A CHARACTERIZATION FOR 2-SELF-CENTERED GRAPHS

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Abstract

A graph is called 2-self-centered if its diameter and radius both equal to 2. In this paper, we begin characterizing these graphs by characterizing edge-maximal 2-self-centered graphs via their complements. Then we split characterizing edge-minimal 2-self-centered graphs into two cases. First, we characterize edge-minimal 2-self-centered graphs without triangles by introducing specialized bi-independent covering (SBIC) and a structure named generalized complete bipartite graph (GCBG). Then, we complete characterization by characterizing edge-minimal 2-self-centered graphs with some triangles. Hence, the main characterization is done since a graph is 2-self-centered if and only if it is a spanning subgraph of some edge-maximal 2-self-centered graphs and, at the same time, it is a spanning supergraph of some edge-minimal 2-self-centered graphs.

Keywords: self-centered graphs, specialized bi-independent covering (SBIC), generalized complete bipartite graphs (GCB).

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1. Introduction

Let $G = (V, E)$ be a connected finite simple graph. For $u, v \in V$ the distance of $u$ and $v$, denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest path between $u$ and $v$. The eccentricity of a vertex $v$, $\text{ecc}(v)$, is $\max\{d(u, v) : u \in V\}$. The maximum and minimum eccentricity of vertices of $G$ are called diameter and radius of $G$ and are denoted by $\text{diam}(G)$ and $\text{rad}(G)$, respectively. Center of a graph $G$ is the subgraph induced by vertices with eccentricity $\text{rad}(G)$. A graph is called self-centered if it is equal to its center, or equivalently, its diameter equals its radius.

A graph $G$ is called $k$-self-centered if $\text{diam}(G) = \text{rad}(G) = k$. The terminology $k$-equi-eccentric graph is also used by some authors. For studies on these graphs see [1–4, 7] and [6].

Clearly, a graph $G$ is 1-self-centered if and only if $G$ is a complete graph. In this paper, we try to characterize 2-self-centered graphs. An edge-maximal 2-self-centered graph can be easily characterized via a condition on its complement. We do this in Section 2, using a lemma which gives a necessary and sufficient condition for a graph to be 2-self-centered. For edge-minimal 2-self-centered graphs we need to divide the discussion into two cases: triangle-free or not. We do these in Section 3. Then, the characterization is done in sight of the following theorem.

**Theorem 1.** A finite graph $G$ is 2-self-centered if and only if there is an edge-minimal 2-self-centered graph $G'$ and an edge-maximal 2-self-centered graph $G''$ such that $G'$ is a spanning subgraph of $G$ while $G$ is itself a spanning subgraph of $G''$.

**Proof.** The proof is clear. Note that $G' \subseteq G$ implies $\text{rad}(G) \geq \text{rad}(G') = 2$ and $G \subseteq G''$ implies $\text{diam}(G) \leq \text{diam}(G'') = 2$. ■

Throughout this paper $G$ is a connected finite simple graph and its complement is denoted by $\overline{G}$. If $G$ is a graph and $e$ is an edge in $G$, then $G \setminus e$ is the graph obtained from $G$ by omitting $e$. Moreover, the graph obtained by adding an edge $e \notin E(G)$ to $G$ is denoted by $G + e$. Whenever two vertices $u$ and $v$ are adjacent, we might write $u \sim v$. For concepts and notations of graph theory, the reader is referred to [5].

2. Edge-Maximal 2-Self-Centered Graphs

In this section, we present a characterization for edge-maximal 2-self-centered graphs. The following lemma is not only essential to do so, but it is also going to be used all over this paper.
Lemma 2. Let $G = (V, E)$ be a graph with $n$ vertices. Then $G$ is 2-self-centered if and only if the following two conditions are true:

(i) $2 \leq \deg(v) \leq n - 2$ for all $v \in V$;
(ii) for each $u, v \in V$ with $uv \notin E$ there is a $w \in V$ such that $uw, wv \in E$.

Proof. The proof is obvious. Note that if $G$ has a vertex $v$ with $\deg(v) = n - 1$ then $\text{rad}(G) = 1$ and if there is a vertex $u$ with $\deg(u) = 1$ then its neighbour should be adjacent to any vertex of $G$, since otherwise $\text{ecc}(u) > 2$.

