BOUNDS ON THE LOCATING ROMAN DOMINATION NUMBER IN TREES

NADER JAFARI RAD AND HADI RAHBANI

Department of Mathematics
Shahrood University of Technology
Shahrood, Iran

e-mail: n.jafarirad@gmail.com

Abstract

A Roman dominating function (or just RDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an RDF $f$ is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF $f$ can be represented as $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. An RDF $f = (V_0, V_1, V_2)$ is called a locating Roman dominating function (or just LRDF) if $N(u) \cap V_2 \neq N(v) \cap V_2$ for any pair $u, v$ of distinct vertices of $V_0$. The locating Roman domination number $\gamma_{LR}(G)$ is the minimum weight of an LRDF of $G$. In this paper, we study the locating Roman domination number in trees. We obtain lower and upper bounds for the locating Roman domination number of a tree in terms of its order and the number of leaves and support vertices, and characterize trees achieving equality for the bounds.

Keywords: Roman domination number, locating domination number, locating Roman domination number, tree.

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1. Introduction

In this paper, we continue the study of a variant of Roman dominating functions, namely, locating Roman dominating functions introduced in [16]. We first present some necessary definitions and notations. For notation and graph theory terminology not given here, we follow [13]. We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of a graph $G$ is called the order of $G$ and is denoted by $n = n(G)$. The
open neighborhood of a vertex \( v \in V \) is \( N(v) = NG(v) = \{ u \in V : uv \in E \} \), and the degree of \( v \), denoted by \( \deg_G(v) \), is the cardinality of its open neighborhood. A leaf of a tree \( T \) is a vertex of degree one, while a support vertex of \( T \) is a vertex adjacent to a leaf. A strong support vertex is a support vertex adjacent to at least two leaves. In this paper, we denote the set of all strong support vertices of \( T \) by \( S(T) \) and the set of leaves by \( L(T) \). We denote \( t(T) = |L(T)| \) and \( s(T) = |S(T)| \). We also denote by \( L(x) \) the set of leaves adjacent to a support vertex \( x \), and denote \( \ell_x = |L(x)| \). If \( T \) is a rooted tree then for any vertex \( v \) we denote by \( T_v \) the subtree rooted at \( v \). A subset \( S \subseteq V \) is a dominating set if every vertex in \( V - S \) has a neighbor in \( S \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \).

The study of locating dominating sets in graphs was pioneered by Slater [21, 22]. For many problems related to graphs, various types of protection sets are studied where the objective is to precisely locate an “intruder”. It is considered that a detection device at a vertex \( v \) is able to determine if the intruder is at \( v \) or if it is in \( N(v) \), but at which vertex in \( N(v) \), it cannot be determined. A locating-dominating set \( D \subseteq V(G) \) is a dominating set with the property that for each vertex \( x \in V(G) - D \) the set \( N(x) \cap D \) is unique. That is, any two vertices \( x, y \) in \( V(G) - D \) are distinguished in the sense that there is a vertex \( v \in V_2 \) with \( |N(v) \cap \{x, y\}| = 1 \). The minimum size of a locating-dominating set for a graph \( G \) is the locating-domination number of \( G \), denoted \( \gamma_L(G) \). The concept of locating domination has been considered for several domination parameters, see for example [4, 5, 6, 8, 9, 11, 12, 14, 15, 18, 23].

For a graph \( G \), let \( f : V(G) \rightarrow \{0, 1, 2\} \) be a function, and let \( (V_0, V_1, V_2) \) be the ordered partition of \( V(G) \) induced by \( f \), where \( V_i = \{ v \in V(G) : f(v) = i \} \) for \( i = 0, 1, 2 \). There is a 1–1 correspondence between the functions \( f : V(G) \rightarrow \{0, 1, 2\} \) and the ordered partitions \( (V_0, V_1, V_2) \) of \( V(G) \). So we will write \( f = (V_0, V_1, V_2) \). A function \( f : V(G) \rightarrow \{0, 1, 2\} \) is a Roman dominating function (or just RDF) if every vertex \( u \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of an RDF \( f \) is \( w(f) = f(V(G)) = \sum_{u \in V(G)} f(u) \). The Roman domination number of a graph \( G \), denoted by \( \gamma_R(G) \), is the minimum weight of an RDF on \( G \). A function \( f = (V_0, V_1, V_2) \) is called a \( \gamma_R \)-function (or \( \gamma_R(G) \)-function when we want to refer \( f \) to \( G \)), if it is an RDF and \( f(V(G)) = \gamma_R(G) \), see [10, 19, 24].

Roman dominating functions with several further conditions have been studied, for example, among other types, see for example [1, 2, 3, 7, 17, 20].

It is known [10] that if \( f = (V_0, V_1, V_2) \) is an RDF in a graph \( G \) then \( V_2 \) is a dominating set for \( G[V_0 \cup V_2] \). Jafari Rad, Rahbani and Volkmann [16] considered Roman dominating functions \( f = (V_0, V_1, V_2) \) with a further condition that for each vertex \( x \in V_0 \), the set \( N(x) \cap V_2 \) is unique. That is, any two vertices \( x, y \) in \( V_0 \) are distinguished in the sense that there is a vertex \( v \in V_2 \) with \( |N(v) \cap \{x, y\}| = 1 \).
An RDF $f = (V_0, V_1, V_2)$ is called a *locating Roman dominating function* (or just LRDF) if $N(u) \cap V_2 \neq N(u) \cap V_2$ for any pair $u, v$ of distinct vertices of $V_0$. The *locating Roman domination number* $\gamma^L_R(G)$ is the minimum weight of an LRDF. Note that $\gamma^L_R(G)$ is defined for any graph $G$, since $(\emptyset, V(G), \emptyset)$ is an LRDF for $G$. We refer to a $\gamma^L_R(G)$-function as an LRDF of $G$ with minimum weight. It is shown in [16] that the decision problem for the locating Roman domination problem is NP-complete for bipartite graphs and chordal graphs. Moreover, several bounds and characterizations are given for the locating Roman domination number of a graph.

In this paper we study the locating Roman domination number in trees. In Section 2, we show that for any tree $T$ of order $n \geq 2$ with $\ell$ leaves and $s$ support vertices, $\gamma^L_R(T) \geq (2n + (\ell - s) + 2)/3$, and characterize all trees that achieve equality for this bound. In Section 3, we show that for any tree $T$ of order $n \geq 2$, with $l$ leaves and $s$ support vertices, $\gamma^L_R(T) \leq (4n + l + s)/5$, and characterize all trees that achieve equality for this bound.

If $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function, then for any vertex $v \in V_2$, we define $pn(v, V_0) = \{u \in V_0 : N(u) \cap V_2 = \{v\}\}$. The following theorem was proved in [4].

**Theorem 1** (Blidia et al. [4]). For any tree $T$ of order $n \geq 2$, $\gamma_L(T) \geq [(n + 1)/3]$.

## 2. Lower Bound

We begin with the following lemma.

