INVERSE PROBLEM ON THE STEINER WIENER INDEX

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Abstract

The Wiener index $W(G)$ of a connected graph $G$, introduced by Wiener in 1947, is defined as $W(G) = \sum_{u,v \in V(G)} d_G(u,v)$, where $d_G(u,v)$ is the distance (the length a shortest path) between the vertices $u$ and $v$ in $G$. For $S \subseteq V(G)$, the Steiner distance $d(S)$ of the vertices of $S$, introduced by Chartrand et al. in 1989, is the minimum size of a connected subgraph of $G$ whose vertex set contains $S$. The $k$-th Steiner Wiener index $SW_k(G)$ of $G$ is defined as $SW_k(G) = \sum_{|S|=k} d(S)$. We investigate the following problem: Fixed a positive integer $k$, for what kind of positive integer $w$ does there exist a connected graph $G$ (or a tree $T$) of order $n \geq k$ such that $SW_k(G) = w$ (or $SW_k(T) = w$)? In this paper, we give some solutions to this problem.

Keywords: distance, Steiner distance, Wiener index, Steiner Wiener index.

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1. Introduction

All graphs in this paper are assumed to be undirected, finite and simple. We refer to [3] for graph theoretical notation and terminology not specified here. A distance is one of basic concepts of graph theory [4]. If $G$ is a connected graph and $u, v \in V(G)$, then the distance $d(u, v) = d_G(u, v)$ between $u$ and $v$ is the length of a shortest path connecting $u$ and $v$. For more details on this subject, see [13].

The Wiener index $W(G)$ of a connected graph $G$ is defined by

$$W(G) = \sum_{u, v \in V(G)} d_G(u, v).$$

Mathematicians have studied this graph invariant since the 1970s in [11]; for details see the surveys [10, 33], the recent papers [2, 7, 14, 17, 15, 20] and the references cited therein. Information on chemical applications of the Wiener index can be found in [27, 28].

The Steiner distance of a graph, introduced by Chartrand et al. in [6] in 1989, is a natural and nice generalization of the concept of the classical graph distance. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is a subgraph $T = (V', E')$ of $G$ that is a tree with $S \subseteq V'$. Let $G$ be a connected graph of order at least 2 and let $S$ be a nonempty set of vertices of $G$. Then the Steiner distance $d(S)$ among the vertices of $S$ (or simply the distance of $S$) is the minimum size of a connected subgraph whose vertex set contains $S$. Note that if $H$ is a connected subgraph of $G$ such that $S \subseteq V(H)$ and $|E(H)| = d(S)$, then $H$ is a tree. Clearly, $d(S) = \min \{|E(T)| : S \subseteq V(T)\}$, where $T$ is a subtree of $G$. Furthermore, if $S = \{u, v\}$, then $d(S) = d(u, v)$ is nothing new, but the classical distance between $u$ and $v$. Clearly, if $|S| = k$, then $d(S) \geq k - 1$. For more details on Steiner distance, we refer to [1, 5, 6, 8, 13, 26].

In [23], we proposed a generalization of the Wiener index concept, using Steiner distance. Thus, the $k$-th Steiner Wiener index $SW_k(G)$ of a connected graph $G$ is defined by

$$SW_k(G) = \sum_{S \subseteq V(G), |S| = k} d(S).$$

For $k = 2$, the Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider $SW_k$ for $2 \leq k \leq n - 1$, but the above definition implies $SW_1(G) = 0$ and $SW_n(G) = n - 1$ for a connected graph $G$ of order $n$. For more details on Steiner Wiener index, we refer to [23, 24, 25].

A chemical application of $SW_k$ was recently reported in [16].
It should be noted that in the 1990s, Dankelmann et al. in [8, 9] studied the average Steiner distance, which is related to our Steiner Wiener index via \( SW_k(G)/\binom{n}{k} \).

The seemingly elementary question: “Which natural numbers are Wiener indices of graphs?” was much investigated in the past; see [12, 19, 21, 29, 31, 32]. We now consider the analogous question for Steiner Wiener indices.

**Problem.** Fixed a positive integer \( k \), for what kind of positive integer \( w \) does there exist a connected graph \( G \) (or a tree \( T \)) of order \( n \geq k \) such that \( SW_k(G) = w \) (or \( SW_k(T) = w \))?

For \( k = 2 \), the authors have nice results in [30, 32], completely solved a conjecture by Lepović and Gutman [22] for trees, which states that for all but 49 positive integers \( w \) one can find a tree with Wiener index \( w \). This is different from our problem for trees, since here we consider graphs or trees with order \( n \).

