Discussiones Mathematicae
Graph Theory 38 (2018) 39–48
doi:10.7151/dmgt.1990

A NOTE ON THE INTERVAL FUNCTION OF A DISCONNECTED GRAPH

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Abstract

In this note we extend the Mulder-Nebeský characterization of the interval function of a connected graph to the disconnected case. One axiom needs to be adapted, but also a new axiom is needed in addition.

Keywords: interval function, transit function, axiomatic characterization, disconnected graph.

2010 Mathematics Subject Classification: 05C12, 05C38, 05C05.

1. Introduction

One of the fundamental notions of metric graph theory is that of the *interval function* $I: V \times V \to 2^V$ of a graph G with vertex set V, where $I(u,v)$ is the set of vertices on shortest paths between u and v in G. The term interval function was coined in [10], which is the first extensive study of this function. Some properties of the interval function have been singled out to define generalizations, such as transit functions, see [11], and the notion of betweenness, see [9]. A main problem in this area is, given a function $R: V \times V \to 2^V$ on a finite set V, what properties should R have such that it is the interval function of some connected graph with vertex set V. Nebeský has obtained many nice such characterizations, see for instance [13–16]. In Proposition 1.1.2 of [10] the first simple properties of the interval function of a connected graph were presented. Nebeský named these five properties the *classical axioms*. These five axioms do not yet characterize the interval function of a connected graph. In [12] an interesting enquiry was undertaken. First the question was addressed how much of the output of a function $R: V \times V \to 2^V$ is determined if we assume that only the five classical axioms are satisfied. Second, the minimal 'road blocks' were determined that prevent such a function to be the interval function of a connected graph. These road blocks were overcome by requiring two additional axioms $(s1)$ and $(s2)$. The axioms are indeed minimal in the sense that weaker axioms than $(s1)$ or $(s2)$ would not do the trick.

So far connectedness has always been an essential part of the research. In this note we want to study the case where we drop the requirement of connectedness. One thing is clear: the first classical axiom has to be weakened to include disconnected graphs. But this is not enough. It turns out that we need a new extra axiom to make things work. In the literature the axioms that we use have been denoted in various ways. For our purposes here we follow that of [12] as closely as possible. For axioms not in that paper, we follow the notation of [11] and [1].

2. Preliminaries

Transit functions were introduced by Mulder in 1998 to generalize three basic notions on discrete structures, namely, interval, convexity and betweenness. This was eventually published in [11]. Let V be a nonempty finite set, and let 2^V be the power set of V. A *transit function on* V is a function $R: V \times V \to 2^V$ that has the following three properties.

$(c1)$ $u \in R(u,v)$, for all u,v in V.
$(c2)$ $R(u,v) = R(v,u)$, for all u,v in V.
$(t3)$ $R(u,u) = \{u\}$, for all u in V.

We call such properties of a function on V *transit axioms*. If G is a graph with vertex set V, then we say that R is a transit function on G. Basically, a transit function on a graph G describes how we can get from vertex u to vertex v: via vertices in $R(u,v)$. In other words, a transit function is a very general notion that unifies many ways how one could move around in a graph and other discrete structures. Transit functions particularly captured attention on discrete structures like graphs, partially ordered sets, hypergraphs, etc. Graph transit functions (transit functions defined on the vertex set of a connected graph) and their associated convexities have been studied extensively from different perspectives. For instance, on betweenness [3, 6, 7, 9, 10] and on convexity [2, 4, 8, 9, 10]. Three well studied transit functions on connected graphs are the interval function [1, 2, 6, 10, 12, 13, 14, 16], the induced-path function [3, 5, 6, 7, 9, 17] and the all-paths function [2]. One of the main problems on transit functions is the characterization of such functions by what we call *transit axioms*, see [11] for many possible questions in this area. Our focus here is the interval function.

The *underlying graph* G_R of a transit function R is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u,v)=\{u,v\}$. Note that, in general, G and G_R need not be isomorphic graphs, see [11].