**Lemma 2.** If $T$ is a tree with $\ell$ leaves and $s$ support vertices, and $f = (V_0, V_1, V_2)$ is a $\gamma^L_R(T)$-function, then $|V_1| \geq \ell - s$.

**Proof.** For any support vertex $x$, $|L(x) \cap V_1| \geq \ell_x - 1$, thus $|V_1| \geq \sum_{x \in S} (\ell_x - 1) = \sum_{x \in S} \ell_x - \sum_{x \in S} 1 = \ell - s$. \hfill \blacksquare

**Theorem 3.** For any tree $T$ of order $n \geq 2$ with $\ell$ leaves and $s$ support vertices, $\gamma^L_R(T) \geq (2n + (\ell - s) + 2)/3$.

**Proof.** Let $T$ be a tree of order $n$, and $f = (V_0, V_1, V_2)$ be a $\gamma^L_R(T)$-function. Let $T_1, T_2, \ldots, T_k$ be the components of $T[V_0 \cup V_2]$, and let $|V(T_i)| = n_i$ for $i = 1, 2, \ldots, k$. Let $D_i = V_2 \cap V(T_i)$ for $i = 1, 2, \ldots, k$. Clearly, $D_i$ is a LDS for $T_i$, and so $\gamma_L(T_i) \leq |D_i|$, for $i = 1, 2, \ldots, k$. By Theorem 1, $|D_i| \geq \gamma_L(T_i) \geq (n_i + 1)/3$ for $i = 1, 2, \ldots, k$. Hence, $(n - |V_1| + k)/3 \leq \sum_{i=1}^k \gamma_L(T_i) \leq \sum_{i=1}^k |D_i| = |V_2|$. Now since $|V_1| \geq \ell - s$ by Lemma 2, we conclude that $\gamma^L_R(T) = |V_1| + 2|V_2| \geq |V_1| + (2(n - |V_1| + k))/3 \geq (2n + |V_1| + 2k)/3 \geq (2n + (\ell - s) + 2)/3$. \hfill \blacksquare
Corollary 4. For any tree $T$ of order $n \geq 2$, $\gamma^L_R(T) \geq (2n + 2)/3$.

We next aim to characterize trees achieving equality in the bound of Theorem 3. For this purpose for each integer $r \geq 0$, we construct a family $\mathcal{T}_r$ of trees as follows.

- Let $\mathcal{T}_0$ be the collection of trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$ ($k \geq 1$) of trees, where $T_1 = P_3$, and $T_{i+1}$ can be obtained recursively from $T_i$ by the following operation for $1 \leq i \leq k - 1$.

**Operation $O_i$.** Join a support vertex of $T_i$ to a leaf of a path $P_3$.

- For $r \geq 1$, let $\mathcal{T}_r$ be the class of trees $T$ that can be obtained from a tree $T_0 \in \mathcal{T}_0$ by adding $r$ leaves to at most $r$ support vertices of $T_0$.

The following lemma plays a key role for the next section.

Lemma 5. Let $T$ be a tree of order $n \geq 3$ with $\gamma^L_R(T) = (2n + 2)/3$. Then

1. $|V_1| = 0$ for every $\gamma^L_R(T)$-function $f = (V_0, V_1, V_2)$.
2. $T$ has no strong support vertex.
3. If $P = x_0 - x_1 - \cdots - x_d$ is a diametrical path of $T$, then $\deg(x_{d-1}) = \deg(x_{d-2}) = 2$, and $x_{d-3}$ is a support vertex.
4. If $P = x_0 - x_1 - \cdots - x_d$ is a diametrical path of $T$, and $T' = T - \{x_{d-1}, x_{d-2}\}$, then $\gamma^L_R(T') = (2|V(T')| + 2)/3$.

**Proof.**

1. Suppose that $f = (V_0, V_1, V_2)$ is a $\gamma^L_R(T)$-function such that $|V_1| > 0$. Let $v \in V_1$. If $v$ is a leaf then by Corollary 4, we have $2n/3 \leq \gamma^L_R(T - v) \leq w(f) - 1 = (2n - 1)/3$, a contradiction. Thus $v$ is not a leaf. Let $T_1, T_2, \ldots, T_k$ ($k \geq 2$) be the components of $T - \{v\}$, and $|V(T_i)| = n_i$ for $i = 1, \ldots, k$. For $i = 1, \ldots, k$, since $f|_{V(T_i)}$ is an LRDF for $T_i$ by Corollary 4 we obtain that

$$\frac{2n+2}{3} \leq \sum_{i=1}^k \frac{2n_i+2}{3} \leq \sum_{i=1}^k \gamma^L_R(T_i) \leq w(f) - 1 = (2n - 1)/3,$$

a contradiction.

2. The result follows from Lemma 2 and part (1).

3. By part (2), $\deg(x_{d-1}) = 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma^L_R(T)$-function. Moreover, by parts (1) and (2) we may assume that $f(u) = 0$ for any leaf $u$, and $f(u) = 2$ for any support vertex $u$. Assume that $\deg(x_{d-2}) \geq 3$. If $x_{d-2}$ is a support vertex then replacing $f(x_d)$ and $f(x_{d-1})$ by 1 yields a $\gamma^L_R(T)$-function, a contradiction to part (1). Thus $x_{d-2}$ is not a support vertex. Then any vertex of $N(x_{d-2}) - \{x_{d-3}\}$ is a support vertex of degree two. If $\deg(x_{d-2}) \geq 4$ then replacing $f(x_d)$ and $f(x_{d-1})$ by 1 yields an LRDF for $T$, a contradiction to part (1). Assume that $\deg(x_{d-2}) = 3$. Observe that $f(x_{d-2}) = 0$. Let $T'$ be the component of $T - x_{d-2} - x_{d-3}$ that contains $x_{d-3}$. By Corollary 4, $\gamma^L_R(T') \geq (2(n - 5) + 2)/3$. But $f|_{V(T')} = \gamma^L_R(T) - 4 = (2n + 2)/3 - 4$, a contradiction. Thus $\deg(x_{d-2}) = 2$. Since $f(x_{d-1}) = 2$, from part (1) we obtain that $f(x_{d-2}) = 0$, and thus $f(x_{d-3}) = 2$. 


