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CONSTANT SUM PARTITION OF SETS OF INTEGERS AND DISTANCE MAGIC GRAPHS

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Abstract

Let $A = \{1, 2, \ldots, tm+tn\}$. We shall say that A has the (m, n, t)-*balanced constant-sum-partition property* ((m, n, t)-*BCSP-property*) if there exists a partition of A into $2t$ pairwise disjoint subsets $A^1, A^2, \ldots, A^t, B^1, B^2, \ldots, B^t$ such that $|A^i| = m$ and $|B^i| = n$, and $\sum_{a\in A^i} a = \sum_{b\in B^j} b$ for $1 \le i \le t$ and $1 \le j \le t$. In this paper we give sufficient and necessary conditions for a set A to have the (m, n, t)-BCSP-property in the case when m and n are both even. We use this result to show some families of distance magic graphs.

Keywords: constant sum partition, distance magic labeling, product of graphs.

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1. Introduction

Let $A = \{1, 2, \ldots, tm + tn\}$. We shall say that A has the (m, n, t)-*balanced constant-sum-partition property* ((m, n, t)-*BCSP-property*) if there exists a partition of A into pairwise disjoint subsets $A^1, A^2, \ldots, A^t, B^1, B^2, \ldots, B^t$ such that $|A^i| = m$ and $|B^i| = n$, and $\sum_{a\in A^i} a = \sum_{b\in B^j} b$ for $1 \le i, j \le t$. A positive integer $\mu = \sum_{a\in A^i} a = \sum_{b\in B^j} b$ is called a *balanced constant*.

All graphs considered in this paper are simple finite graphs. Given a graph G, we denote its order by $|G|$, its size by $||G||$, its vertex set by $V(G)$ and the edge set by $E(G)$. The *neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the *degree* $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x.

Distance magic labeling (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $l\colon V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called *magic constant*) such that $w(x) = \sum_{y \in N_G(x)} l(y) = k$ for every $x \in V$. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph* (see [29]). It was proved recently that the magic constant is unique ([27]).

The concept of distance magic labeling has been motivated by the construction of magic rectangles. Magic rectangles are a natural generalization of magic squares which have long intrigued mathematicians and the general public [17]. A *magic* (m, n)*-rectangle* S is an $m \times n$ array in which the first mn positive integers are placed so that the sum over each row of S is constant and the sum over each column of S is another (different if $m \neq n$) constant. Harmuth proved the following theorem.

Theorem 1 [19, 20]. *For $m, n > 1$ there is a magic (m, n)-rectangle S if and only if $m \equiv n \mod 2$ and $(m, n) \neq (2, 2)$.*

As in the case of magic squares, we can construct a distance magic complete m partite graph with each part size equal to n by labeling the vertices of each part by the columns of the magic rectangle. Moreover, observe that constant sum partition of $\{1, 2, \ldots, n\}$ leads to complete multipartite distance magic labeled graphs. For instance, the partition $\{1, 4\}$, $\{2, 3\}$ of the set $\{1, 2, 3, 4\}$ with constant sum 5 leads to distance magic labeling of the complete bipartite graph $K_{2,2}$, see [6]. Beena proved the following.

Theorem 2 [6]. *Let m and n be two positive integers such that $m \leq n$. The complete bipartite graph $K_{m,n}$ is a distance magic graph if and only if*

- $m + n \equiv 0$ *or* $3 \pmod 4$, *and*
- *either* $n \leq \lfloor (1 + \sqrt{2})m - \frac{1}{2} \rfloor$ *or* $2(2n + 1)^2 - (2m + 2n - 1)^2 = 1$.

Moreover, Kotlar recently gave necessary and sufficient conditions for complete 4-partite graph to be distance magic (see [22]). He also posted the following open problem.

Problem 1.1 [22]. Let n, k and $p_1, p_2, \ldots, p_k$ be positive integers such that $p_1 + p_2 + \cdots + p_k = n$ and $\binom{n+1}{2}/k$ is an integer. When is it possible to find a partition of the set $\{1, 2, \ldots, n\}$ into k subsets of sizes $p_1, p_2, \ldots, p_k$, respectively, such that the sum of the elements in each subset is $\binom{n+1}{2}/k$?

