UPPER BOUNDS FOR THE STRONG CHROMATIC INDEX OF HALIN GRAPHS

ZIYU HU
Department of Mathematical Sciences
Florida Atlantic University
e-mail: azuth.hu@gmail.com

KO-WEI LIH
Institute of Mathematics, Academia Sinica
e-mail: makwlih@sinica.edu.tw

AND

DAPHNE DER-FEN LIU
Department of Mathematics
California State University Los Angeles
e-mail: dliu@calstatela.edu

Abstract

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The strong chromatic index of a graph $G$, denoted by $\chi'_s(G)$, is the minimum number of vertex induced matchings needed to partition the edge set of $G$. Let $T$ be a tree without vertices of degree 2 and have at least one vertex of degree greater than 2. We construct a Halin graph $G$ by drawing $T$ on the plane and then drawing a cycle $C$ connecting all its leaves in such a way that $C$ forms the boundary of the unbounded face. We call $T$ the characteristic tree of $G$. Let $G$ denote a Halin graph with maximum degree $\Delta$ and characteristic tree $T$. We prove that $\chi'_s(G) \leq 2\Delta + 1$ when $\Delta \geq 4$. In addition, we show that if $\Delta = 4$ and $G$ is not a wheel, then $\chi'_s(G) \leq \chi'_s(T) + 2$. A similar result for $\Delta = 3$ was established by Lih and Liu [21].

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1. Introduction

Let $G$ be a simple graph. The distance between two edges $e$ and $e'$ in $G$ is the minimum $k$ for which there is a sequence $e = e_0, e_1, \ldots, e_k = e'$ of distinct edges such that for $1 \leq i \leq k$, $e_{i-1}$ and $e_i$ share an end vertex. A strong edge-coloring of a graph is a function that assigns to each edge a color such that any two edges with distance at most two must receive different colors. A strong $k$-edge-coloring is a strong edge-coloring using $k$ colors. The strong chromatic index of a graph $G$, denoted by $\chi'_s(G)$, is the minimum $k$ such that $G$ admits a strong $k$-edge-coloring. The pre-image of each color in a strong edge-coloring is an induced matching. Thus, the strong chromatic index is also the minimum number of vertex induced matchings needed to partition the edge set of $G$.

Denote the maximum degree of a graph $G$ by $\Delta(G)$ (or, simply by $\Delta$ when $G$ is clear in the context). A trivial upper bound is that $\chi'_s(G) \leq 2\Delta(G)^2 - 2\Delta(G) + 1$. Fouquet and Jolivet [13] established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta(G)^2 - 2\Delta(G)$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [24]. The following conjecture was posed by Erdős and Nešetřil [10, 11].

**Conjecture 1.** For any graph $G$ of maximum degree $\Delta$,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

For graphs with maximum degree $\Delta(G) = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [18], independently. For $\Delta(G) = 4$, while Conjecture 1 asserts that $\chi'_s(G) \leq 20$, Horák [17] obtained $\chi'_s(G) \leq 23$ and Cranston [8] proved $\chi'_s(G) \leq 22$. For general graphs $G$ with maximum degree $\Delta$, Molloy and Reed [22] showed that $\chi'_s(G) \leq 1.998\Delta^2$. Most recently, this bound has been improved by Bruhn and Joos [4] to $1.93\Delta^2$.

Strong edge-coloring for planar graphs has been investigated by many authors. Fouquet and Jolivet [13, 14] first studied strong edge-coloring for cubic planar graphs. Let $G$ be a planar graph with maximum degree $\Delta$ and girth $g$. Faudree et al. [12] proved that $\chi'_s(G) \leq 4\Delta + 4$. Bensmail et al. [2] established the bound $\chi'_s(G) \leq 3\Delta + 1$ for $g \geq 6$. Hudák et al. [19] showed $\chi'_s(G) \leq 3\Delta$ if $g \geq 7$, and the bound is sharp for some subcubic (that is, $\Delta \leq 3$) planar graphs. Furthermore, Hocquard et al. [16] showed that $\chi'_s(G) \leq 9$ for subcubic planar graphs $G$ which do not contain cycles of lengths 4 or 5. DeOrsey et al. [9] recently reduced this bound to $\chi'_s(G) \leq 5$ if $g \geq 30$. For planar graphs with large girth, Borodin and Ivanova [3] established a rather tight bound $\chi'_s(G) \leq 2\Delta - 1$ if $g \geq 40[\Delta/2] + 1$; Chang et al. [7] further confirmed that the bound also holds if $g \geq 10\Delta + 46$. Clearly, the bound $\chi'_s(G) \leq 2\Delta - 1$ becomes sharp when $G$ contains two adjacent vertices of maximum degree $\Delta$. 
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By definition, a trivial lower bound of $\chi'_s(G)$ for a graph $G$ would be $\sigma(G)$, where
\[
\sigma(G) := \max\{\deg_G(u) + \deg_G(v) - 1 \mid uv \in E(G)\}.
\]
If $G$ has no edges, then define $\sigma(G) = 0$. It is known and easy to verify that for a tree $T$, we have $\chi'_s(T) = \sigma(T)$. Wu and Lin [25] proved that if $\sigma(G) \leq 4$ and $G$ is not isomorphic to the graph of the 5-cycle connecting two non-adjacent vertices, then $\chi'_s(G) \leq \chi'_s(T) + 2$.