Suppose now that $x_{d-3}$ is not a support vertex. Assume that $\deg(x_{d-3}) = 2$. Clearly, we may assume that $f(x_{d-4}) = 0$, since otherwise replacing $f(x_d)$ and $f(x_{d-1})$ by $1$ yields an $\gamma^L_R(T)$-function, a contradiction. By the same reason, we obtain that $N(x_{d-4}) \cap V_2 = \{x_{d-3}\}$. So $x_{d-4}$ is neither a support vertex nor adjacent to a support vertex. Let $T_0, T_1, T_2, \ldots, T_l$ be the components of $T - x_{d-4}$, where $T_0$ contains $x_{d-3}$. Clearly, $f|_{V(T_i)}$ is an LRDF for $T_i$, and by Corollary 4, $w(f|_{V(T_i)}) \geq \gamma^L_R(T_i) \geq (2|V(T_i)| + 2)/3$ for $i = 1, 2, \ldots, l$. Thus

$$
\frac{(2n - 8)}{3} \leq \frac{(2(n - 5) + 2l)}{3} = \sum_{i=1}^{l} (2|V(T_i)| + 2)/3 \leq \sum_{i=1}^{l} \gamma^L_R(T_i)
$$

$$
\leq \sum_{i=1}^{l} w(f|_{V(T_i)}) = w(f) - 4 = (2n + 2)/3 - 4 = (2n - 10)/3
$$

a contradiction. Thus $\deg(x_{d-3}) \geq 3$. Let $a_1$ be a leaf of $T$ such that the $d(x_{d-3}, a_1)$ is minimum and the shortest path from $a_1$ to $x_{d-3}$ does not intersect $P$. Clearly, $d(x_{d-3}, a_1) \in \{2, 3\}$. Assume that $d(x_{d-3}, a_1) = 2$. Let $b_1 \in N(a_1) \cap N(x_{d-3})$. Thus $\deg(b_1) = 2$ by part (2). Then $f(b_1) = 2$, and so replacing $f(a_1)$ and $f(b_1)$ by $1$ yields a $\gamma^L_R(T)$-function, a contradiction. Thus $d(x_{d-3}, a_1) = 3$. Therefore, any vertex of $N(x_{d-3}) - \{x_{d-4}\}$ has degree two and is adjacent to a support vertex of degree two. Let $N(x_{d-3}) - \{x_{d-4}, x_{d-2}\} = \{c_1, \ldots, c_k\}$, where $k = \deg(x_{d-3}) - 2$. Then $c_i$ is adjacent to a support vertex $b_i$ with $\deg(b_i) = 2$, for $i = 1, 2, \ldots, k$. Let $a_i$ be the leaf adjacent to $b_i$ for $i = 1, 2, \ldots, k$. Then $f(b_i) = 2$ and $f(a_i) = f(c_i) = 0$ for $i = 1, 2, \ldots, k$. Note that we may assume that $f(x_{d-4}) = 0$, since otherwise replacing $f(x_{d-1})$ and $f(x_d)$ by $1$ yields a $\gamma^L_R(T)$-function, a contradiction. Thus $x_{d-4}$ is neither a support vertex nor adjacent to a support vertex. By the same reason, $N(x_{d-4}) \cap V_2 = \{x_{d-3}\}$. Let $T_0, T_1, T_2, \ldots, T_l$ be the components of $T - x_{d-4}$, where $T_0$ contains $x_{d-3}$. Clearly, $f|_{V(T_i)}$ is an LRDF for $T_i$, and by Corollary 4, $w(f|_{V(T_i)}) \geq \gamma^L_R(T_i) \geq (2|V(T_i)| + 2)/3$ for $i = 1, 2, \ldots, l$. Thus

$$
\frac{(2n - 6k - 8)}{3} \leq \frac{2/3 + 2/3(n - 3k - 5)}{3} \leq \sum_{i=1}^{l} \frac{(2|V(T_i)| + 2)}{3} \leq \sum_{i=1}^{l} w(f|_{V(T_i)}) = w(f) - 2(k + 1) - 2
$$

$$
= (2n + 2)/3 - 2k - 4 = (2n - 6k - 10)/3
$$

a contradiction.

(4) By part (3), $\deg(x_{d-1}) = \deg(x_{d-2}) = 2$ and $x_{d-3}$ is a support vertex.

Let $f = (V_0, V_1, V_2)$ be a $\gamma^L_R(T)$-function. As seen earlier, $|V_1| = 0$, $f(x_d) = f(x_{d-2}) = 0$ and $f(x_{d-1}) = 2$. Therefore, $f|_{T'}$ is an LRDF for $T'$. By Corollary 4, $(2|V(T')| + 2)/3 \leq \gamma^L_R(T') \leq w(f|_{T'}) = \gamma^L_R(T) - 2 = (2n + 2)/3 - 2 = (2|V(T')| + 2)/3$. Therefore, $\gamma^L_R(T') = (2|V(T')| + 2)/3$. 

\[\blacksquare\]
We are now ready to characterize trees achieving equality in the bound of Theorem 3.

**Theorem 6.** For a tree $T$ of order $n \geq 2$ with $\ell$ leaves and $s$ support vertices, $\gamma^f_R(T) = (2n + (\ell - s) + 2)/3$ if and only if $T = K_2$ or $T \in T_k$ for some integer $k \geq 0$.

**Proof.** Let $T \neq K_2$ be a tree of order $n$ with $\ell$ leaves and $s$ support vertices. We proceed with two claims.

**Claim 1.** $\gamma^f_R(T) = (2n + 2)/3$ if and only if $T \in T_0$.

**Proof.** Assume that $\gamma^f_R(T) = (2n + 2)/3$. We show by induction on $n$ that $T \in T_0$. For the base step of the induction it is easy to see that $P_4$ is the smallest tree $T$ for which $\gamma^f_R(T) = (2n + 2)/3$. Assume that any tree $T'$ of order $5 < n' < n$ and such that $\gamma^f_R(T') = (2n' + 2)/3$ belongs to $T_0$. Let $P = x_0 - x_1 - \cdots - x_d$ be a diametrical path of $T$. By Lemma 5(3), $\text{deg}(x_d-1) = \text{deg}(x_d-2) = 2$, and $x_d-3$ is a support vertex. Let $T_1 = T - \{x_d, x_{d-1}, x_{d-2}\}$. By Lemma 5(4), $\gamma^f_R(T_1) = (2|V(T_1)| + 2)/3$. By the inductive hypothesis, $T_1 \in T_0$. Hence $T$ is obtained from $T_1$ by Operation $O_1$, and thus $T \in T_0$. For the converse it is sufficient to show that if $\gamma^f_R(T_i) = (2|V(T_i)| + 2)/3$ and $T_{i+1}$ is obtained from $T_i$ by the operation $O_1$, then $\gamma^f_R(T_{i+1}) = (2|V(T_{i+1})| + 2)/3$, and then the result follows by an induction on the number of operations performed to construct a tree $T \in T_0$. Let $\gamma^f_R(T_i) = (2|V(T_i)| + 2)/3$, and $T_{i+1}$ be obtained from $T_i$ by joining a support vertex $v \in V(T_i)$ to the leaf $x$ of a path $P_3 : xyz$. Let $f$ be a $\gamma^f_R(T_i)$-function. By Lemma 5(1) and (2), we may assume that $f(v) = 2$. Then $g : V(T_{i+1}) \rightarrow \{0, 1, 2\}$ defined by $g(x) = g(z) = 0, g(y) = 2$ and $g(u) = f(u)$ for any $u \in V(T_i)$, is an LRDF for $T_{i+1}$. By Corollary 4, $(2|V(T_{i+1})| + 2)/3 \leq \gamma^f_R(T_{i+1}) \leq w(g) = \gamma^f_R(T) + 2 = (2|V(T_i)| + 2)/3 + 2 = (2|V(T_{i+1})| + 2)/3$. Therefore, $\gamma^f_R(T_{i+1}) = (2|V(T_{i+1})| + 2)/3$. □

**Claim 2.** $\gamma^f_R(T) = (2n + (\ell - s) + 2)/3$, with $\ell \neq s$, if and only if $T \in T_k$ for some integer $k \geq 1$.