A similar problem was also considered in [2, 7, 9, 12, 14, 23, 24]. Namely, a non-increasing sequence $\langle m_1, \ldots, m_k \rangle$ of positive integers is said to be *n-realizable* if the set $\{1, 2, \ldots, n\}$ can be partitioned into k mutually disjoint subsets X_1, $X_2, \ldots, X_k$ such that $\sum_{x \in X_i} x = m_i$ for each $1 \leq i \leq k$. The study of n-realizable sequences was motivated by the ascending subgraph decomposition problem posed by Alavi, Boals, Chartrand, Erdős and Oellerman [1], which asks for a decomposition of a given graph G of size $\binom{n+1}{2}$ by subgraphs $H_1, H_2, \ldots, H_n$, where H_i has size i and is a subgraph of H_{i+1} for each $i = 1, 2, \ldots, n-1$. These authors conjectured that a forest of stars of size $\binom{n+1}{2}$ with each component having at least n edges admits an ascending subgraph decomposition by stars. This is equivalent to the fact that every non-increasing sequence $\langle m_1, \ldots, m_k \rangle$ with $\sum_{i=1}^{k} m_i = \binom{n+1}{2}$ and $m_k \geq n$ is n-realizable, a result which was proved by Ma, Zhou and Zhou [25]. Although the general ascending subgraph decomposition conjecture is unsolved so far, some partial results have been obtained [10, 11, 13].

We recall two out of four standard graph products (see [21]). Both, the *lexicographic product* $G \circ H$ and the *direct product* $G \times H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \circ H$ if and only if either g is adjacent to g' in G or $g = g'$ and h is adjacent to h' in H;
- $G \times H$ if g is adjacent to g' in G and h is adjacent to h' in H.

The graph $G \circ H$ is also called the *composition* and denoted by $G[H]$ (see [18]). The product $G \times H$, also known as *Kronecker product*, *tensor product*, *categorical product* and *graph conjunction*, is the most natural graph product.

Some graphs which are distance magic among (some) products can be found in [3, 4, 6, 8, 16, 26, 28].

The following problem was posted in [5].

Problem 1.2 [5]**.** If G is non-regular graph, determine if there is a distance magic labeling of $G \circ C_4$.

Anholcer and Cichacz proved the following.

Theorem 3 [3]**.** *Let m and n be integers such that $1 \leq m < n$. Then $K_{m,n} \circ C_4$ is distance magic if and only if the following conditions hold.*

(1) *The numbers*

$$a = \frac{(m+n)(4m+4n+1)(2m-1)}{4mn-m-n}$$

and

$$b = \frac{(m+n)(4m+4n+1)(2n-1)}{4mn-m-n}$$

are integers.

(2) *There exist integers $p, q, t \geq 1$ such that*

$$\begin{aligned} p+q &= (b-a), \\ 4n &= pt, \\ 4m &= qt. \end{aligned}$$

Moreover, they showed that a product $C_3^{(t)} \circ C_4$ is not distance magic, where $C_3^{(t)}$, called a *Dutch Windmill Graph*, is the graph obtained by taking $t > 1$ copies of the cycle graph C_3 with a vertex in common [15]. We prove that also the product $C_3^{(t)} \times C_4$ is not distance magic.

Thus we state a problem similar to Problem 1.2 for direct product.

Problem 1.3. If G is a non-regular graph, determine if there is a distance magic labeling of $G \times C_4$.

The paper is organized as follows. In the next section we focus on sets having an (m, n, t)-BCSP-property. We give the necessary and sufficient conditions for a set $A = \{1, 2, \ldots, tm+tn\}$ to have the (m, n, t)-BCSP-property in the case when m and n are both even. In the third section we generalize the Beena's result ([6]) by showing necessary and sufficient conditions for t copies of $K_{m,n}$ ($tK_{m,n}$) to be distance magic, if m and n are both even. We use this result to give necessary and sufficient conditions for the direct product $K_{m,n} \times C_4$ to be distance magic.