Let G be a graph with vertex set V and distance function d, and let u and v be vertices of G. The *interval* between u and v is the set

$$I_G(u,v)=\{\, x \mid d(u,x)+d(x,v)=d(u,v)\},$$

that is, the set of vertices lying on shortest paths between u and v. When no confusion arises we write I instead of I_G. Then $I\colon V\times V\to 2^V$ is the *interval function* of G. The interval function on a connected graph is the prime example of a transit function.

In [9] a *betweenness* was introduced that can be phrased in the terminology of transit functions. It is a transit function R on V satisfying the following two additional axioms, called the *betweenness axioms*.

($b1$) $x\in R(u,v), x\neq v \Rightarrow v\notin R(u,x)$, for all u,v,x in V.

($c3$) $x\in R(u,v)\Rightarrow R(u,x)\subseteq R(u,v)$, for all u,v,x in V.

Clearly, the interval function of a connected graph is a betweenness.

In Proposition 1.1.2 of [10] the first simple properties of the interval function of a connected graph were presented. These five properties are the above transit axioms ($c1$), ($c2$), the betweenness axiom ($c3$), and the following two axioms.

($c4$) $x\in R(u,v)\Rightarrow R(u,x)\cap R(x,v)=\{x\}$, for all u,v,x in V.

($c5$) $x\in R(u,v)$, $y\in R(u,x)\Rightarrow x\in R(y,v)$, for all u,v,x,y in V.

Because they played a key role in his many studies of the interval function, Nebeský named these properties the *five classical axioms*. In [1] it was shown that a transit function satisfying the five classical axioms is a betweenness. Also in [1] examples were presented that show that the five classical axioms are independent, that is, for each axiom we have a function that does not satisfy this axiom but satisfies the other four axioms. As said above, the five classical axioms do not yet characterize the interval function of a connected graph.

In [13] two additional axioms were introduced that, together with the five classical axioms, characterize the interval function of a connected graph.

($s1$) If $R(u,\bar{u}) = \{u,\bar{u}\}$, $R(v,\bar{v}) = \{v,\bar{v}\}$, $\bar{u},\bar{v} \in R(u,v)$ and $u \in R(\bar{u},\bar{v})$, then $v \in R(\bar{u},\bar{v})$, for all $u,\bar{u},v,\bar{v}$ in V with $u \neq \bar{u}$ and $v \neq \bar{v}$.

($s2$) If $R(u,\bar{u}) = \{u,\bar{u}\}$, $R(v,\bar{v}) = \{v,\bar{v}\}$, $\bar{u} \in R(u,v)$, $\bar{v} \notin R(u,v)$, and $v \notin R(\bar{u},\bar{v})$, then $\bar{u} \in R(u,\bar{v})$, for all $u,\bar{u},v,\bar{v}$ in V with $u \neq \bar{u}$ and $v \neq \bar{v}$.

To get a feeling for these two axioms, let R be the interval function of a connected graph G. In the case of ($s1$), let u and v be vertices at distance k in G, let $\bar{u}$ be a neighbor of u in $I(u,v)$ and $\bar{v}$ be a neighbor of v in $I(u,v)$ such that $d(\bar{u},\bar{v}) = k$ as well. Such a situation can be found for instance in an even isometric cycle of G. In the case of ($s2$), let v and $\bar{v}$ be two adjacent vertices, and let u be a vertex at distance k from both v and $\bar{v}$, and let $\bar{u}$ be a neighbor of u at distance $k-1$ from both v and $\bar{v}$.

The main result in [12] is the following theorem, which gives a characterization of the interval function of a connected graph in terms of transit axioms.

Theorem A. *Let $R: V \times V \to 2^V$ be a function on the nonempty set V with underlying graph G. Then G is connected and R is the interval function of G if and only if R satisfies the five classical axioms and ($s1$) and ($s2$).*

In [12] it was shown that these axioms are minimal in the sense that the five classical axioms are all needed, and weaker versions of ($s1$) and/or ($s2$) do not suffice.

3. The Disconnected Case

So far, all the papers on characterizing the interval function deal exclusively with connected graphs. What happens on disconnected graphs? If u and v are vertices in different components, then by definition, we have $I(u,v) = \emptyset$. Hence axiom ($c1$) does not hold anymore. But a weaker axiom still holds.