Remark 3. If we show that for a graph $G$ item (ii) of Lemma 2 holds and no vertex is adjacent to all vertices, then we can deduce that no vertex has degree 1 and therefore $G$ is 2-self-centered.

A 2-self-centered graph $G$ is said to be edge-maximal if there are no non-adjacent $u, v \in V(G)$ such that $G + uv$ is 2-self-centered. The following theorem is a characterization for edge-maximal 2-self-centered graphs.

Theorem 4. Let $G$ be a 2-self-centered graph. Then $G$ is edge-maximal if and only if $G$ is disconnected and each connected component of $G$ is a star with at least two vertices.

Proof. Let $H_1, \ldots, H_r$ be the connected components of $G$, where $r$ is a positive integer. At first, note that each $H_i$ should be a tree with at least two vertices. To see this, if $H_i$ has only one vertex $v$, then the degree of $v$ in $G$ is zero and thus its degree should be $n - 1$ in $G$ which contradicts to (i) of Lemma 2. Furthermore, if the connected component $H_i$ is not a tree, then there is an edge $e$ with end vertices $u_0$ and $v_0$ in $H_i$ which is not a cut edge. Let $H = G + e$. Since $G$ is edge-maximal, $H$ cannot be 2-self-centered. Using Lemma 2, we can deduce that the degree of $u_0$ or $v_0$ in $G$ must be $n - 2$. This means that the degree of $u_0$ or $v_0$ in $G$ is 1 and consequently $e$ is a cut edge, a contradiction.

Now, we show that each connected component $H_i$ is a star. Let $u$ be a vertex with maximum degree $k$ in $H_i$. If $k = 1$ then $H_i$ is $K_{1,1}$. Let $k \geq 2$. If $H_i$ is not $K_{1,k}$ then one of the neighbours of $u$, say $v$, has a neighbour $w \neq u$. Let $e'$ be the edge between $u$ and $v$ in $G$ and $H' = G + e'$. Since $G$ is edge-maximal, $H'$ cannot be 2-self-centered. Using Lemma 2, we can again deduce that the degree of $u$ or $v$ in $G$ should be $n - 2$. This means that the degree of $u$ or $v$ in $G$ is 1; which is a contradiction.

Conversely, suppose that $G$ is a disconnected graph whose connected components are all stars, each of which has at least two vertices. Then, $2 \leq \deg(v) \leq n - 2$ for all $v \in V(G)$ and whenever $u$ and $v$ are two non-adjacent vertices of $G$, there must be a $w \in V(G)$ such that $u$ and $v$ are both adjacent to $w$. Therefore, by Lemma 2 $G$ is a 2-self-centered graph. Moreover, since every connected component of $G$ is a star with at least two vertices, adding an edge between two
non-adjacent vertices in $G$ makes the complement to have a singleton as a connected component, which means that the resulted graph is not 2-self-centered.

3. Edge-Minimal 2-Self-Centered Graphs

A 2-self-centered graph $G$ is said to be edge-minimal if for each $e \in E(G)$, $G \setminus e$ is not a 2-self-centered graph. In this section, we determine all edge-minimal 2-self-centered graphs. To do so, let at first suppose that $G$ is disconnected.

**Proposition 5.** Let $G$ be a graph. Then $G$ is an edge-minimal 2-self-centered graph such that $G$ is disconnected if and only if it is the complete bipartite graph $K_{k,\ell}$ for some $k, \ell \geq 2$.

**Proof.** Let $H_1, \ldots, H_r$ be the connected components of $\overline{G}$, where $r \geq 2$. At first we prove that each $H_i$ is a clique in $\overline{G}$, or in another word, each $H_i$ is an independent set in $G$. Let $e$ be an edge in $G$ between two vertices $u$ and $v$ of $H_i$. If $H = G \setminus e$, then edge minimality of $G$ implies that $H$ cannot be 2-self-centered.

Let $u'$ and $v'$ be two non-adjacent vertices of $H$. Then $u'$ and $v'$ are belonged to a connected component $H_j$ of $\overline{G}$. Let $w'$ be any vertex of $H_{j'}$, where $j' \neq j$. Thus $u'w', w'v' \in E(H)$. This shows that $H$ satisfies part (ii) of Lemma 2.