If a Halin graph $G$ is different from a wheel, then $\chi'_s(G) \leq \chi'_s(T) + 2$.

A Halin graph is a plane graph $G$ constructed as follows. Let $T$ be a tree with at least 4 vertices, called the characteristic tree of $G$. All vertices of $T$ are either of degree 1, called leaves, or of degree at least 3. We draw $T$ on the plane. Let $C$ be a cycle, called the adjoint cycle of $G$, connecting all leaves of $T$ in such a way that $C$ forms the boundary of the unbounded face. We usually write $G = T \cup C$ to reveal the characteristic tree and the adjoint cycle. For $n \geq 3$, the wheel $W_n$ with $n + 1$ vertices is a particular Halin graph whose characteristic tree is the complete bipartite graph $K_{1,n}$ (called a star). A graph is said to be cubic if the degree of every vertex is 3. For $h \geq 1$, a cubic Halin graph $Ne_h$, called a necklace, was introduced in [23]. Its characteristic tree $T$ consists of the path $v_0, v_1, \ldots, v_h, v_{h+1}$ and leaves $v'_1, v'_2, \ldots, v'_h$ such that the unique neighbor of $v'_i$ in $T$ is $v_i$ for $1 \leq i \leq h$ and vertices $v_0, v'_1, \ldots, v'_h, v_{h+1}$ are connected in this order to form the adjoint cycle $C_{h+2}$.

Lai, Lih and Tsai [20] proved the following result.

**Theorem 2** [20]. If a Halin graph $G = T \cup C$ is different from a certain necklace $Ne_2$ and any wheel $W_n$, $n \not\equiv 0$ (mod 3), then $\chi'_s(G) \leq \chi'_s(T) + 3$.

For cubic Halin graphs, Lih and Liu improved the above bound as follows.

**Theorem 3** [21]. A cubic Halin graph $G$ different from $Ne_2$ or $Ne_4$ satisfies $\chi'_s(G) \leq 7$.

The exact values of $\chi'_s(G)$ for special families of cubic Halin graphs were determined by Shiu and Tam [24] and by Chang and Liu [6].

For a Halin graph $G = T \cup C$ with maximum degree $\Delta$, since $\chi'_s(T) \leq 2\Delta - 1$, the bound in Theorem 2 implies that $\chi'_s(G) \leq 2\Delta + 2$. We improve this bound and establish a result similar to Theorem 3 for Halin graphs of maximum degree 4.

**Theorem 4.** Let $G$ be a Halin graph with maximum degree $\Delta \geq 4$. Then $\chi'_s(G) \leq 2\Delta + 1$.

**Theorem 5.** Let $G = T \cup C$ be a Halin graph with maximum degree $\Delta = 4$, and let $G$ be different from a wheel. Then $\chi'_s(G) \leq \chi'_s(T) + 2$. 
Both bounds in Theorems 4 and 5 are sharp. Consider the graph $G$ in Figure 1. A strong edge-coloring of $G$ must use at least 7 colors on the edges incident to $u$ or $v$. Let these colors be $\{1, 2, \ldots, 7\}$. Next, since the edges $w_1$ and $w_2$ must use colors different from $\{1, 2, \ldots, 7\}$, at least 8 colors are needed. Assume that we only have 8 colors. Then $w_1$ and $w_2$ must be colored by the same new color, say color 8. This implies that the four edges $e_1, e_2, e_3, e_4$ shown in Figure 1 only have three admissible colors, from the set $\{5, 6, 7\}$, which is a contradiction as these edges must receive different colors. Hence $\chi'_s(G) \geq 9$. By coloring $e_1, e_2, e_3, e_4$ with colors 5, 6, 7, 9 and the last edge with color 4, it follows that $\chi'_s(G) = 9$. This example shows that both bounds in Theorems 4 and 5 are sharp.

![Figure 1. An example showing sharp bounds of Theorems 4 and 5.](image)

### 2. Proof of Theorem 4

A **double star** is a tree with exactly two non-leaf vertices. Denote by $D_{a,b}$ a double star, where $a, b$ are the degrees of the two non-leaf vertices and $a \leq b$. Prior to the proof of Theorem 4, we quote several known results as follows.