($c1^*$) If $R(u,v) \neq \emptyset$ then $u \in R(u,v)$, for all u,v in V.

An immediate question arises: what are the consequences if we replace axiom ($c1$) by ($c1^*$). In the literature various implications were derived, where ($c1$) together with some of the axioms from Section 2 implied one of the other axioms. Here are some pertinent instances.

$(i1)$: $(c1)$ and $(c4) \Rightarrow (t3)$ (obvious).

$(i2)$: $(c1)$ and $(b1) \Rightarrow (t3)$ (see [1]).

$(i3)$: $(c1)$, $(c2)$, $(t3)$ and $(c5) \Rightarrow (c4)$ (see [12]).

It turns out that none of these implications holds when we replace $(c1)$ by $(c1^*)$, see Example 1 for the counterpart of implications $(i1)$ and $(i2)$, and Example 2 for the counterpart of implication $(i3)$.

Example 1. $R(u,v) = \emptyset$, for all u and v in V.
Clearly, R satisfies $(c1^*)$, $(c2)$, $(c3)$, $(c4)$, $(c5)$, $(b1)$, $(s1)$ and $(s2)$, but R does not satisfy $(t3)$.

Example 2. Let $V = \{u,v,w\}$. Define $R(u,v) = R(v,u) = R(u,w) = R(w,u) = \emptyset$, $R(v,w) = R(w,v) = V$, and $R(x,x) = \{x\}$ for all x in V.
It is easy to see that R satisfies axioms $(c1^*), (c2), (c3)$, $(c5)$, $(s1)$, $(s2)$ and $(t3)$ but does not satisfy axiom $(c4)$: we have $u \in R(v,w)$, but $R(v,u) \cap R(u,w) = \emptyset$.

In [1] it was proved that $(c1)$, $(c2)$ and $(c4)$ imply $(b1)$. In this case the implication still holds when we replace $(c1)$ by $(c1^*)$. This implication is an essential first step for the disconnected case.

Lemma 3. *Axioms $(c1^*)$, $(c2)$ and $(c4)$ imply axiom $(b1)$.*

Proof. Let $R: V \times V \to 2^V$ be a function on V satisfying axioms $(c1^*)$, $(c2)$ and $(c4)$. Assume that x is in $R(u,v)$ with $x \neq v$. By $(c4)$, we have $R(u,x) \cap R(x,v) = \{x\}$. From $(c1^*)$ and $(c2)$, it follows that v lies in $R(x,v)$. Since $v \neq x$, we have that v is not in $R(u,x)$. ∎

For the disconnected case, it becomes important to determine for a function R on V what the components are in G_R.

Lemma 4. *Let $R: V \times V \to 2^V$ be a function on V satisfying the axioms $(c1^*)$, $(c2)$, $(c3)$ and $(c4)$. If $R(u,v) \neq \emptyset$, then the set $R(u,v)$ induces a connected subgraph in G_R.*

Proof. By Lemma 3, R satisfies $(b1)$. Let u and v be vertices with $R(u,v) \neq \emptyset$. By $(c1^*)$ and $(c2)$, we have that u and v are in $R(u,v)$. Hence, if $u \neq v$, then $|R(u,v)| \geq 2$.

We prove the lemma by induction on $|R(u,v)|$. If $|R(u,v)| = 1$, then necessarily $u = v$, so we have $R(u,v) = R(u,u) = \{u\}$.

Assume that $|R(u,v)| = 2$. If $u = v$, then there would exist a vertex x in $R(u,u)$ distinct from u. By $(c4)$, we would have $R(u,x) \cap R(x,v) = \{x\}$. So we would have $R(u,x) \neq \emptyset$, whence $u \in R(u,x)$ by $(c1^*)$. But by $(b1)$, we would have that u, being equal to v, is not in $R(u,x)$. This contradiction shows that $u \neq v$. Hence we have $R(u,v) = \{u,v\}$, so that u and v are adjacent in G_R.