Since $H$ is not 2-self-centered, Lemma 2 implies that the degree of $u$ or $v$ in $H$ is 1. Let the degree of $u$ in $H$ be 1. Thus $u$ has a neighbour $w$ in $H$. This implies that all other vertices of $G$ are in $H_i$. We know that $v$ is also in $H_i$. Thus $H_i$ contains all vertices except $w$ and $w$ is itself a component. Hence, the degree of $w$ in $G$ is $n-1$ which contradicts to Lemma 2.

Now we show that $r = 2$. Let $r \geq 3$. Choose $x, y$ and $z$ in three different components. Let $e = xy$ and $H = G \setminus e$. Due to the existence of $z$, $H$ is clearly 2-self-centered which contradicts the edge-minimality of $G$.

Conversely, the complete bipartite graph $K_{k,\ell}$ for $k, \ell \geq 2$ is an edge-minimal 2-self-centered graph such that its complement is disconnected. 

For those 2-self-centered graphs that have connected complements, Proposition 5 is not useful. So, we may develop the characterization in some separate propositions for them, or, we can prove a more general statement which covers this case as a special case. In this paper, we do the later one, for which some preliminaries are needed.

**Definition 6.** Let $G$ be a 2-self-centered graph. A vertex $x$ in $G$ is called critical for $u$ and $v$ if $uv \notin E$ and $x$ is the only common neighbour of $u$ and $v$.

**Lemma 7.** Let $G$ be an edge minimal 2-self-centered graph with no critical vertex for any pair of vertices. Then $G$ is triangle-free. Furthermore, every triangle-free 2-self-centered graph is edge-minimal.
**Proof.** Suppose in contrary that there are $u, v, w \in V(G)$ such that $uv, vw, wu \in E(G)$. If $\deg(u) = \deg(v) = \deg(w) = 2$, then $G$ is itself a triangle which contradicts to $\text{rad}(G) = 2$. If $\deg(u) = \deg(v) = 2$, then $\text{diam}(G) = 2$ implies that all other vertices of $G$ are neighbours of $w$. Thus $\deg(w) = n - 1$ which contradicts the fact that $\text{rad}(G) = 2$. Hence, at most one of the vertices $u$, $v$ and $w$ has degree 2. Suppose that $\deg(u)$ and $\deg(v)$ are both greater than 2.

Let $e = uv$ and $H = G \setminus e$. Then edge-minimality of $G$ implies that $H$ is not a 2-self-centered graph. Since $\deg_G(u)$ and $\deg_G(v)$ are at least 3, this happens only if there are two vertices $x$ and $y$ such that $d_H(x, y) > 2$. Since $d_G(x, y) \leq 2$ it can be deduced that $\{x, y\} \cap \{u, v\} \neq \emptyset$. The cases $x = u$ and $y = v$ cannot happen at the same time because we have the path $x \sim w \sim y$ in $H$. If $x = u$ and $y$ is a vertex other than $v$, then there is a path $xt, ty$ in $G$ for some vertex $t$, since $v$ is not critical for $u$ and $y$. For the case $y = v$ and any other vertex $x$ the argument is similar. Therefore, $H$ is a 2-self-centered graph, a contradiction.

Moreover, let $G$ be triangle-free. If $G$ is not edge-minimal, then there is an edge $e$ with ends $u$ and $v$ such that $G \setminus e$ is still a 2-self-centered graph. Thus there is a path of length 2 between $u$ and $v$ in $G \setminus e$. This gives a triangle in $G$. □

Nevertheless, there are examples of edge-minimal 2-self-centered graphs possessing some critical vertices with or without triangles.

**Example 8.** Let $G$ be the graph with vertex set $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and edge set $E = \{01, 23, 12, 14, 15, 23, 36, 37, 46, 57, 67\}$. Then $G$ is an edge-minimal 2-self-centered graph possessing the critical vertex 6 for the vertices 4 and 7, with a triangle on 3, 6, 7, see Figure 1.

