofs. ■

**Theorem 16.** *For a cubic brick  $G$  with  $d(P(G)) \leq 18$ , if  $\varphi(P(G)) \leq 12$ , then the 3PM-property holds.*

**Proof.** By Lemma 14,  $|E(G)| = 3d \leq 54$ . By Theorem 15, we need only consider the case  $\varphi(P(G)) = 12$ . Then the equations (4) and (5) hold for  $k = 12$ . Suppose, to the contrary, that the 3PM-property does not hold.

By Lemma 8(i), each edge-combination contains from three to six  $c$ -elements of  $c_1, c_2, \dots, c_{12}$ . Moreover, by Lemma 8(iii), the 12  $c$ -elements are divided into three sets, each of which has sum  $\frac{1}{3}$ . Note that the only partitions  $\{k_1, k_2, k_3\}$  of integer 12, where  $k_1 + k_2 + k_3 = 12$ ,  $3 \leq k_i \leq 6$ , are  $\{3, 3, 6\}$ ,  $\{3, 4, 5\}$ , and  $\{4, 4, 4\}$ . So we consider three cases as follows.

*Case 1.* There is a  $\{3, 3, 6\}$ -partition of  $S^c$ . By Lemma 8(i), there are six  $c$ -elements being  $\frac{1}{9}$  and no other  $c_i$  is greater than  $\frac{1}{9}$ . So we may assume that

$c_1 = c_2 = \cdots = c_6 = \frac{1}{9}$ ,  $c_7 + c_8 + \cdots + c_{12} = \frac{1}{3}$ , and  $\frac{1}{9} \geq c_7 \geq \cdots \geq c_{12}$ . Similarly to Case 1 in the proof of Theorem 13, we have the following subcases.

*Subcase 1.1.*  $c_7 = c_8 = \frac{1}{9}$  and  $c_9 + c_{10} + c_{11} + c_{12} = \frac{1}{9}$ . Since each 3-combination  $\{c_i, c_j, c_k\}$  taken from  $\{c_1, c_2, \dots, c_8\}$  is an edge-combination, we have  $\mathcal{E}(S^c) \geq \binom{8}{3} > 54 \geq |E(G)|$ , contradicting Lemma 7.

*Subcase 1.2.*  $c_7 = \frac{1}{9}$  and  $c_8 + c_9 + c_{10} + c_{11} + c_{12} = \frac{2}{9}$ . Then there are  $\binom{7}{3}$  3-combinations  $\{c_i, c_j, c_k\}$  taken from  $\{c_1, c_2, \dots, c_7\}$ , each of which is an edge-combination. For each  $\{c_i, c_j\}$  chosen from  $\{c_1, c_2, \dots, c_7\}$ ,  $c_i + c_j + c_8 < \frac{1}{3}$ . By Lemma 8(ii),  $\{c_i, c_j, c_8\}$  can be extended to an edge-combination. Therefore,  $\mathcal{E}(S^c) \geq \binom{7}{3} + \binom{7}{2} > 54 \geq |E(G)|$ , as required.

*Subcase 1.3.*  $c_i < \frac{1}{9}$  for  $7 \leq i \leq 12$  and  $c_7 \geq c_8 \geq \cdots \geq c_{12}$ . For a pair of given  $i, j$  ( $1 \leq i, j \leq 6$ ),  $c_i + c_j + c_k < \frac{1}{3}$  for any  $k$  with  $7 \leq k \leq 12$ , and so  $\{c_i, c_j, c_k\}$  can be extended to an edge-combination  $S_k$ . Note that  $S_k \setminus \{c_i, c_j\} \subseteq \{c_7, c_8, \dots, c_{12}\}$ . It is impossible that  $|S_k| = 6$ , for otherwise there would be a  $c_l \geq \frac{1}{9}$  for  $7 \leq l \leq 12$ . So  $4 \leq |S_k| \leq 5$ .

If there is an  $S_k$  with  $|S_k| = 5$ , then the sum of the three  $c$ -elements of  $S_k \setminus \{c_i, c_j\}$  is  $\frac{1}{9}$  and the sum of the remaining three  $c$ -elements in  $\{c_7, c_8, \dots, c_{12}\}$  is  $\frac{2}{9}$ , say  $c_{k_1} + c_{k_2} + c_{k_3} = \frac{2}{9}$ . Since the sum of any two  $c$ -elements of  $c_{k_1}, c_{k_2}, c_{k_3}$  is more than  $\frac{1}{9}$ ,  $S_{k_1}, S_{k_2}$ , and  $S_{k_3}$  are different. If each  $S_k$  has exactly four  $c$ -elements, say  $S_k = \{c_i, c_j, c_k, c_{r(k)}\}$ , then there are at least three different  $S_k$ 's.

Now we evaluate the number of edge-combinations. There are  $\binom{6}{3}$  3-combinations taken from  $\{c_1, c_2, \dots, c_6\}$ , each of which is an edge-combination. Moreover, each 2-combination  $\{c_i, c_j\}$  taken from  $\{c_1, c_2, \dots, c_6\}$  corresponds to at least three edge-combinations. Therefore, we have  $\mathcal{E}(S^c) \geq \binom{6}{3} + 3\binom{6}{2} > 54 \geq |E(G)|$ , as needed.

*Case 2.* There is a  $\{4, 4, 4\}$ -partition of  $S^c$ . Then every edge-combination has cardinality 4. In this context, there are  $\binom{12}{3}$  3-combinations  $\{c_i, c_j, c_k\}$ , at most  $\binom{4}{3}$  of which correspond to a common edge-combination. Therefore,  $\mathcal{E}(S^c) \geq \binom{12}{3}/4 > 54 \geq |E(G)|$ , as needed.

*Case 3.* Every 3-partition of  $S^c$  is a  $\{3, 4, 5\}$ -partition. Then for any vertex  $v$  of  $G$ , the three edges of  $G$  incident with  $v$  lie in exactly three, four, and five perfect matchings of  $M_1, M_2, \dots, M_{12}$ , respectively. Let  $N_1, N_2, N_3$  be the sets of edges which lie in exactly three, four, and five perfect matchings of  $M_1, M_2, \dots, M_{12}$ , respectively. Then we see that  $N_1, N_2$ , and  $N_3$  are three pairwise disjoint perfect matchings of  $G$ , a contradiction to the assumption that 3PM-property fails. This completes the proof.  $\blacksquare$

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