2. Constant Sum Partition

Theorem 4. *Let m and n be two positive integers such that $m \leq n$. If the set $A = \{1, 2, \ldots, tm+tn\}$ has the (m, n, t)-BCSP-property, then the conditions hold:*

- $m + n \equiv 0 \pmod 4$ *or* $tm + tn \equiv 3 \pmod 4$, *and*
- $1 = 2(2tn + 1)^2 - (2tm + 2tn + 1)^2$ *or* $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2}-1}{2t}$.

Proof. Suppose that $A^1, A^2, \ldots, A^t, B^1, B^2, \ldots, B^t$ is an (m, n, t)-constant sum partition of the set A. Let $A^i = \{a_0^i, a_1^i, \ldots, a_{m-1}^i\}$ and $B^i = \{b_0^i, b_1^i, \ldots, b_{n-1}^i\}$ for $i = 1, 2, \ldots, t$. Since for the balanced constant μ we have $\mu = \sum_{i=0}^{m-1} a_i^j = \sum_{l=0}^{n-1} b_l^j$, for $j = 1, 2, \ldots, t$, it is easy to observe that

$$\mu = \frac{1}{2t} \sum_{i=1}^{tn+tm} i = \frac{(tm + tn)(tm + tn + 1)}{4t},$$

which implies that $m + n \equiv 0 \pmod 4$ or $tm + tn \equiv 3 \pmod 4$. Notice that $\sum_{i=0}^{m-1} \sum_{j=1}^{t} a_i^j \leq \sum_{i=1}^{tm} (i + tn) = \frac{tm(tm+2tn+1)}{2}$, thus $\mu \leq \frac{m(tm+2tn+1)}{2}$. This implies $(m + n)(tm + tn + 1) \leq 2m(tm + 2tn + 1)$ and therefore

$$\left[tm + \left(tn + \frac{1}{2}\right)\right]^2 \geq t^2n^2 + tn + \left(tn + \frac{1}{2}\right)^2 = \frac{(2tn + 1)^2}{2} - \frac{1}{4}.$$

That is

$$1 \geq 2(2tn+1)^2 - (2tm+2tn+1)^2.$$

Therefore, $1 = 2(2tn+1)^2 - (2tm+2tn+1)^2$ or $m \geq (\sqrt{2}-1)n + \frac{\sqrt{2}-1}{2t}$. ■

Theorem 5. *Let m and n be two positive integers such that $m \leq n$. If the conditions hold:*

- $m + n \equiv 0 \pmod 4$ *or* $tm + tn \equiv 3 \pmod 4$, *and*
- $1 = 2(2tn+1)^2 - (2tm+2tn+1)^2$,

then the set $A = \{1, 2, \ldots, tm+tn\}$ has the (m, n, t)-BCSP-property.

Proof. Using the same arguments as in the proof of Theorem 4, the condition $1 = 2(2tn+1)^2 - (2tm+2tn+1)^2$ relates to the solution when the tm elements in $A^1 \cup A^2 \cup \cdots \cup A^t$ have to be the tm largest integers $1+tn, 2+tn, \ldots, tn+tm$ (because then $\sum_{i=0}^{m-1} \sum_{j=1}^{t} a_i^j = \sum_{i=1}^{tm}(i+tn) = \frac{tm(tm+2tn+1)}{2}$), whereas the tn elements in $B^1 \cup B^2 \cup \cdots \cup B^t$ have to be the tn smallest integers $1, 2, \ldots, tn$ and $\mu = \frac{m(tm+2tn+1)}{2} = \frac{n(tn+1)}{2}$. Notice that if m or n is odd, then t is odd since the constant μ is an integer.

If m is odd, then there exists a magic (t, m)-rectangle by Theorem 1. Let $a_{i,j}$ be an (i, j)-entry of the (t, m)-rectangle, $0 \leq i \leq t-1$ and $0 \leq j \leq m-1$. Notice that $\sum_{j=0}^{m-1} a_{i,j} = \frac{m(1+tm)}{2}$. Let $a_j^i = a_{i,j} + tn$, for $j = 0, 1, \ldots, m-1$ and $i = 0, 1, \ldots, t-1$.

If n is odd, then there exists a magic (t, n)-rectangle by Theorem 1. Let $b_{i,j}$ be an (i, j)-entry of the (t, n)-rectangle, $0 \leq i \leq t-1$ and $0 \leq j \leq n-1$. Notice that $\sum_{i=0}^{n-1} b_{i,j} = \frac{n(1+tn)}{2}$. Let $b_j^i = b_{i,j}$, for $j = 0, 1, \ldots, n-1$ and $i = 0, 1, \ldots, t-1$.