**Proof.** Assume that $\gamma^f_R(T) = (2n + (\ell - s) + 2)/3$, and $\ell \neq s$. Let $f = (V_0, V_1, V_2)$ be a $\gamma^f_R(T)$-function. For any support vertex $x$, $f(u) = 1$ for at least $\ell_x - 1$ leaves $u \in N(x)$ by Lemma 2. Let $T'$ be a tree obtained from $T$ by removing $\ell_x - 1$ leaves $u$ of any strong support vertex $x$ with $f(u) = 1$. Then $f|_{T'}$ is a LRDF for $T'$, and so $\gamma^f_R(T') \leq \gamma^f_R(T) - (l - s) = (2(n - (l - s) + 2))/3 = (2|V(T')| + 2)/3$. Corollary 4 implies that $\gamma^f_R(T') = (2|V(T')| + 2)/3$. Now Claim 1 implies that $T' \in T_0$, and so $T \in T_k$, where $k = l - s$. Conversely, let $T \in T_k$ for some integer $k \geq 1$. Thus $T$ is obtained from a tree $T' \in T_0$ by adding $k$ leaves to at most $k$ support vertices of $T'$. By Claim 1, $\gamma^f_R(T') = (2|V(T')| + 2)/3$. Let $f'$ be a $\gamma^f_R(T')$-function. We extend $f'$ to a LRDF for $T$ by assigning 1 to any vertex of
Proof. Let \( T \) be obtained from a tree \( T' \) by joining a leaf \( v \) of \( T' \) to the leaf \( a \) of a path \( P_5 : abcd \). If \( f = (V_0, V_1, V_2) \) is a \( \gamma_R^L(T') \)-function, then \( g = (V_0 \cup \{a, c, e\}, V_1, V_2 \cup \{b, d\}) \) is an LRDF for \( T \), and so \( \gamma_R^L(T) \leq \gamma_R^L(T') + 4 \). Let \( h = (V_0, V_1, V_2) \) be a \( \gamma_R^L(T) \)-function. If \( a \not\in V_2 \), then \( h(a) + h(b) + h(c) + h(d) + h(e) = 4 \) and \( h_{\gamma R}^V(T') \) is an LRDF for \( T' \), so \( \gamma_R^L(T') \leq \gamma_R^L(T) - 4 \). If \( a \in V_2 \), then \( h(a) + h(b) + h(c) + h(d) + h(e) = 5 \), so \( \gamma_R^L(T') \leq w(h_{\gamma R}^V(T')) + 1 = \gamma_R^L(T) - 4 \). Thus \( \gamma_R^L(T) = \gamma_R^L(T') + 4 \).

Similarly the following is verified.

Lemma 8. Let \( T' \) be a tree with a vertex \( w \) of degree at least two and \( \gamma_R^L(T' - w) \geq \gamma_R^L(T') \). If \( T \) is obtained from \( T' \) by joining \( w \) to the center of a path \( P_3 \), then \( \gamma_R^L(T) = \gamma_R^L(T') + 8 \).

Theorem 9. For any tree \( T \) of order \( n \geq 2 \), with \( \ell \) leaves and \( s \) support vertices, \( \gamma_R^L(T) \leq (4n + \ell + s)/5 \).

Proof. We use an induction on the order \( n = n(T) \) of a tree \( T \). The base step is obvious for \( n \leq 4 \). Assume that for any tree \( T' \) of order \( n' < n \), with \( \ell' \) leaves and \( s' \) support vertices, \( \gamma_R^L(T') \leq (4n' + \ell' + s')/5 \). Now consider the tree \( T \) of order \( n \geq 5 \), with \( \ell \) leaves and \( s \) support vertices. Assume that \( T \) has a strong support vertex \( v \), and \( u \) is a leaf adjacent to \( v \). Let \( T' = T - u \). Clearly, \( \gamma_R^L(T) \leq \gamma_R^L(T') + 1 \). By the induction hypothesis, \( \gamma_R^L(T) \leq \gamma_R^L(T') + 1 \leq (4n' + \ell' + s')/5 + 1 = (4(n - 1) + (l - 1) + s)/5 + 1 = (4n + \ell + s)/5 \). Next assume that \( T \) has an edge \( e = uv \) with \( \text{deg}(u) \geq 3 \) and \( \text{deg}(v) \geq 3 \). Let \( T_1 \) and \( T_2 \) be the components of \( T - e \), with \( u \in V(T_1) \) and \( v \in V(T_2) \). Assume that \( T_i \) has order \( n_i \), \( \ell_i \) leaves and \( s_i \) support vertices, for \( i = 1, 2 \). By the induction hypothesis, \( \gamma_R^L(T) \leq \gamma_R^L(T_1) + \gamma_R^L(T_2) \leq (4n_1 + \ell_1 + s_1)/5 + (4n_2 + \ell_2 + s_2)/5 = (4n + \ell + s)/5 \). Thus the following claims hold.

Claim 1. \( T \) has no strong support vertex.

Claim 2. For each edge \( e = uv \), \( \text{deg}(u) \leq 2 \) or \( \text{deg}(v) \leq 2 \).
We root $T$ at a leaf $x_0$ of a diametrical path $x_0x_1 \cdots x_d$ from $x_0$ to a leaf $x_d$ farthest from $x_0$. By Claim 1, $d \geq 3$. If $d = 3$ then $T$ is a double-star, and it can be easily seen that $\gamma_{L}(T) = (4n + \ell + s)/5$. Thus assume that $d \geq 4$.

By Claim 1, $\deg(x_{d-1}) = 2$. Assume that $\deg(x_{d-2}) \geq 3$. Assume that $x_{d-2}$ is a support vertex. Let $u$ be the unique leaf adjacent to $x_{d-2}$. Let $T' = T - u$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}(T') + 1 \leq (4(n-1) + (\ell - 1) + (s - 1))/5 + 1 < (4n + \ell + s)/5$. Thus assume that $x_{d-2}$ is not a support vertex. Let $u$ be a child of $x_{d-2}$ different from $x_{d-1}$. By Claim 1, $\deg(u) = 2$. Let $v$ be the child of $u$, and $T'' = T - \{u, v\}$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}(T'') + 2 \leq (4(n-2) + (\ell - 1) + s - 1)/5 + 2 = (4n + \ell + s)/5$. We thus assume that $\deg(x_{d-2}) = 2$.