Now let $|R(u,v)| > 2$. Take any vertex x in $R(u,v)$ with $x \neq u, v$. Such a vertex must exist. Then, by $(c4)$, we have $R(u,x) \cap R(x,v) = \{x\}$. So $R(u,x) \neq \emptyset$ as well as $R(x,v) \neq \emptyset$. By $(c3)$, we have $R(u,x) \subseteq R(u,v)$ as well as $R(x,v) \subseteq R(u,v)$. By $(b1)$ and $(c2)$, we have $u \notin R(v,x) = R(x,v)$ and $v \notin R(u,x)$, and hence $|R(u,x)| < |R(u,v)|$ and $|R(x,v)| < |R(u,v)|$. So, by the induction hypothesis, the sets $R(u,x)$ and $R(x,v)$ induce connected subgraphs in G_R. Hence there is a path from x to u as well as to v in $R(u,x) \cup R(x,v) \subseteq R(u,v)$. Since this is true for every x in $R(u,v)$, it follows that the set $R(u,v)$ induces a connected subgraph in G_R. ∎

An immediate consequence of this lemma is that $R(u,v) = \emptyset$, for any two vertices u and v in different components of G_R. Note that the converse need not be true, as is shown by the following example.

Example 5. Let $V = \{u,v,w\}$ and define R on V as follows: $R(x,x) = \{x\}$ for all x in V, $R(u,v) = R(v,u) = \{u,v\}$, $R(v,w) = R(w,v) = \{v,w\}$ and $R(u,w) = R(w,u) = \emptyset$.

It is easy to see that R satisfies $(c1^*), (c2), (c3), (c4)$, $(c5)$. Moreover, we have $R(u,w) = \emptyset$, but u, w are in the same component of G_R.

In the proof of Lemma 4 we came across an important detail. If $R(u,u) \neq \emptyset$ then $R(u,u) = \{u\}$. But, as Example 1 shows, $R(u,u)$ might be empty, even though all other axioms in Theorem A are satisfied. So we also need an axiom to avoid this case. We call it $(t3^*)$ because it is a weaker axiom than $(t3)$, but still serves the purpose.

$(t3^*)$ $R(u,u) \neq \emptyset$, for all u in V.

Note that the function R in Example 5 satisfies $(t3^*)$, but R is not the interval function of its underlying graph G_R. This graph is the path $u \to v \to w$, whereas $R(u,w) = \emptyset \neq \{u,v,w\}$. So we still need an extra axiom.

(Compt) If $R(u,v) \neq \emptyset$ and $R(v,w) \neq \emptyset$, then $R(u,w) \neq \emptyset$.

Clearly, the interval function of a graph satisfies this axiom, in the connected case as well as in the disconnected case. If R satisfies the three axioms $(c2)$, $(t3^*)$ and (Compt) then we can define the equivalence relation $\sim_R$ on V by

$$u \sim_R v \text{ if } R(u,v) \neq \emptyset.$$

Thus we get the following lemma as an immediate consequence of Lemma 4.

Lemma 6. *Let $R\colon V \times V \to 2^V$ be a function on V satisfying the axioms $(c1^*)$, $(c2)$, $(c3)$, $(c4)$, $(t3^*)$ and (Compt). Then the components of G_R are the subgraphs induced by the equivalence classes of $\sim_R$.*

Now we are ready for the characterization of the interval function that also holds for disconnected graphs.

Theorem 7. *Let $R: V \times V \to 2^V$ be a function on the finite nonempty set V. Then R satisfies the axioms $(c1^*)$, $(c2)$, $(c3)$, $(c4)$, $(c5)$, $(s1)$, $(s2)$, $(t3^*)$ and $(Compt)$ if and only if R is the interval function of G_R.*

Proof. Clearly, the interval function of a graph $G = (V, E)$ satisfies all the axioms and its underlying graph is G.