If m is even, then $a_{2j}^i = tn + i\frac{m}{2} + j + 1$, $a_{2j+1}^i = tn + tm - i\frac{m}{2} - j$, for $j = 0, 1, \ldots, m/2 - 1$ and $i = 0, 1, \ldots, t-1$.

If n is even, then $b_{2j}^i = i\frac{n}{2} + j + 1$, $b_{2j+1}^i = tn - i\frac{n}{2} - j$, for $j = 0, 1, \ldots, n/2 - 1$ and $i = 0, 1, \ldots, t-1$. ■

Theorem 6. *Let m and n be two positive even integers such that $m \leq n$. The set $A = \{1, 2, \ldots, tm+tn\}$ has the (m, n, t)-BCSP-property if and only if the conditions hold:*

- $m + n \equiv 0 \pmod 4$, *and*
- $1 = 2(2tn+1)^2 - (2tm+2tn+1)^2$ *or* $m \geq (\sqrt{2}-1)n + \frac{\sqrt{2}-1}{2t}$.

Proof. The necessity is obvious by Theorem 4. Suppose now that m and n are positive even integers satisfying above assumptions. We can also assume that $m \geq (\sqrt{2}-1)n + \frac{\sqrt{2}-1}{2t}$ (which in these case is equivalent to $m > -n + \frac{\sqrt{2(2tn+1)^2-1}-1}{2t}$ since $\sqrt{2}(2tn+1) > \sqrt{2(2tn+1)^2-1}$), by Theorem 5.

Let us partition the set A into t disjoint sets $V_i = \{i + 2tj, 2t - i + 1 + 2tj, j \in \{0, 1, \ldots, \frac{m+n-2}{2}\}\}$ for $i \in \{1, \ldots, t\}$ with cardinality $m+n$. For every $a \in V_i$ let $\overline{a}$ denote the element in V_i such that $a + \overline{a} = tm + tn + 1$. Observe that for every element $a \in V_i$ there exists $\overline{a} \in V_i$. The sum of integers in each set V_i is $K = \frac{(1+tm+tn)(m+n)}{2}$. Obviously, a balanced constant is $\mu = \frac{K}{2}$.

Let W_i be the sequence of m greatest integers in V_i for every $i \in \{1, \ldots, t\}$, so $W_i = (tn + i, tm + tn - (m-2)t - i + 1, \ldots, tm + tn - 2t + i, tm + tn - i + 1)$. Denote the j-th element in a sequence W_i by w_i^j. Then for each i we obtain that

$$\sum_{j=1}^{m} w_i^j = \frac{m(1 + tm + 2tn)}{2} =: S.$$

Since $m > -n + \frac{\sqrt{2(2tn+1)^2-1}-1}{2t}$, observe that $S - \mu > 0$. Hence, there exist nonnegative integers k and d such that $S - \mu = km + d$, where $0 \leq d < m$. Therefore, $S - \mu = \frac{tm^2}{4} + \frac{tmn}{2} + \frac{m-n}{4} - \frac{tn^2}{4} = km + d \leq \frac{tmn}{2}$, since $m \leq n$. Hence, we obtain that $k \leq \frac{tn}{2}$. Furthermore, $tn - k > 0$.

If $d = 0$ we create sets $A_1, \ldots, A_t$ putting $A_i = \{w_i^1 - k, \ldots, w_i^m - k\}$. Note that $A_i \cap A_j = \emptyset$ for every $i \neq j$. Moreover, $\sum_{a \in A_i} a = S - mk = \mu$ for $i \in \{1, \ldots, t\}$.