![Figure 1. The graph $G$ of Example 8.](image)

**Example 9.** Let $H$ be a graph constructed in the following way: consider the graph $K_{3,3}$ with two vertices $y$ and $z$ in different parts connected by the edge $e$. Omit $e$ and add a vertex $x$ with two edges $xy$ and $xz$ to obtain $H$. Then $H$ is an edge minimal 2-self-centered graph possessing the critical vertex $x$ for the vertices $y$ and $z$, without any triangle.
Definition 10. A graph $G$ is called to have a *specialized bi-independent covering* via $(A_r, B_s)$ if

(i) $G$ is triangle-free,

(ii) there are two families $A_r = \{A_1, \ldots, A_r\}$ and $B_s = \{B_1, \ldots, B_s\}$ of not necessarily distinct independent subsets of $G$ such that we have $V(G) = \bigcup_{i=1}^{r} A_i = \bigcup_{j=1}^{s} B_j$,

(iii) for all $u, v \in V(G)$ if $d(u, v) \geq 3$, then there is an $1 \leq i \leq r$ such that $u, v \in A_i$ or there is $1 \leq j \leq s$ such that $u, v \in B_j$,

(iv) for all $u \in V(G)$ and $i \in \{1, \ldots, r\}$ if $d(u, A_i) \geq 2$, then there is a $j \in \{1, \ldots, s\}$ such that $A_i \cap B_j = \emptyset$ and $u \in B_j$, and

(v) for all $u \in V(G)$ and $j \in \{1, \ldots, s\}$ if $d(u, B_j) \geq 2$, then there is an $i \in \{1, \ldots, r\}$ such that $A_i \cap B_j = \emptyset$ and $u \in A_i$.

To make it easy, we shorten the name “specialized bi-independent covering” to SBIC. It is straightforward to check that every triangle-free graph $G$ has two families of independent sets $A_r, B_s$ such that $G$ has a SBIC via $(A_r, B_s)$. To see this, fix two independent coverings of $G$, and by adding enough independent sets to them, we can always satisfy items (iii) to (v) of Definition 10.

We need the following definition to complete our characterization of triangle-free 2-self-centered graphs.

Definition 11. A graph $G$ is called an $X$-generalized complete bipartite, denoted by $\text{GCB}_X(k, \ell, A_r, B_s)$, if $X$ has an SBIC via $(A_r, B_s)$ and $G$ is constructed in the following way:

1. $V(G) = K \cup L \cup Y \cup Z \cup V(X)$ where $|K| = k$, $|L| = \ell$, $Y = \{y_1, \ldots, y_r\}$ and $Z = \{z_1, \ldots, z_s\}$.

2. $a \sim t$ for all $a \in K$ and $t \in L \cup Y$.

3. $b \sim t$ for all $b \in L$ and $t \in K \cup Z$.

4. $y_i \sim t$ for all $t \in A_i$ and $1 \leq i \leq r$.

5. $z_j \sim t$ for all $t \in B_j$ and $1 \leq j \leq s$.

6. $y_i \sim z_j$ if and only if $A_i \cap B_j = \emptyset$.

Moreover, there are some special cases that must be treated separately:
(7) If \( k = 0 \) then every member of \( Y \) has a neighbor in \( Z \) and for all \( i, j \in \{1, \ldots, r\} \) we have \( A_i \cap A_j \neq \emptyset \) or there is a \( p \in \{1, \ldots, s\} \) such that \( A_i \cap B_p = A_j \cap B_p = \emptyset \).

(8) If \( \ell = 0 \) then every member of \( Z \) has a neighbor in \( Y \) and for all \( i, j \in \{1, \ldots, r\} \) we have \( A_i \cap A_j \neq \emptyset \) or there is a \( p \in \{1, \ldots, s\} \) such that \( A_i \cap B_p = A_j \cap B_p = \emptyset \).

(9) If \( r = 0 \) then \( k \neq 0 \) and if \( s = 0 \) then \( \ell \neq 0 \).

(10) \( r = s = 0 \) if and only if \( X = \emptyset \) and \( k, \ell \geq 2 \).

(11) If \( |X| = 1 \) then at least one of \( k \) or \( \ell \) is non-zero.

**Proposition 12.** Any generalized complete bipartite graph is a triangle-free 2-self-centered graph.

**Proof.** Let \( G = \text{GCB}_X(k, \ell, A_r, B_s) \) and \( t = |X| \). Then \( n := |V(G)| = k + \ell + r + s + t \). We show that \( G \) has no vertex of degree \( n - 1 \) and then we show that item (ii) of Lemma 2 holds for \( G \). Then by Remark 3 we deduce that \( G \) is 2-self-centered. By proving that \( G \) has no triangle and using Lemma 7, we actually show that \( G \) is also edge-minimal.