Conversely, let R be a function on V satisfying all the axioms. Let G_R be its underlying graph, and let I be the interval function of G_R. Let W be an equivalence class of $\sim_R$. Then, by Lemma 6, the set W induces a component of G_R. Let $I|_W$ be the restriction of I to W. The restriction $R|_W$ of R to W is a function on W that satisfies $(c1)$, $(c2)$, $(c3)$, $(c4)$, $(c5)$, $(s1)$ and $(s2)$, so, by Theorem A, $R|_W$ is the interval function $I|_W$ of the subgraph of G_R induced by W. Let u_1 and u_2 be vertices in different components of G_R, and hence in different equivalence classes. Then, by definition, we have $I(u_1, u_2) = \emptyset = R(u_1, u_2)$. So $R = I$. ∎

4. Independence of the Axioms

Finally we need to show that all the axioms in Theorem 7 are independent, and hence necessary. Example 1 satisfies trivially also $(Compt)$ but not $(t3^*)$. So axiom $(t3^*)$ is independent. Example 2 also satisfies $(Compt)$. So $(c4)$ is independent. In [12] Mulder and Nebeský gave two examples that showed the independence of, respectively, $(s1)$ and $(s2)$ from the other axioms in Theorem A. These examples also satisfy trivially axioms $(t3^*)$ and $(Compt)$. So also $(s1)$ and $(s2)$ are independent. We refer the reader to [12] for the details.

Example 8. Let V be a set with $|V| \geq 3$, and let z be a fixed vertex in V. Define R on V by $R(x, x) = \{x\}$ for all x in V, and $R(x, y) = \{z\}$ for all distinct x and y in V.

Clearly, R satisfies $(c2)$, $(c3)$, $(c4)$, $(c5)$, $(t3^*)$ and $(Compt)$, but not $(c1^*)$. Since there are no edges in G_R, axioms $(s1)$ and $(s2)$ are trivially satisfied. So axiom $(c1^*)$ is independent.

In the other examples most of the work goes into checking whether $(s1)$ and $(s2)$ are satisfied. So let us first have a closer look at these two axioms. We presume axioms $(c1^*)$, $(c2)$, $(t3^*)$ and $(c4)$. We consider two edges $u\bar{u}$ and $v\bar{v}$. If $\{u, \bar{u}\} = \{v, \bar{v}\}$, then $(s1)$ and $(s2)$ are trivially satisfied. Assume that the two edges have exactly one vertex in common. First consider $(s1)$. Then $\bar{u}, \bar{v} \in R(u, v)$ implies that u and v are in $R(u, v)$ as well. Hence uv is not an edge. So the only way that $u\bar{u}$ and $v\bar{v}$ have a vertex in common is that $\bar{u} = \bar{v}$. Note that we have $\{u, \bar{u}\} = R(u, \bar{u}) = R(\bar{u}, u)$. If we would have $u \in R(\bar{u}, \bar{v}) = R(\bar{u}, \bar{u})$, then $(c4)$ would imply that $R(\bar{u}, u) \cap R(u, \bar{u}) = \{u\}$. This contradiction shows that u is not in $R(\bar{u}, \bar{v})$. Hence $(s1)$ is satisfied. Next consider $(s2)$. Assume that $\bar{u}$ is in $R(u, v)$ and $\bar{v}$ is not in $R(u, v)$. Then $\bar{v} \neq u, v$, so, for the two edges to have a vertex in common, we must have $\bar{u} = v$. Now v lies in $R(\bar{u}, \bar{v})$. Hence $(s2)$ is satisfied. So below we only need to check $(s1)$ and $(s2)$ in the case of two disjoint edges $u\bar{u}$ and $v\bar{v}$. In the case of $(s1)$, we only need to check the sets $R(x, y)$ that contain two disjoint edges.

Example 3 trivially does not satisfy $(Compt)$. With the above considerations it is also trivial that this example satisfies $(s1)$ and $(s2)$. So also $(Compt)$ is an independent axiom. To show the independence of $(c2)$ and $(c3)$ we present two new examples.

Example 9. Let $V = \{u, v, w\}$ and let R be the function on V defined as follows: $R(x, x) = \{x\}$ for all x in V, $R(u, w) = \{u\}$, $R(w, u) = \{w\}$, $R(u, v) = R(v, u) = \{u, v\}$ and $R(v, w) = R(w, v) = \{v, w\}$.