Assume that $\deg(x_{d-3}) \geq 3$. Assume that $x_{d-3}$ is a support vertex. Let $u$ be the unique leaf adjacent to $x_{d-3}$. Let $T' = T - u$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}(T') + 1 \leq (4(n-1) + (\ell - 1) + s - 1)/5 + 1 < (4n + \ell + s)/5$. Thus assume that $x_{d-3}$ is not a support vertex. Let $u$ be a child of $x_{d-3}$ different from $x_{d-2}$. Assume that $u$ is a support vertex. By Claim 1, $\deg(u) = 2$. Let $v$ be the child of $u$. Let $T'' = T - \{u, v\}$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}(T'') + 2 \leq (4(n-2) + (\ell - 1) + s - 1)/5 + 2 = (4n + \ell + s)/5$. Thus assume that $u$ is not a support vertex. Thus any child of $u$ is a support vertex of degree two by Claim 1. Furthermore, since $\deg(x_{d-3}) \geq 3$, we deduce that $d \geq 6$, and this implies that $x_{d-5} \neq x_0$. Let $\deg(x_{d-3}) = k + 1$. By Claim 2, $\deg(x_{d-4}) = 2$. Let $T'' = T - T_{x_{d-4}}$. Assume that $T''$ has $n'$ vertices, $\ell'$ leaves and $s'$ support vertices. By the inductive hypothesis, $\gamma_{L}(T'') \leq (4n' + \ell' + s')/5$. But $\ell' \leq \ell - k + 1$, $s' \leq s - k + 1$, and $n' = n - 3k - 2$. Let $f$ be a $\gamma_{L}(T''$)-function. We extend $f$ to an LRDF for $T$ by assigning 2 to $x_{d-4}$ at any vertex of $T_{x_{d-4}}$ at distance two from $x_{d-3}$, and 0 to any other vertex of $T_{x_{d-4}}$. Thus $\gamma_{L}(T) \leq \gamma_{L}(T'') + 2k + 2 \leq (4n' + \ell' + s')/5 + 2k + 2 \leq (4n + \ell + s - 4k + 4)/5 \leq (4n + \ell + s)/5$. Thus assume that $\deg(x_{d-3}) = 2$.

Assume that $\deg(x_{d-4}) \geq 3$. As before, we can assume that $x_{d-4}$ is not a support vertex, and is not adjacent to a support vertex of degree two. By Claim 2, $\deg(x_{d-5}) = 2$, and also any child of $x_{d-4}$ has degree two. If there is a leaf $u \neq x_d$ of $T_{x_{d-5}}$ at distance four from $x_{d-4}$ then any internal vertex in the path from $u$ to $x_{d-4}$ has degree two, since $u$ plays the same role of $x_d$. Thus any leaf $u$ of $T_{x_{d-4}}$ is at distance 3 or 4 from $X_{d-4}$, and any internal vertex in the path from $u$ to $x_{d-4}$ has degree two. Let $k_1$ be the number of leaves of $T_{x_{d-5}}$ at distance four from $x_{d-4}$, and $k_2$ be the number of leaves of $T_{x_{d-5}}$ at distance three from $x_{d-4}$. Then $\deg(x_{d-4}) = k_1 + k_2 + 1$. Since $\deg(x_{d-4}) \geq 3$, we obtain that $d \geq 7$, and this implies that $x_{d-6} \neq x_0$. Let $T'' = T - T_{x_{d-5}}$. Assume that $T''$ has $n'$ vertices, $\ell'$ leaves and $s'$ support vertices. By the inductive hypothesis, $\gamma_{L}(T'') \leq (4n' + \ell' + s')/5$. But $\ell' \leq \ell - k_1 - k_2 + 1$, $s' \leq s - k_1 - k_2 + 1$, and $n' = n - 4k_1 - 3k_2 - 2$. Let $f$ be a $\gamma_{L}(T'')$-function. We extend $f$ to an LRDF for $T$ by assigning 2 to $x_{d-4}$ and any vertex of $T_{x_{d-5}}$ at distance two from $x_{d-4}$,
1 to any vertex of $T_{x_{d-5}}$ at distance four from $x_{d-4}$, and 0 to any other vertex of $T_{x_{d-5}}$. Thus $\gamma^R_L(T) \leq \gamma^R_L(T') + 3k_1 + 2k_2 + 2 \leq (4n' + \ell' + s')/5 + 3k_1 + 2k_2 + 2 \leq (4n + \ell + s - 3k_1 - 4k_2 + 4)/5 < (4n + \ell + s)/5$.

Thus assume that $\deg(x_{d-4}) = 2$. Let $T' = T - T_{x_{d-5}}$. Assume that $T'$ has $n'$ vertices, $\ell'$ leaves and $s'$ support vertices. By the inductive hypothesis, $\gamma^R_L(T') \leq (4n' + \ell' + s')/5$. But $\ell' \leq \ell$, $s' \leq s$, and $n' = n - 5$. Let $f$ be a $\gamma^R_L(T')$-function. We extend $f$ to an LRDF for $T$ by assigning 2 to $x_{d-4}$ and $x_{d-1}$, and 0 to $x_{d-4}, x_{d-2}$ and $x_d$. Thus $\gamma^R_L(T) \leq \gamma^R_L(T') + 4 \leq (4n' + \ell' + s')/5 + 4 \leq (4n + \ell + s)/5$. \hfill \Box

We next aim to characterize trees achieving equality for the bound of Theorem 3. A vertex $w$ of degree at least two in a tree $T$ is called a special vertex if the following conditions hold:

1. If $f(w) = 2$ for a $\gamma^R_L(T)$-function $h = (V_0, V_1, V_2)$, then $\partial n(w, V_0) \neq \emptyset$.
2. If $f(w) = 1$ for a $\gamma^R_L(T)$-function $h = (V_0, V_1, V_2)$, then $\partial n(w) \cap V_2 = \emptyset$.

Let $T$ be the collection of trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$ ($k \geq 1$) of trees, where $T_1 = P_4$, and $T_{i+1}$ can be obtained recursively from $T_i$ by one of the following operations for $1 \leq i \leq k - 1$.

**Operation $O_1$.** Assume that $w$ is a support vertex of $T_i$. Then $T_{i+1}$ is obtained from $T_i$ by adding a leaf to $w$.

**Operation $O_2$.** Assume that $w$ is a leaf of $T_i$. Then $T_{i+1}$ is obtained from $T_i$ by joining $w$ to a leaf of a path $P_2$.

**Operation $O_3$.** Assume that $w$ is a special vertex of $T_i$. Then $T_{i+1}$ is obtained from $T_i$ by joining $w$ to a leaf of a path $P_2$.

**Operation $O_4$.** Assume that $w$ is a vertex of $T_i$ of degree at least two and $\gamma^R_L(T_i - w) \geq \gamma^R_L(T_i)$. Then $T_{i+1}$ is obtained from $T_i$ by joining $w$ to the center of a path $P_3$.