For \( a \in K \), \( \deg(a) = \ell + r = n - s - k - t \). Thus if \( r = s = 0 \) then by item (10) of Definition 11 we have \( k \geq 2 \) and hence \( \deg(a) \leq n - 2 \). If \( r \) or \( s \) is non-zero, then by item (10) we have \( t \neq 0 \) and therefore we have \( \deg(a) \leq n - 2 \) (by items (2) and (9) of Definition 11, and because no element of \( K \) is adjacent to a vertex of \( X \)).

For \( b \in L \), by a similar proof to the case \( a \in K \) we can deduce that \( \deg(b) \leq n - 2 \).

For \( y_i \in Y \), if \( \ell \neq 0 \) then \( \deg(y_i) \leq n - 2 \) because no element of \( L \) is adjacent to \( y_i \). If \( \ell = 0 \) then either \( y_i \) is not adjacent to all vertices of \( X \) or if \( y_i \) is adjacent to all vertices of \( x \) then it is not adjacent to \( z_j \) for some \( j \in \{1, \ldots, s\} \) (which its existence is supported by item (8) of Definition 11), each of which cases yields to \( \deg(y_i) \leq n - 2 \).

For \( z_j \in Z \) we have the same argument to \( y_i \in Y \).

Finally, for each \( x \in X \), item (11) of Definition 11 guarantees that \( \deg(x) \leq n - 2 \) whenever \( X \) has only one vertex. So, assume that \( t \geq 2 \). Therefore, there are two possibilities: either there is \( \hat{x} \in X \) such that \( x \) is not adjacent to \( \hat{x} \), or \( x \) is adjacent to all other vertices of \( X \). If the former case is true then \( \deg(x) \leq n - 2 \). For the later case, since \( x \) is not in any independent set with other vertices of \( X \), we have there is some \( i \in \{1, \ldots, r\} \) and \( j \in \{1, \ldots, s\} \) such that \( \{x\} \cap A_i = \{x\} \cap B_j = \emptyset \). Thus, by items (4) and (5) of Definition 11, we have \( x \) is not adjacent to \( y_i \) and \( z_j \) and hence \( \deg(x) \leq n - 2 \).

To show that (ii) of Lemma 2 is also satisfied, we should choose two vertices \( u \) and \( v \) in \( G \) and show that whenever they are not adjacent, they have at least
one common neighbour. There are 15 different ways for choosing \( u \) and \( v \) from \( G = K \cup L \cup Y \cup Z \cup X \).

If \((u, v) \in (K \times L) \cup (K \times Y) \cup (L \times Z)\) then \( u \) and \( v \) are adjacent to each other.

If \((u, v) \in (K \times K) \cup (L \times L)\) then there is a path of length 2 between \( u \) and \( v \) via one of the sets \( L \cup Y \) or \( K \cup Z \).

If \((u, v) \in (Y \times Y)\) then if \( k \neq 0 \) there is a path of length 2 between \( u \) and \( v \) via any member of \( K \). If \( k = 0 \) then, by item (7) of Definition 11, we have either there is a \( z_y \in Z \) which is a common neighbour of \( u \) and \( v \), or, \( u \) and \( v \) are both adjacent to a vertex \( x \in X \). The case \((u, v) \in (Z \times Z)\) is also similar.

If \((u, v) \in (L \times Y)\) then if \( k \neq 0 \) there is a path of length 2 between \( u \) and \( v \) via any member of \( K \). If \( k = 0 \) then, by item (7) of Definition 11, we have every member of \( Y \) has a neighbour in \( Z \), so \( Z \) is non-empty and \( v \) has a neighbour in \( Z \), namely \( z \). Since \( u \) is also adjacent to \( z \) by item (3) of Definition 11, there is a path of length 2 between \( u \) and \( v \). The case \((u, v) \in (K \times Z)\) is also similar.

If \((u, v) \in Y \times Z\) then \( u = y_i \) and \( v = z_j \) for some \( i \) and \( j \). If \( A_i \cap B_j \neq \emptyset \) then we can choose a \( c \) in \( A_i \cap B_j \) such that there is a path of length 2 between \( u \) and \( v \) via \( c \). If \( A_i \cap B_j = \emptyset \) then \( u \) is adjacent to \( v \), by item (6) of Definition 11.