2. **The Cases** \( k = n \) **and** \( k = n - 1 \)

At first, let us consider the case \( k = n \).

If \( G \) is a connected graph or a tree of order \( n \), then for \( k = n \), \( SW_k(G) = n - 1 \). Thus the following result is immediate.

**Proposition 2.1.** For a positive integer \( w \), there exists a connected graph \( G \) or a tree \( T \) of order \( n \) such that \( SW_n(G) = w \) or \( SW_n(T) = w \) if and only if \( w = n - 1 \).

For the case \( k = n - 1 \), we need the following results in [23].

**Lemma 2.2** [23]. Let \( T \) be a tree of order \( n \), possessing \( p \) pendant vertices. Then

\[
SW_{n-1}(T) = n(n-1) - p
\]

irrespective of any other structural detail of \( T \).

**Lemma 2.3** [23]. Let \( K_n \) be the complete graph of order \( n \), and let \( k \) be an integer such that \( 2 \leq k \leq n \). Then

\[
SW_k(K_n) = \binom{n}{k} (k - 1).
\]

**Lemma 2.4** [23]. Let \( G \) be a connected graph of order \( n \), and let \( k \) be an integer such that \( 2 \leq k \leq n \). Then

\[
\binom{n}{k} (k - 1) \leq SW_k(G) \leq (k - 1) \binom{n + 1}{k + 1}.
\]

Moreover, the lower bound is sharp.
From the previous results, we can derive the following proposition.

**Proposition 2.5.** For a positive integer \( w \), there exists a connected graph \( G \) of order \( n \) such that \( SW_{n-1}(G) = w \) if and only if \( n^2 - 2n \leq w \leq n^2 - n - 2 \).

**Proof.** By Lemma 2.4, if \( G \) is a connected graph of order \( n \), then

\[
n(n - 2) \leq SW_{n-1}(G) \leq (n + 1)(n - 2).
\]

Therefore, \( n^2 - 2n \leq w \leq n^2 - n - 2 \).

By Lemma 2.3, \( SW_{n-1}(K_n) = n^2 - 2n \).

Let \( T \) be a tree of order \( n \) with \( p \) pendant vertices with \( 2 \leq p \leq n - 1 \). By Lemma 2.2, \( SW_{n-1}(T) = n^2 - n - p \), and thus for any integer \( w \) with \( n^2 - n - (n - 1) \leq w \leq n^2 - n - 2 \), there exists a tree \( T \) of order \( n \) such that \( SW_{n-1}(T) = w \). \( \blacksquare \)

From the proof of Proposition 2.5 the next result immediately follows.

**Proposition 2.6.** For a positive integer \( w \), there exists a tree \( T \) of order \( n \) such that \( SW_{n-1}(T) = w \) if and only if \( n^2 - 2n + 1 \leq w \leq n^2 - n - 2 \).

3. **The Case** \( k = n - 2 \)

Similarly to Lemma 2.2, we can derive the following result.

**Lemma 3.1.** Let \( T \) be a tree of order \( n \), possessing \( p \) pendant vertices. Let \( q \) be the number of vertices of degree 2 in \( T \) that are adjacent to a pendant vertex. Then

\[
(1) \quad SW_{n-2}(T) = \frac{1}{2} \left( n^3 - 2n^2 + n - 2np + 2p - 2q \right).
\]

**Proof.** For any \( S \subseteq V(T) \) and \( |S| = n - 2 \), let \( \bar{S} = \{u, v\} \). If \( d_T(u) = d_T(v) = 1 \), then \( d_T(S) = n - 3 \), and this case contributes to \( SW_{n-2} \) by

\[
\sum_{u,v \in S, d_T(u) = d_T(v) = 1} d_T(S) = \binom{p}{2} (n - 3).
\]

If \( d_T(u) \geq 2 \) and \( d_T(v) \geq 2 \), then \( d_T(S) = n - 1 \), and this case contributes to \( SW_{n-2} \) by

\[
\sum_{u,v \in S, d_T(u) \geq 2, d_T(v) \geq 2} d_T(S) = \binom{n - p}{2} (n - 1).
\]