**Lemma 10.** If $\gamma^R_L(T_i) = (4n(T_i) + \ell(T_i) + s(T_i))/5$, and $T_{i+1}$ is obtained from $T_i$ by Operation $O_j$, for $j = 1, 2, 3, 4$, then $\gamma^R_L(T_{i+1}) = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$.

**Proof.** Let $\gamma^R_L(T_i) = (4n_i + \ell_i - 2 + s_i)/5$, where $n_i = n(T_i)$, $\ell_i = \ell(T_i)$ and $s_i = s(T_i)$. Assume that $T_{i+1}$ is obtained from $T_i$ by Operation $O_1$. Let $T_{i+1}$ be obtained from $T_i$ by adding a leaf $v$ to a support vertex $w$ of $T_i$. Then $\gamma^R_L(T_{i+1}) \leq \gamma^R_L(T_i) + 1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma^R_L(T_{i+1})$-function, without loss of generality, we may assume that $v \in V_1$. Then $f = (V_0, V_1 - \{v\}, V_2)$ is an LRDF for $T_i$, implying that $\gamma^R_L(T_i) \leq \gamma^R_L(T_{i+1}) - 1$. Thus $\gamma^R_L(T_{i+1}) = \gamma^R_L(T_i) + 1$. Now $\gamma^R_L(T_{i+1}) = (4n(T_i) + \ell(T_i) + s(T_i))/5 + 1 = (4(n(T_i) + 1) + (\ell(T_i) + 1) + s(T_i))/5 = (4n(T_i) + \ell(T_{i+1}) + s(T_{i+1}))/5$.

Next assume that $T_{i+1}$ is obtained from $T_i$ by Operation $O_2$. By Lemma 7, $\gamma^R_L(T_{i+1}) = \gamma^R_L(T_i) + 4$. Now $\gamma^R_L(T_{i+1}) = (4n(T_i) + \ell(T_i) + s(T_i))/5 + 4 = (4(n(T_i) + 5) + \ell(T_i) + s(T_i))/5 = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$. 


Now assume that \( T_{i+1} \) is obtained from \( T_i \) by Operation \( O_3 \). Let \( T_{i+1} \) be obtained from \( T_i \) by joining a special vertex \( v \) of \( T_i \) to the leaf \( a \) of a path \( P_2 : ab \).

Suppose that \( \gamma_R^l(T_{i+1}) = \gamma_R^l(T_i) + 1 \). Let \( h \) be a \( \gamma_R^l(T_{i+1}) \)-function. Assume that \( h(a) = 2 \). Clearly, we may assume that \( h(b) = 0 \). If \( h(v) \neq 0 \), then \( h|_{V(T_i)} \) is an LRDF for \( T_i \) of weight less than \( \gamma_R^l(T_i) \), a contradiction. Thus \( h(v) = 0 \). Since \( h \) is an LRDF for \( T_{i+1} \), there is a vertex \( w \in N(v) - \{a\} \) such that \( h(w) = 2 \).

Now \( h' \) defined on \( V(T_i) \) by \( h'(v) = 1 \) and \( h'(x) = h(x) \) otherwise, is an LRDF for \( T_i \). Clearly, \( h' \) is a \( \gamma_R^l(T_i) \)-function. This is a contradiction, since \( v \) is a special vertex of \( T_i \). If \( h(a) = 1 \), then \( h(b) = 1 \) and we can replace \( h(a) \) by 2 and \( h(b) \) by 0, and as before, get a contradiction. Thus \( h(a) = 0 \). If \( h(b) = 2 \), then we replace \( h(a) \) by 2 and \( h(b) \) by 0, and as before, get a contradiction. Thus \( h(b) = 1 \), and so \( h(v) = 2 \). Thus \( h|_{V(T_i)} = (V_0, V_1, V_2) \) is a \( \gamma_R^l(T_i) \)-function with \( \gamma(R, V_1) = 2 \). Clearly, we may assume that \( \gamma(R, V_0, V_0) = 0 \). This is a contradiction, since \( v \) is a special vertex of \( T_i \). Thus \( \gamma_R^l(T_{i+1}) = \gamma_R^l(T_i) + 2 \). Now \( \gamma_R^l(T_{i+1}) = (4(n(T_i)) + 2) + (\ell(T_i) + 1) + (s(T_i) + 1))/5 = (4n(T_i) + \ell(T_i + 1) + s(T_{i+1}))/5 \).

Finally assume that \( T_{i+1} \) is obtained from \( T_i \) by Operation \( O_4 \). By Lemma 8, \( \gamma_R^l(T_{i+1}) = \gamma_R^l(T_i) + 8 \). Now \( \gamma_R^l(T_{i+1}) = (4n(T_i) + \ell(T_i) + s(T_i))/5 + 8 = (4n(T_i) + 9) + (\ell(T_i) + 2) + (s(T_i) + 2))/5 = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5 \).

By a simple induction on the operations performed to construct a tree \( T \in \mathcal{T} \) and Lemma 10 we obtain the following.

**Lemma 11.** For any tree \( T \in \mathcal{T} \) of order \( n \geq 2 \) with \( \ell \) leaves and \( s \) support vertices, \( \gamma_R^l(T) = (4n + \ell + s)/5 \).

**Theorem 12.** For a tree \( T \) of order \( n \geq 2 \) with \( \ell \) leaves and \( s \) support vertices, \( \gamma_R^l(T) = (4n + \ell + s)/5 \) if and only if \( T = K_{1,n-1} \) or \( T \in \mathcal{T} \).

**Proof.** We use an induction on the order \( n \) of a tree \( T \neq K_{1,n-1} \) with \( \ell \) leaves, \( s \) support vertices and \( \gamma_R^l(T) = (4n + \ell + s)/5 \) to show that \( T \in \mathcal{T} \). Since \( T \neq K_{1,n-1} \), for the basic step consider a path \( P_3 \), and note that \( P_3 \in \mathcal{T} \).

Assume that any tree \( T \) of order \( n' < n \), with \( \ell' \) leaves, \( s' \) support vertices and \( \gamma_L(T') = (4n' + \ell' + s')/5 \) belongs to \( \mathcal{T} \). Let \( n = n(T) \geq 5 \).

Assume that \( T \) has a support vertex \( u \) with \( \deg(u) \geq 3 \). Let \( v \) be a leaf adjacent to \( u \), and \( T' = T - v \). We can easily see that \( \gamma_R^l(T) = \gamma_R^l(T') + 1 \). If \( u \) is not a strong support vertex, then \( \gamma_R^l(T) \leq \gamma_R^l(T') + 1 = (4(n - 1) + \ell - 1 + s - 1)/5 + 1 < (4n + \ell + s)/5 \), a contradiction. Thus \( u \) is a strong support vertex. Then \( \gamma_R^l(T') = \gamma_R^l(T) - 1 = (4n + \ell + s)/5 - 1 = (4n - 1) + (\ell - 1 + s)/5 \).