If \((u, v) \in X \times X\) then either \( d_X(u, v) \leq 2 \) or by item (iii) of Definition 10 there is an \( i \in \{1, \ldots, r\} \) or a \( j \in \{1, \ldots, s\} \) such that both \( u \) and \( v \) are adjacent to \( y_i \) or \( z_j \).

If \((u, v) \in (K \times X) \cup (L \times X)\) then there is an \( i \in \{1, \ldots, r\} \) or a \( j \in \{1, \ldots, s\} \) such that \( v \) is adjacent to \( y_i \) and \( z_j \). Then, since \( u \) is adjacent to \( y_i \) or \( z_j \), we have \( d_G(u, v) = 2 \).

If \((u, v) \in (Y \times X)\) then there is an \( i \in \{1, \ldots, r\} \) such that \( u = y_i \). Then, either \( d(v, A_i) \leq 1 \) which means that \( d(u, v) \leq 2 \), or, if \( d(v, A_i) \geq 2 \) then by item (iv) of Definition 10 there is a \( j \in \{1, \ldots, s\} \) such that \( A_i \cap B_j = \emptyset \) and \( v \in B_j \). Hence by items (5) and (6) of Definition 11 we have \( z_j \) is adjacent to \( u \) and \( v \). The case \((u, v) \in (Z \times X)\) is also similar.

So, for each of 15 ways of choosing \( u \) and \( v \) from vertices of \( G \) we have \( d(u, v) \leq 2 \).

We finally show that \( G \) is triangle-free. On contrary, suppose that \( u, v, w \) are vertices of a triangle in \( G \). The case that none of \( u, v \) and \( w \) is a vertex of \( X \) cannot happen because \( K \cup Z \) and \( L \cup Y \) are independent sets. Since \( X \) is triangle-free, \( u, v \) and \( w \) are not all together vertices of \( X \). Meanwhile, if only two vertices of \( \{u, v, w\} \) are in \( X \), then the third is not adjacent to the other two because they cannot be in the same independent set in \( X \). So, at most one of \( \{u, v, w\} \) is a vertex of \( X \). Let for instance \( w \) be a vertex of \( X \). Then \( u \) and \( v \) are not members of \( Y \) or \( Z \) at the same time, because otherwise they are not adjacent together. The case that one of \( u \) and \( v \) is in \( Y \) and the other in \( Z \) is also impossible because it is contrary to item (6) of Definition 11.
Theorem 13. A graph $G$ is a triangle-free 2-self-centered graph if and only if there are positive integers $k, \ell, r, s$ and a graph $X$ which has a SBIC via $(A_r, B_s)$ such that $G = GCB_X(k, \ell, A_r, B_s)$.

Proof. Let $Y'$ be a maximal independent subset of $G$, let $Z'$ be a maximal independent subset of $G \setminus Y'$ and let $X = G \setminus (Y' \cup Z')$. Suppose that $K$ (respectively $L$) is the set of all vertices in $Z'$ (respectively $Y'$) which are not adjacent to any member of $X$ and put $Y = Y' \setminus L, Z = Z' \setminus K$.

Let $a \in K$ and $y' \in Y'$. We claim that $ay' \in E$. Suppose on the contrary that $ay' \notin E$. Since diam($G$) = 2 there is a $u$ in $G$ such that $au, uy' \in E$. The vertex $u$ cannot be in $Y'$ or $Z'$ since $Y'$ and $Z'$ are independent sets. Hence $u \in X$. This contradicts to the definition of $K$.

A similar argument shows that each member of $L$ is adjacent to each member of $Z'$.

Let $k = |K|, \ell = |L|, r = |Y|, s = |Z|, Y = \{y_1, \ldots, y_r\}$ and $Z = \{z_1, \ldots, z_s\}$. Now put $A_i = N_X(y_i)$ and $B_j = N_X(z_j)$. We show that $A_i$'s and $B_j$'s are independent subsets of $X$ and $X$ has a SBIC via $(A_r, B_s)$.