Suppose that \( d_T(u) = 1 \) and \( d_T(v) \geq 2 \). If \( d_T(u) = 1, d_T(v) = 2 \) and \( uv \in E(G) \), then \( d_T(S) = n - 3 \). If \( d_T(u) = 1, d_T(v) \geq 3 \) and \( uv \in E(T) \), then
Inverse Problem on the Steiner Wiener Index 87
d\(T(S) = n - 2\). If \(d_T(u) = 1\), \(d_T(v) \geq 2\) and \(uv \notin E(T)\), then \(d_T(S) = n - 2\). Therefore, this case contributes to \(SW_{n-2}\) by

\[
\sum_{u,v \in S, d_T(u)=1, d_T(v)\geq2} d_T(S) = \sum_{u,v \in S, d_T(u)=1, d_T(v)=2} d_T(S) + \sum_{u,v \in S, d_T(u)=1, d_T(v)\geq3} d_T(S) + \sum_{u,v \in S, uv \notin E(T)} d_T(S)
\]

\[= q(n - 3) + (p - q)(n - 2) + p(n - p - 1)(n - 2).\]

From the above argument, we have

\[
SW_{n-2}(T) = \binom{p}{2}(n - 3) + \binom{n-p}{2}(n - 1) + q(n - 3) + (p - q)(n - 2) + p(n - p - 1)(n - 2)
\]

\[= \frac{1}{2} (n^3 - 2n^2 + n - 2np + 2p - 2q). \]

Li et al. obtained the following sharp lower and upper bounds of \(SW_k(T)\) for a tree \(T\).

**Lemma 3.2** [23]. Let \(T\) be a tree of order \(n\), and let \(k\) be an integer such that \(2 \leq k \leq n\). Then

\[
\binom{n-1}{k-1}(n - 1) \leq SW_k(T) \leq (k - 1) \binom{n+1}{k+1}.
\]

Moreover, among all trees of order \(n\), the star \(S_n\) minimizes the Steiner Wiener \(k\)-index, whereas the path \(P_n\) maximizes the Steiner Wiener \(k\)-index.

For trees, we have the following result.

**Theorem 3.3.** For a positive integer \(w\), there exists a tree \(T\) of order \(n\) (\(n \geq 5\)), possessing \(p\) pendant vertices, such that \(SW_{n-2}(T) = w\) if and only if \(w = \frac{1}{2} (n^3 - 2n^2 + n - 2np + 2p - 2q)\), where \(q\) is the number of vertices of degree \(2\) in \(T\) that are adjacent to a pendant vertex, and one of the following holds:

1. \(2 \leq q \leq \left\lfloor \frac{n-1}{2} \right\rfloor\) and \(q \leq p \leq n - q - 1\);
2. \(q = 1\) and \(3 \leq p \leq n - 2\);
3. \(q = 0\) and \(4 \leq p \leq n - 1\).

**Proof.** Suppose that \(w = \frac{1}{2} (n^3 - 2n^2 + n - 2np + 2p - 2q)\), where \(0 \leq q \leq \left\lfloor \frac{n-1}{2} \right\rfloor\), \(q \leq p \leq n - q - 1\). Let \(K_{1,p-1}\) be a star of order \(p\), and let \(v\) be the center of \(K_{1,p-1}\). Then \(K_{1,p-1}^*\) is a graph obtained from \(K_{1,p-1}\) by picking up \(q - 1\) edges and then replacing each of them by a path of length \(2\). Note that \(K_{1,p-1}^*\) is a
subdivision of $K_{1,p-1}$. Let $G$ be a graph obtained by $K_{1,p-1}^*$ and a path $P_{n-p-q+2}$ by identifying $v$ and one endvertex of the path. Clearly, $G$ is a tree of order $n$ with $p$ pendant vertices, and there are exactly $q$ vertices of degree 2 in $T$ such that each of them is adjacent to a pendant vertex. From Lemma 3.1, we have $SW_{n-2}(T) = \frac{1}{2}(n^3 - 2n^2 + n - 2np + 2p - 2q) = w$, as desired.

Conversely, for any tree $T$ of order $n$ ($n \geq 5$) with $p$ pendant vertices, from Lemma 3.1, $SW_{n-2}(T) = \frac{1}{2}(n^3 - 2n^2 + n - 2np + 2p - 2q)$. We now show that $p,q$ satisfy one of (1), (2), (3). Clearly, $p \geq 2$, $0 \leq q \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and $q \leq p$.