By the inductive hypothesis, \( T' \in \mathcal{T} \). Hence \( T \) is obtained from \( T' \) by Operation \( O_1 \). Thus we assume that the following claim holds.

**Claim 1.** Any support vertex of \( T \) is of degree two.

We root \( T \) at a leaf \( x_0 \) of a diametrical path \( x_0x_1 \cdots x_d \) from \( x_0 \) to a leaf \( x_d \) farthest from \( x_0 \). Clearly, \( d \geq 3 \). Since \( n > 4 \) and \( T \) has no strong support vertex,
we find that \( d \geq 4 \). Clearly, \( \deg(x_1) = \deg(x_{d-1}) = 2 \). Assume that \( d = 4 \). If \( \deg(x_2) = 2 \) then \( T = P_5 \), and \( \gamma^L_R(T) = 4 < (4n + \ell + s)/5 \), a contradiction. Thus \( \deg(x_2) > 2 \). By Claim 1, \( x_2 \) is not a support vertex. Then \( T \) has \( \deg(x_2) \) support vertices of degree two, and we can see that \((L(T) \cup \{x_2\}, \emptyset, S(T))\) is an LRDF for \( T \), implying that \( \gamma^L_R(P_7) = 6 < (4(7) + 2 + 2)/5 \), a contradiction. Thus \( \deg(x_{d-6}) \geq 2 \). Since \( \gamma^L_R(P_8) = 7 < (4(8) + 2 + 2)/5 \), we find that \( \deg(x_{d-6}) \geq 2 \). Since \( \gamma^L_R(P_k) = 7 < (4(8) + 2 + 2)/5 \), we find

We show that \( \deg(x_{d-2}) = 2 \). Assume that \( 3 \leq \deg(x_{d-2}) = k + 1 \). By Claim 1, \( x_{d-2} \) is not a support vertex. Thus any child of \( x_{d-2} \) is a support vertex of degree two. Let \( T' = T - x_{d-2} \), and let \( f = (V_0, V_1, V_2) \) be a \( \gamma^L_R(T') \)-function. If \( k_2 = 0 \) then \( h = (V_0 \cup S(T_{x_{d-2}}) \cup \{x_{d-2}\}, V_1, V_2 \cup S(T_{x_{d-3}}) \cup \{x_{d-3}\}) \) is an LRDF for \( T \), implying by Theorem 9 that \( \gamma^L_R(T) \leq 2 + 2k + 2k_2 + 2 < (4n + \ell + s)/5 \), a contradiction. Thus assume that \( k_2 > 0 \). Let \( u \) be a leaf at distance two from \( x_{d-3} \) and \( v \) the father of \( u \). Then \( h = (V_0 \cup V(T_{x_{d-2}}) - (S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}, u\}), V_1 \cup \{v\}, V_2 \cup S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}\}) \) is an LRDF for \( T \), implying by Theorem 9 that \( \gamma^L_R(T) \leq 2k_1 + 2k_2 + 1 < (4n + \ell + s)/5 \), a contradiction. We deduce that \( \deg(x_{d-4}) = 2 \). Assume that \( k_2 = 0 \). Let \( T' = T - x_{d-4} \), and let \( f = (V_0, V_1, V_2) \) be a \( \gamma^L_R(T') \)-function. Then \( h = (V_0 \cup V(T_{x_{d-2}}) - S(T_{x_{d-4}}), V_1, V_2 \cup S(T_{x_{d-3}} - \{v\})) \) is an LRDF for \( T \), implying by Theorem 9 that \( \gamma^L_R(T) \leq 2 + 2k_1 + 2k_2 + 2 < (4n + \ell + s)/5 \), a contradiction. Thus assume that \( k_2 > 0 \). Let \( T' = T' - x_{d-3} \), and let \( f = (V_0, V_1, V_2) \) be a \( \gamma^L_R(T') \)-function. Let \( u \) be a leaf at distance two from \( x_{d-3} \) and \( v \) the father of \( u \). Then \( h = (V_0 \cup V(T_{x_{d-2}}) - (S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}, u\}), V_1 \cup \{v\}, V_2 \cup S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}\}) \) is an LRDF for \( T \), implying by Theorem 9 that \( \gamma^L_R(T) \leq 2k_1 + 2k_2 + 1 < (4n + \ell + s)/5 \), a contradiction. We conclude that \( \deg(x_{d-3}) = 2 \). Assume that \( \deg(x_{d-3}) = 2 \). If \( \deg(x_{d-5}) \geq 3 \), then let \( T' = T - x_{d-4} \) and \( f = (V_0, V_1, V_2) \) be a \( \gamma^L_R(T') \)-function. Then \( h = (V_0 \cup \{x_{d-4}, x_{d-3}, x_{d-5}\}, V_1, V_2 \cup \{x_{d-3}, x_{d-5}\}) \) is a LRDF function for \( T \). Hence \( \gamma^L_R(T) \leq 2 + 4 < (4n + \ell + s)/5 \), a contradiction. Thus \( \deg(x_{d-5}) = 2 \). Since \( \gamma^L_R(P_k) = 6 < (4(7) + 2 + 2)/5 \), we find that \( \deg(x_{d-5}) \geq 2 \). Since \( \gamma^L_R(P_k) = 7 < (4(8) + 2 + 2)/5 \), we find
that \( \deg(x_{d-7}) \geq 2 \). Thus \( x_{d-6} \) is not a support vertex. By Lemma 7, \( \gamma^{L}_R(T') = \gamma^{R}_L(T) - 4 = (4n + \ell + s)/5 - 4 = (4(n-5) + \ell + s)/5 = (4n + \ell(T') + s(T'))/5 \). By the inductive hypothesis, \( T' \in \mathcal{T} \). Now \( T \) is obtained from \( T' \) by Operation 2.