It is easy to see that R satisfies axioms $(c1^*)$, $(c3)$, $(c4)$, $(c5)$, $(s1)$, $(s2)$, $(t3^*)$ and $(Compt)$, but that R does not satisfy $(c2)$, since $R(u, w) \neq R(w, u)$.

Example 10. Let $V = \{w, x, y, z\}$ and let R be the function defined as follows: $R(w, y) = \{w, x, y\}$, $R(w, z) = \{w, y, z\}$, $R(x, z) = \{x, y, z\}$, and $R(p, q) = \{p, q\}$ for every other distinct pair p, q in V, and $R(p, p) = \{p\}$ for all p in V.

Note that G_R is the path $w \to x \to y \to z$. It is easy to see that R satisfies axioms $(c1^*)$, $(c2)$, $(c4)$, $(c5)$, $(t3^*)$ and $(Compt)$. Clearly, R does not satisfy axiom $(c3)$, since y is in $R(w, z)$ but $R(w, y) \not\subseteq R(w, z)$. We still have to check that R satisfies $(s1)$ and $(s2)$. First, there is no set $R(p, q)$ containing two disjoint edges, so $(s1)$ is satisfied. For $(s2)$ we have to consider two disjoint edges $u\bar{u}$ and $v\bar{v}$. If $u = w$, then $\bar{u} = x$. To get $\bar{u} \in R(u, v)$, we must have $v = y$. Hence $\bar{v} = z$ and v lies in $R(\bar{u}, \bar{v})$, so that $(s2)$ is satisfied. If $u = x$, then, to get the two disjoint edges, we must have $\bar{u} = w$ and v is y or z. Now $\bar{u}$ cannot anymore be in $R(u, v)$. So $(s2)$ is satisfied. If $u = y$, then $\bar{u} = z$, and $(s2)$ is satisfied. Finally, if $u = z$, then $\bar{u} = y$ and $v = x$. Now $\bar{v} = w$ and $\bar{v} \notin R(u, v)$, but $v \in R(\bar{u}, \bar{v})$, so again $(s2)$ is satisfied.

In [1] an example is given to show that $(c5)$ is independent of the other four classical axioms. The underlying graph in this example is the 4-fan F_4, which consists of a path $P = w \to x \to y \to z$ on four vertices and an extra vertex s adjacent to all vertices on the path. The function R is the induced path function on F_4, that is, $R(u, v)$ consists of all vertices on induced paths between u and v. In [1] it is shown that R satisfies $(c1)$, $(c2)$, $(c3)$ and $(c4)$ but not $(c5)$, Obviously R satisfies $(t3^*)$ and $(Compt)$. What is left is to check that it also satisfies $(s1)$

and $(s2)$. First take $(s1)$. We have to take two disjoint edges $u\bar{u}$ and $v\bar{v}$ such that $\bar{u}, \bar{v} \in R(u,v)$. There are three possible choices for u and v, *viz.* $\{u,v\} = \{w,y\}$, $\{u,v\} = \{w,z\}$, and $\{u,v\} = \{x,z\}$. In all cases $u \notin R(\bar{u},\bar{v})$. So $(s1)$ is satisfied. In the case of $(s2)$, to get $\bar{u} \in R(u,v)$, the vertices u and v cannot be adjacent. To get $\bar{v} \notin R(u,v)$, two choices remain: *viz.* $u = w$ and $v = y$ or $u = z$ and $v = x$. Without loss of generality let $u = w$ and $v = y$, so that $\bar{v} = z$. To get $v \notin R(\bar{u},\bar{v})$ we have $\bar{u} = s$. But now $\bar{u} \in R(w,z) = R(u,\bar{v})$. So also $(s2)$ is satisfied.

Thus we have shown the independence of all axioms in Theorem 7. Due to the 'minimality' result of Mulder-Nebeský in [12] we know that the axiom set in Theorem 7 is also such a minimal set: no weaker axioms would do the trick.

Acknowledgement

The first author was supported by NBHM-DAE grant (No. 2/48(9)/2014/ 299 NBHM(R.P)/R&D II/14364, 17th November, 2014) of Govt. of India.

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Received 21 May 2016
Revised 13 September 2016
Accepted 14 September 2016