**Claim 1.** $p + q \leq n - 1$.

**Proof.** Assume, to the contrary, that $p + q = n$. Then $T$ is a path of order $n$. Since $n \geq 5$, it follows that there exists a vertex of degree 2 having no adjacent pendant vertex, which contradicts to $p + q = n$. \hfill \Box

If $q \geq 2$, then it follows from Claim 1 and $q \leq p$ that $q \leq p \leq n - q - 1$. If $q = 1$, then it follows from Claim 1 that $2 \leq p \leq n - 2$. Furthermore, if $p = 2$, then $T$ is a path of order $n$. Since $n \geq 5$, it follows that $q = 2$, a contradiction. If $q = 0$, then it follows from Claim 1 that $2 \leq p \leq n - 1$. Furthermore, if $p = 2$, then $T$ is a path of order $n$. Since $n \geq 5$, it follows that $q = 2$, a contradiction. If $p = 3$, then $T$ is a tree of order $n$. Since $n \geq 5$, it follows that $q \geq 1$, a contradiction. \hfill \Box

4. The Case for General $k$

For trees, we have the following result.

**Theorem 4.1.** Let $T$ be a graph obtained from a path $P_t$ and a star $S_{n-t+1}$ by identifying a pendant vertex of $P_t$ and the center $v$ of $S_{n-t+1}$, where $1 \leq t \leq n-1$ and $k \leq n$. Then

$$SW_k(T) = t \binom{n-1}{k} - \binom{t}{k+1} - \binom{n}{k+1} + \binom{n-t+1}{k+1} + (k-1)\binom{n}{k}.$$  

**Proof.** For any $S \subseteq V(T)$ and $|S| = k$, if $S \subseteq V(S_{n-t+1}) - v$, then $d_G(S) = k$. There are $\binom{n-t}{k}$ such subsets, contributing to $SW_k$ by $k\binom{n-t}{k}$. If $S \subset V(P_t)$, then it contributes to $SW_k$ by $(k-1)\binom{t+1}{k+1}$ from Lemma 3.2. Suppose that $S \cap V(P_t) \neq \emptyset$ and $S \cap (V(S_{n-t+1}) - v) \neq \emptyset$. Let $|S \cap V(S_{n-t+1}) - v| = i, |S \cap V(P_t)| = k-i$ and $P_t = u_1u_2\cdots u_t$, where $v = u_1$. Without loss of generality, let $S \cap V(P_t) = \{u_{j_1}, u_{j_2}, \ldots, u_{j_{k-i}}\}$, where $1 \leq j_1 < j_2 < \cdots < j_{k-i} \leq t$. Then $k-i \leq j_{k-i} \leq t$. Let $j_{k-i} = j$. Then $d_G(S) = i + j - 1$, and $k-i \leq j \leq t$. Once the vertex $u_j$ is chosen, we have $\binom{j-2}{k-i-1}$ ways to choose $u_{j_1}, u_{j_2}, \ldots, u_{j_{k-i-1}}$. In
this case, we contribute to $SW_k$ by

$$X = \sum_{i=1}^{k-1} \left( \begin{array}{c} n-t \\ i \end{array} \right) \left[ \sum_{j=k-i}^{t} \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right) (j+i-1) \right].$$

Since

$$\left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right) (j+i-1) = \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right) j + \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right) (i-1)$$

$$= (k-i) \left( \begin{array}{c} j \\ k-i \end{array} \right) + (i-1) \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right),$$

it follows that

$$\sum_{j=k-i}^{t} \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right) (j+i-1) = (k-i) \sum_{j=k-i}^{t} \left( \begin{array}{c} j \\ k-i \end{array} \right) + (i-1) \sum_{j=k-i}^{t} \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right)$$

$$= (k-i) \left( \begin{array}{c} t+1 \\ k-i+1 \end{array} \right) + (i-1) \left( \begin{array}{c} t \\ k-i \end{array} \right),$$

and hence

$$X = \sum_{i=1}^{k-1} \left( \begin{array}{c} n-t \\ i \end{array} \right) \left[ \sum_{j=k-i}^{t} \left( \begin{array}{c} j-1 \\ k-i-1 \end{array} \right) (j+i-1) \right]$$

$$= \sum_{i=1}^{k-1} \left( \begin{array}{c} n-t \\ i \end{array} \right) \left[ (k-i) \left( \begin{array}{c} t+1 \\ k-i+1 \end{array} \right) + (i-1) \left( \begin{array}{c} t \\ k-i \end{array} \right) \right]$$