Next assume that \( \deg(x_{d-4}) \geq 3 \). By Claim 1, \( x_{d-4} \) is not a support vertex. Suppose that there is a leaf \( v \) of \( T_{x_{d-4}} \) at distance two from \( x_{d-4} \). Let \( u \) be the father of \( v \). Clearly, \( \deg(u) = 2 \). Let \( T' = T - \{u, v\} \). Suppose that there is a \( \gamma^{R}_L(T') \) function \( f = (V_0, V_1, V_2) \) such that \( f(x_{d-4}) = 2 \) and \( pn(x_{d-4}, V_0) = \emptyset \). Then \( (V_0 \cup \{u\}, V_1 \cup \{v\}, V_2) \) is an LRDF for \( T \), and so \( \gamma^{R}_L(T) \leq \gamma^{R}_L(T') + 1 < (4n + \ell + s)/5 \), a contradiction. Thus there is no \( \gamma^{R}_L(T') \) function \( f = (V_0, V_1, V_2) \) with \( f(x_{d-4}) = 2 \) and \( pn(x_{d-4}, V_0) = \emptyset \). Suppose that there is a \( \gamma^{R}_L(T') \) function \( f = (V_0, V_1, V_2) \) with \( f(x_{d-4}) = 1 \), and \( N(x_{d-4}) \cap V_2 \neq \emptyset \). Then \( (V_0 \cup \{v, x_{d-4}\}, V_1 - \{x_{d-4}\}, V_2 \cup \{u\}) \) is an LRDF for \( T' \), and so \( \gamma^{R}_L(T) \leq \gamma^{R}_L(T') + 1 < (4n + \ell + s)/5 \), a contradiction. Thus \( x_{d-4} \) is a special vertex of \( T' \). Clearly, \( \gamma^{R}_L(T') + 1 \leq \gamma^{R}_L(T) \leq \gamma^{R}_L(T') + 2 \). Suppose that \( \gamma^{R}_L(T) = \gamma^{R}_L(T') + 1 \). Let \( h \) be a \( \gamma^{R}_L(T) \)-function. Assume that \( h(u) = 2 \). Clearly, we may assume that \( h(v) = 0 \). If \( h(x_{d-4}) \neq 0 \), then \( h|_{V(T')} \) is an LRDF for \( T' \) of weight less than \( \gamma^{R}_L(T') \), a contradiction. Thus \( h(x_{d-4}) = 0 \). Since \( h \) is an LRDF for \( T \), there is a vertex \( w \in N(x_{d-4}) - \{u\} \) such that \( h(w) = 2 \). Now \( h' \) defined on \( V(T') \) by \( h'(x_{d-4}) = 1 \) and \( h'(x) = h(x) \) otherwise, is an LRDF for \( T' \). Clearly, that \( h' \) is a \( \gamma^{R}_L(T') \)-function. This is a contradiction, since \( x_{d-4} \) is a special vertex of \( T' \). If \( h(u) = 1 \), then \( h(v) = 1 \) and we can replace \( h(u) \) by 2 and \( h(v) \) by 0, and as before, get a contradiction. Then \( h(u) = 0 \). If \( h(v) = 2 \), then we replace \( h(u) \) by 2 and \( h(v) \) by 0, and as before, get a contradiction. Thus \( h(v) = 1 \), and so \( h(x_{d-4}) = 2 \). Thus \( h|_{V(T')} = (V_0, V_1, V_2) \) is a \( \gamma^{R}_L(T') \)-function with \( pn(x_{d-4}, V_0) = \emptyset \). This is a contradiction, since \( x_{d-4} \) is a special vertex of \( T' \). Thus \( \gamma^{R}_L(T) = \gamma^{R}_L(T') + 2 \). Now \( \gamma^{R}_L(T') = \gamma^{R}_L(T) - 2 = (4n + \ell + s)/5 - 2 = (4(n-2) + \ell - 1 + s - 1)/5 = (4n + \ell(T') + s(T'))/5 \). By the inductive hypothesis, \( T' \in \mathcal{T} \). Thus \( T \) is obtained from \( T' \) by Operation \( O_3 \).

Now, we assume that any leaf of \( T_{x_{d-4}} \) has degree three or four from \( x_{d-4} \). If there is a leaf \( v \) of \( T_{x_{d-4}} \) at distance four from \( x_{d-4} \), then any internal vertex in the path from \( v \) to \( x_{d-4} \) is of degree two, since \( v \) plays the role of \( x_d \). Moreover, by Claim 1, if \( v \) is a leaf of \( T_{x_{d-4}} \) at distance three from \( x_{d-4} \), then any internal vertex in the path from \( v \) to \( x_{d-4} \) is of degree two. Let \( k_1 \) be the number of leaves of \( T_{x_{d-4}} \) at distance four from \( x_{d-4} \) and \( k_2 \) be the number of leaves of \( T_{x_{d-3}} \) at distance three from \( x_{d-4} \). Note that \( \deg(x_{d-5}) = k_1 + k_2 + 1 \). Suppose that \( \deg(x_{d-5}) = 2 \). Let \( T' = T - T_{x_{d-5}} \) and \( f = (V_0, V_1, V_2) \) be a \( \gamma^{R}_L(T') \)-function. Then \( h = (V_0 \cup W, V_1 \cup Z, V_2 \cup U) \) is a LRDF for \( T \), where \( W \) is the set of vertices of \( T_{x_{d-4}} \) at distance one or three of \( x_{d-4} \), \( Z \) is the set of vertices at distance four from \( x_{d-4} \), and \( U \) contains \( x_{d-4} \) and all vertices at distance two from \( x_{d-4} \). Hence \( \gamma^{R}_L(T) \leq \gamma^{R}_L(T') + 3k_1 + 2k_2 + 2 < (4n + \ell + s)/5 \), a contradiction. Thus \( \deg(x_{d-5}) \geq 3 \). Let \( T'' = T - T_{x_{d-4}} \) and \( f = (V_0, V_1, V_2) \) be a \( \gamma^{R}_L(T') \)-function.
Then $h = (V_0 \cup W, V_1 \cup Z, V_2 \cup U)$ is a $\gamma_R^L(T)$-function, where $W$ is the set of vertices at distance one or three of $x_{d-4}$, $Z$ is the set of vertices at distance four from $x_{d-4}$, and $U$ contains $x_{d-4}$ and vertices at distance two from $x_{d-4}$. Hence $\gamma_R^L(T) \leq \gamma_R^L(T') + 3k_1 + 2k_2 + 2$. If $k_2 \neq 0$ or $k_1 \geq 3$, then $\gamma_R^L(T) \leq \gamma_R^L(T') + 3k_1 + 2k_2 + 2 < (4n + \ell + s)/5$, a contradiction. Thus $k_2 = 0$ and $k_1 = 2$.

By Lemma 8, $\gamma_R^L(T) = \gamma_R^L(T') + 8$. Thus $\gamma_R^L(T') = \gamma_R^L(T) - 8 = (4n + \ell + s)/5 - 8 = (4(n - 9) + (\ell - 2) + (s - 2))/5 = (4n(T') + \ell(T') + s(T'))/5$. By the inductive hypothesis, $T' \in T$. Suppose $\gamma_R^L(T' - x_{d-5}) < \gamma_R^L(T')$. Let $g$ be a $\gamma_R^L(T' - x_{d-5})$-function. We extend $g$ to an LRDF for $T$ by assigning 0 to $x_{d-5}$ and the vertices of $T_{x_{d-4}}$ at distance one or three from $x_{d-4}$, 2 to $x_{d-4}$ and the vertices of $T_{x_{d-4}}$ at distance two. Thus $\gamma_R^L(T) \leq \gamma_R^L(T' - x_{d-5}) + 8 < \gamma_R^L(T') + 8 < (4n + \ell + s)/5$, a contradiction. Hence $\gamma_R^L(T' - x_{d-5}) \geq \gamma_R^L(T')$. Now $T$ is obtained from $T'$ by Operation $O_4$. The converse follows by Lemma 11.

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