Let $B_i' = \{\overline{w_i^1 - k}, \ldots, \overline{w_i^m - k}\}$. Observe that the set

$$B = A \setminus \left(\bigcup_{i=1}^{t} A_i \cup \bigcup_{i=1}^{t} B_i' \right)$$

has cardinality $t(n-m)$. Indeed, we can part it into $\frac{t(n-m)}{2}$ pairs with type $\{a, \overline{a}\}$ (see Example 7). Then we part the set B into t disjoint subsets $B_1'', \ldots, B_t''$ with cardinality $n - m$ so that the elements of every set B_i'' create exactly $\frac{n-m}{2}$ pairs with type $\{a, \overline{a}\}$. Let $B_i = B_i' \cup B_i''$ for $i \in \{1, \ldots, t\}$. Then each set B_i contains n elements and $B_i \cap B_j = \emptyset$ for $i \neq j$. Furthermore,

$$\sum_{b \in B_i} b = \frac{(n-m)(tm + tn + 1)}{2} + m(tm + tn + 1) - \mu = \mu.$$

Example 7. Let $m = n = t = 2$. Then $A = \{1, 2, \ldots, 8\}$, $S = 13$, $\mu = 9$. Since $V_1 = \{1, 4, 5, 8\}$ and $V_2 = \{2, 3, 6, 7\}$, we have $W_1 = \{5, 8\}$ and $W_2 = \{6, 7\}$. Observe that $d = 0$ and then $A_1 = \{3, 6\}$, $A_2 = \{4, 5\}$, $B_1' = \{6, 3\}$ and $B_2' = \{5, 4\}$. Therefore $B = \{1, 2, 7, 8\}$ and elements of it create two pairs with type $\{a, \overline{a}\}$, namely $\{1, 8\}$ and $\{2, 7\}$.

If $d > 0$ we create sets A_i as follows. We subtract 1 from each of the first d labels: $A_i = \{w_i^1 - k - 1, \ldots, w_i^d - k - 1, w_i^{d+1} - k, \ldots, w_i^m - k\}$ for

$i \in \{1, \ldots, t\}$. Then $\sum_{a \in A_i} a = S - mk - d = \mu$ for $i \in \{1, \ldots, t\}$. Then $B'_i = \{\overline{w_i^1 - k - 1}, \ldots, \overline{w_i^d - k - 1}, \overline{w_i^{d+1} - k}, \ldots, \overline{w_i^m - k}\}$ and elements of a set $B = A \setminus (\bigcup_{i=1}^t A_i \cup \bigcup_{i=1}^t B'_i)$ create $\frac{t(n-m)}{2}$ pairs with type $\{a, \overline{a}\}$. As above, we part the set B into t disjoint subsets $B''_1, \ldots, B''_t$ with cardinality $n - m$ so that the elements of every set B''_i create exactly $\frac{n-m}{2}$ pairs with type $\{a, \overline{a}\}$ and define pairwise disjoint sets $B_i = B'_i \cup B''_i$ for $i \in \{1, \ldots, t\}$. Each set B_i contains n elements and $\sum_{b \in B_i} b = \mu$.

Hence A has the (m, n, t)-BCSP-property. ■

Notice that although the numbers $m = 3$, $n = 6$, $t = 3$ satisfy the necessary conditions of Theorem 4, they do not satisfy the sufficient conditions either of Theorem 5 or 6. Let $A_1 = \{10, 26, 27\}$, $A_2 = \{14, 24, 25\}$, $A_3 = \{18, 22, 23\}$, $B_1 = \{1, 4, 7, 13, 17, 21\}$, $B_2 = \{2, 5, 8, 12, 16, 20\}$, $B_3 = \{3, 6, 9, 11, 15, 19\}$. Thus, the set $A = \{1, 2, \ldots, 27\}$ has the (3, 6, 3)-BCSP-property. Therefore, we conclude this section by stating the following.

Conjecture 2.1. *Let m and n be two positive integers such that $m \leq n$. The set $A = \{1, 2, \ldots, tm + tn\}$ has the (m, n, t)-BCSP-property if and only if the conditions hold:*

- $m + n \equiv 0 \pmod 4$ *or* $tm + tn \equiv 3 \pmod 4$, *and*
- $1 = 2(2tn + 1)^2 - (2tm + 2tn + 1)^2$ *or* $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2}-1}{2t}$.

Recall that the conjecture is true for $t = 1$ by Theorem 2. Moreover, one can verify that the conjecture is also true for $t = 2$ (see e.g. [22], Theorem 2).

3. Distance Magic Graphs

We obtain the following corollaries by Theorem 6.