$$= \sum_{i=1}^{k-1} \left( \begin{array}{c} n-t \\ i \end{array} \right) (k-i) \left( \begin{array}{c} t+1 \\ k-i+1 \end{array} \right) + \sum_{i=1}^{k-1} \left( \begin{array}{c} n-t \\ i \end{array} \right) (i-1) \left( \begin{array}{c} t \\ k-i \end{array} \right)$$

$$= \sum_{i=1}^{k-1} (k-i) \left( \begin{array}{c} t \\ k-i+1 \end{array} \right) \left( \begin{array}{c} n-t \\ i \end{array} \right) + \sum_{i=1}^{k-1} (k-i) \left( \begin{array}{c} t \\ k-i \end{array} \right) \left( \begin{array}{c} n-t \\ i \end{array} \right)$$

$$+ \sum_{i=1}^{k-1} (i-1) \left( \begin{array}{c} t \\ k-i \end{array} \right) \left( \begin{array}{c} n-t \\ i \end{array} \right).$$
\[
\begin{align*}
&= \sum_{i=1}^{k-1} (k-i) \binom{t}{k-i+1} \binom{n-t}{i} + (k-1) \sum_{i=1}^{k-1} \binom{t}{k-i} \binom{n-t}{i} \\
&= \sum_{i=1}^{k-1} (k-i) \binom{t}{k-i+1} \binom{n-t}{i} + (k-1) \left[ \binom{n}{k} - \binom{t}{k} - \binom{n-t}{k} \right].
\end{align*}
\]

Let
\[
Y = \sum_{i=1}^{k-1} (k-i) \binom{t}{k-i+1} \binom{n-t}{i}.
\]

Then
\[
\begin{align*}
Y &= \sum_{i=1}^{k-1} (k-i+1) \binom{t}{k-i+1} \binom{n-t}{i} - \sum_{i=1}^{k-1} \binom{t}{k-i+1} \binom{n-t}{i} \\
&= t \sum_{i=1}^{k-1} \binom{t-1}{k-i} \binom{n-t}{i} - \sum_{i=1}^{k-1} \binom{t}{k+1-i} \binom{n-t}{i} \\
&= t \left[ \binom{n-1}{k} - \binom{t-1}{k} - \binom{n-t}{k} \right] \\
&\quad - \left[ \binom{n}{k+1} - \binom{t}{k+1} - t \binom{n-t}{k+1} - \binom{n-t}{k+1} \right],
\end{align*}
\]

and hence
\[
SW_k(T) = (k-1) \binom{t+1}{k+1} + k \binom{n-t}{k} + X
\]
\[
= (k-1) \binom{t+1}{k+1} + k \binom{n-t}{k} + Y + (k-1) \left[ \binom{n}{k} - \binom{t}{k} - \binom{n-t}{k} \right]
\]
\[
= (k-1) \binom{t+1}{k+1} + k \binom{n-t}{k} + t \left[ \binom{n-1}{k} - \binom{t-1}{k} - \binom{n-t}{k} \right] \\
&\quad - \left[ \binom{n}{k+1} - \binom{t}{k+1} - t \binom{n-t}{k+1} - \binom{n-t}{k+1} \right] \\
&\quad + (k-1) \left[ \binom{n}{k} - \binom{t}{k} - \binom{n-t}{k} \right].
\]
Inverse Problem on the Steiner Wiener Index

\[
\begin{align*}
= (k - 1) \left( \frac{t}{k + 1} \right) + (k - 1) \left( \frac{t}{k} \right) + k \left( \frac{n - t}{k} \right) + t \left( \frac{n - 1}{k} \right) - t \left( \frac{t - 1}{k} \right) \\
- t \left( \frac{n - t}{k} \right) - \left( \frac{n}{k + 1} \right) + \left( \frac{t}{k + 1} \right) + t \left( \frac{n - t}{k} \right) + \left( \frac{n - t}{k + 1} \right) \\
+ (k - 1) \left( \frac{n}{k} \right) - (k - 1) \left( \frac{t}{k + 1} \right) - (k - 1) \left( \frac{n - t}{k} \right)
\end{align*}
\]

\[
= (k - 1) \left( \frac{t}{k + 1} \right) + k \left( \frac{n - t}{k} \right) + t \left( \frac{n - 1}{k} \right) - t \left( \frac{t - 1}{k} \right) \\
- \left( \frac{n}{k + 1} \right) + \left( \frac{t}{k + 1} \right) + \left( \frac{n - t}{k + 1} \right) + (k - 1) \left( \frac{n}{k} \right) \\
= k \left( \frac{t}{k + 1} \right) + \left( \frac{n - t}{k} \right) + t \left( \frac{n - 1}{k} \right) - \left( \frac{t}{k + 1} \right) - \left( \frac{n}{k + 1} \right) + \left( \frac{n - t}{k + 1} \right) + (k - 1) \left( \frac{n}{k} \right)
\]

The following corollary is immediate from Theorem 4.1.