Let $x$ be an arbitrary member of $X$. Since $Y'$ and $Z'$ are maximal independent, there should be neighbours for $x$ in $Y'$ and $Z'$. We know that these neighbours are in $Y$ and $Z$. Let $y_i$ and $z_j$ be adjacent to $x$. Thus $x \in A_i$ and $x \in B_j$. This shows that $X = \bigcup_{i=1}^r A_i = \bigcup_{j=1}^s B_j$.

Each $A_i$ and each $B_j$ is independent, since $G$ is triangle-free. Moreover, if $y_i$ and $z_j$ are not adjacent to each other, then since diam($G$) = 2, there should be an $x \in X$ with $y_ix, xz_j \in E(X)$. Thus $x \in A_i \cap B_j$. If $y_i$ is adjacent to $z_j$ then there must not be such an $x$, so we have $y_i \sim z_j$ if and only if $A_i \cap B_j = \emptyset$.

Furthermore, $X$ is triangle-free since $X$ is a subgraph of the triangle-free graph $G$.

Items (iii), (iv) and (v) of Definition 10 must hold because $G$ is a triangle-free 2-self-centered graph. Therefore, $X$ has an SBIC via $(A_r, B_s)$.

Items (1) to (6) of Definition 11 have already hold. Moreover, items (7) to (11) of Definition 11 must also hold because $G$ is a triangle-free 2-self-centered graph. Hence $G = GCB_X(k, \ell, A_r, B_s)$.

Since the converse is evident by Proposition 12, we are done with the proof. ■

The reader should note that every complete bipartite graph $K_{k,\ell}$ with $k, \ell \geq 2$ is a generalized complete bipartite graph $GCB(k, \ell, \emptyset, \emptyset)$.

Now, we can consider edge-minimal 2-self-centered graphs with some triangles. We need the following procedure to proceed.

Procedure 14. Let $G$ be a graph, $u, v, w$ form a triangle in $G$ and suppose that $v$ is a critical vertex for $u$ and $v_1, \ldots, v_p$ and/or $u$ is a critical vertex for $v$ and $u_1, \ldots, u_q$. Remove the edge $uv$ and add edges $uv_1, \ldots, uv_p$ and $vu_1, \ldots, vu_q$. 
The following theorem characterizes edge-minimal 2-self-centered graphs with triangles, which completes the characterization of all 2-self-centered graphs.

**Theorem 15.** Let $G$ be a graph. Then $G$ is an edge-minimal 2-self-centered graph with some triangle if and only if the following two conditions are true:

(i) for each edge of every triangle in $G$, at least one end-vertex is a critical vertex (for the other end-vertex of that edge and some other vertices of $G$), and

(ii) iteration of Procedure 14 on $G$ (at most to the number of triangles of $G$) transforms $G$ to a triangle-free 2-self-centered graph.

**Proof.** Assume that $u, v, w$ form a triangle in $G$. Since $G$ is edge-minimal, if we omit the edge $uv$, then the resulting graph is not 2-self-centered. This shows that $u$ or $v$ is a critical vertex. Let $u$ be a critical vertex. Thus there are vertices $u_1, \ldots, u_q$ such that $u$ is the common neighbour of $v$ and each of the $u_i$’s. Moreover, if $v$ is also a critical vertex for $u$ and some other vertices, then we suppose that $v_1, \ldots, v_p$ are the vertices such that $v$ is a common neighbour of $u$ and each of the $v_j$’s.

If we omit $uv$ and add edges $u_1v, \ldots, u_qv, uv_1, \ldots, uv_p$, then the resulting graph $G'$ is clearly 2-self-centered and the number of triangles of $G'$ is less than the number of triangles of $G$. To see this, note that edges of a triangle on $u, v$ and $w$ are omitted and no new triangle is added. In contrary, suppose that we have a new triangle. Then it should be of the form $u_i, v, t$ (or $v_j, u, s$) which contradicts to the fact that $u$ (or $v$) is a critical vertex for $u_i$ and $v$ (for $v_j$ and $u$).

If $G$ has still some triangle then we can proceed this process. Therefore, we finally transform $G$ into a triangle-free 2-self-centered graph.

Conversely, if the two conditions are true for a graph $G$ with some triangles, then $G$ is an edge-minimal 2-self-centered graph because condition (ii) guarantees that $G$ is 2-self-centered while condition (i) obligates $G$ to be edge-minimal.

**References**


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