Corollary 1. *Let m and n be two positive even integers such that $m \leq n$. The graph $tK_{m,n}$ is distance magic if and only if the conditions hold:*

- $m + n \equiv 0 \pmod 4$, *and*
- $1 = 2(2tn + 1)^2 - (2tm + 2tn + 1)^2$ *or* $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2}-1}{2t}$.

Let $K_{m[a],n[b]} \cong K_{\underbrace{m, \ldots, m}_{a}, \underbrace{n, \ldots, n}_{b}}$.

Corollary 2. *Let m and n be two positive even integers such that $m \leq n$. The graph $K_{m[t],n[t]}$ is distance magic if and only if the conditions hold:*

- $m + n \equiv 0 \pmod 4$, *and*
- $1 = 2(2tn + 1)^2 - (2tm + 2tn + 1)^2$ *or* $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2}-1}{2t}$.

Corollary 3. Let m and n be two positive integers such that $m \leq n$. The graph $K_{m,n} \times C_4$ is a distance magic graph if and only if the following conditions hold:

- $m + n \equiv 0 \pmod 2$, and
- $1 = 2(8n+1)^2 - (8m+8n+1)^2$ or $m \geq (\sqrt{2}-1)n + \frac{\sqrt{2}-1}{8}$.

Proof. Since $K_{m,n} \times C_4 \cong 2K_{2m,2n}$ we are done by Theorem 2. ■

We now show that there does not exist a distance magic labeling for $C_3^{(t)} \times C_4$.

Theorem 8. *The graph $C_3^{(t)} \times C_4$ is not a distance magic graph.*

Proof. Let $C_3^{(t)}$ have the central vertex x and let vertices y_i, z_i belong to ith copy of a cycle C_3. Let $C_4 = v^0v^1v^2v^3v^0$. Suppose that l is a distance magic labeling of the graph $H = C_3^{(t)} \times C_4$ and $k = w(x)$, for all vertices $x \in V\left(C_3^{(t)} \times C_4\right)$. Let

- $l(x, v^0) + l(x, v^2) = s_1$,
- $l(x, v^1) + l(x, v^3) = s_2$,
- $l(y_i, v^0) + l(y_i, v^2) = a_i^1$,
- $l(y_i, v^1) + l(y_i, v^3) = a_i^2$,
- $l(z_i, v^0) + l(z_i, v^2) = b_i^1$,
- $l(z_i, v^1) + l(z_i, v^3) = b_i^2$,

for $0 \leq i \leq t-1$.

Since $k = a_i^1 + s_2 = b_i^1 + s_2$ and $k = a_i^2 + s_1 = b_i^2 + s_1$, we observe that $l(y_i, v^0) + l(y_i, v^2) = l(z_i, v^0) + l(z_i, v^2) = a_1$ and $l(y_i, v^1) + l(y_i, v^3) = l(z_i, v^1) + l(z_i, v^3) = a_2$ for $0 \leq i \leq t-1$. Furthermore, since $k = w(x, v^0) = 2ta_2 = w(x, v^1) = 2ta_1$, we have $a_1 = a_2 = a$ and hence $s_1 = s_2 = s$.

Notice that $2s + 4ta = \sum_{x \in V(H)} l(x) = \sum_{i=1}^{8t+4} i = (4t+2)(8t+5)$. Since $k = 2ta = a + s$, we obtain that $(4t-1)a = (2t+1)(8t+5)$. Recall that a needs to be an integer, hence $(4t-1)$ needs to divide $(22t+5)$. Therefore we obtain that $t \in \{1, 2\}$. Suppose that $t = 2$ then $|V(H)| = 20$, $a = 15$, $s = 45$, then $l(x, v^i) = 15$ for some $i = 0, 1, 2, 3$ and thus $l(x, v^{i+2}) = 30 > 20$, a contradiction. ■

Notice that if we want to find the values of m and n such that $K_{m,n} \times C_4$ is a distance magic graph we need to solve the Diophantine equation

$$(1) \qquad \alpha = 2(4n+1)^2 - (4m+4n+1)^2$$

for some integer $\alpha \leq 1$. For instance if $\alpha = 1$, then the equation (1) is a Pell's equation, thus for example $K_{102,246} \times C_4$ is a distance magic graph.

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