**Corollary 4.2.** For a positive integer \( w \), there exists a tree \( T \) of order \( n \) such that

\[
SW_k(T) = w
\]

if

\[
w = t \left( \frac{n - 1}{k} \right) - \left( \frac{t}{k + 1} \right) - \left( \frac{n}{k + 1} \right) + \left( \frac{n - t + 1}{k + 1} \right) + (k - 1) \left( \frac{n}{k} \right),
\]

where \( 1 \leq t \leq n - 1 \) and \( k \leq n \).

For general graphs, we have the following.

**Theorem 4.3.** Let \( G \) be a graph obtained from a clique \( K_{n-r} \) and a star \( S_{r+1} \) by identifying a vertex of \( K_{n-r} \) and the center \( v \) of \( S_{r+1} \). For \( k \leq r \leq n - 1 - k \),

\[
SW_k(G) = (n - 1) \left( \frac{n - 1}{k - 1} \right) - \left( \frac{n - r - 1}{k} \right).
\]
Proof. For any $S \subseteq V(G)$ and $|S| = k$, if $S \subseteq V(K_{n-r})$, then $d_G(S) = k - 1$. There are $\binom{n-r}{k}$ such subsets, contributing to $SW_k$ by $(k-1)^{\binom{n-r}{k}}$. If $S \subseteq V(S_{r+1}) - v$, then $d_G(S) = k$. There are $\binom{k}{r}$ such subsets, contributing to $SW_k$ by $k^{\binom{k}{r}}$. Suppose that $S \cap V(K_{n-r}) \neq \emptyset$ and $S \cap (V(S_{r+1}) - v) \neq \emptyset$. If $v \in S$, then $d_G(S) = k - 1$. There are $\binom{n-r}{k-r-1}\binom{k}{r}$ such subsets, contributing to $SW_k$ by $(k-1)\sum_{x=1}^{k-1}\binom{n-r-1}{k-x-1}\binom{r}{x}$. If $v \notin S$, then $d_G(S) = k$. There are $\binom{n-r}{k-x}\binom{r}{x}$ such subsets, contributing to $SW_k$ by $k\sum_{x=1}^{k-1}\binom{n-r-1}{k-x}\binom{r}{x}$. Then

$$SW_k(G) = (k-1)^{\binom{n-r}{k}} + k^{\binom{k}{r}} + (k-1)\sum_{x=1}^{k-1}(n-r-1)\binom{r}{x}$$

$$+ k\sum_{x=1}^{k-1}\binom{n-r-1}{k-x-1}\binom{r}{x}$$

$$= (k-1)^{\binom{n-r}{k}} + k^{\binom{k}{r}} + (k-1)\left[\binom{n-1}{k-1} - \binom{n-1-r}{k-1}\right]$$

$$+ k\left[\binom{n-1}{k} - \binom{n-1-r}{k}\right]$$

$$= (k-1)^{\binom{n-r}{k}} + (n-1)^{\binom{n-1}{k-1}} - (k-1)\binom{n-1-r}{k-1} - k\binom{n-1-r}{k}$$

$$= (n-1)^{\binom{n-1}{k-1}} + (k-1)\binom{n-r-1}{k} - k\binom{n-1-r}{k}$$

$$= (n-1)^{\binom{n-1}{k-1}} - \binom{n-1-r}{k},$$

as desired.

The following corollary is immediate from Theorems 4.1 and 4.3.

**Corollary 4.4.** For a positive integer $w$, there exists a connected graph $G$ of order $n$ such that $SW_k(G) = w$ if $w$ satisfies one of the following conditions.
Inverse Problem on the Steiner Wiener Index

(1) \[ w = t \binom{n-1}{k} - \binom{t}{k+1} - \binom{n}{k+1} + \binom{n-t+1}{k+1} + (k-1) \binom{n}{k}, \text{ where } 1 \leq t \leq n-1 \text{ and } \ k \leq n. \]

(2) \[ w = (n-1)\binom{n-1}{k-1} - \binom{n-r-1}{k}, \text{ where } k \leq r \leq n-1-k \text{ and } k \leq n. \]

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References


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