**Lemma 6** [20]. Let $G = T \cup C$ be a Halin graph. If $T = D_{a,b}$ is a double star with $a \leq b$, then

$$\chi'_s(G) = \begin{cases} 
\chi'_s(T) + 4 & \text{if } a = b = 3; \\
\chi'_s(T) + 2 & \text{if } a = 3 \text{ and } b \geq 4; \\
\chi'_s(T) + 1 & \text{if } a \geq 4.
\end{cases}$$

If $T = K_{1,k}$ (that is, $G$ is a wheel $W_k$), then

$$\chi'_s(W_k) = \begin{cases} 
k + 3 & \text{if } k \equiv 0 \pmod{3}; \\
k + 5 & \text{if } k = 5; \\
k + 4 & \text{otherwise}.
\end{cases}$$
Lemma 7 [23]. Suppose \( h \geq 1 \). Then
\[
\chi'_s(N_{eh}) = \begin{cases} 
6 & \text{if } h \text{ is odd;} \\
7 & \text{if } h \geq 6 \text{ and } h \text{ is even;} \\
8 & \text{if } h = 4; \\
9 & \text{if } h = 2.
\end{cases}
\]

**Proof of Theorem 4.** Let \( G = T \cup C \) be a Halin graph with \( \Delta(G) \geq 4 \). If \( T \) is a star or a double star, by Lemma 6, the conclusion of Theorem 4 follows. Assume that \( T \) is neither a star nor a double star. We proceed by induction on \( |C| \), the length of \( C \). The shortest length of \( C \) is 6. Three possible graphs along with their corresponding strong edge-colorings satisfying the desired upper bounds are shown in Figure 2. So the result follows.

![Figure 2. All Halin graphs with \(|C| = 6\) and \(\Delta(G) = 4\).](image)

Assume \( |C| \geq 7 \). Let \( P = u_0, u_1, \ldots, u_l \) be a longest path in \( T \) with length \( l \). As \( T \) is neither a star nor a double star, so \( l \geq 4 \). Without loss of generality, we assume \( \deg_G(u_{l-1}) \geq \deg_G(u_1) \).

Denote \( u_1 = v, u_2 = u, u_3 = w \), and label the \( k \geq 2 \) leaf neighbors of \( v \) as \( v_1, v_2, \ldots, v_k \). Since \( P \) is a longest path in \( T \), it is easy to see that \( v_1, v_2, \ldots, v_k \) must be on the adjoint cycle \( C \). Let \( x_1, x_2, y_1, y_2 \) be vertices on \( C \), where \( x_1 \) is adjacent to \( v_1 \) and \( x_2 \); \( y_1 \) is adjacent to \( v_k \) and \( y_2 \). Let \( x_3 \) and \( y_3 \) be vertices not on \( C \), where \( x_1x_3 \) and \( y_1y_3 \) are edges in \( T \) (see Figure 3).

Since \( G \) is a Halin graph and \( u \) is a vertex of degree at least 3, there exists a path \( P' \) in \( T \) from \( u \) to \( x_1 \) or from \( u \) to \( y_1 \) with \( P \cap P' = \{u\} \). Without loss of generality, we shall assume that \( P' \) is from \( u \) to \( y_1 \). By our assumption that \( P \) is a longest path, it must be that \( |P'| \leq 2 \). Thus, either \( u = y_3 \) or \( u \) is adjacent to \( y_3 \).

In the following, we denote by \( G' = T' \cup C' \) the Halin graph obtained by adding some new edges to an induced subgraph of \( G \) such that \( |C'| < |C| \) and \( \Delta(G') \leq \Delta(G) \). If \( \Delta(G') \geq 4 \), then \( \chi'(G') \leq 2\Delta(G) + 1 \) holds because \( T' \) is a star or double star (see the beginning of the proof) or by the inductive hypothesis as \( |C'| < |C| \). If \( \Delta(G') = 3 \), then \( \chi'_s(G') \leq 9 \leq 2\Delta(G) + 1 \) by Theorem 2,
Lemma 6, and because $\Delta(G) \geq 4$. In the following case analysis these steps will be repeatedly used, while may not be mentioned explicitly all the time.

![Figure 3. The neighborhood around one end of the longest path $P$.](image)

We call $G'$ a reduction of $G$. Depending on various situations, different types of $G'$ are created. In the corresponding figures, the dashed lines represent new edges added in $G'$, and dark vertices represent the vertices that are temporarily deleted from $G$.

Let $\psi$ be a strong edge-coloring of $G'$ using the minimum number of colors. A strong edge-coloring $\phi$ of $G$ is obtained as follows. We color the edges that are in both $G$ and $G'$ by the same colors used in $\psi$, i.e., let $\phi(e) = \psi(e)$ for every $e \in E(G) \cap E(G')$. For edges in $e \in E(G) \setminus E(G')$, we develop different coloring schemes for different cases, and in each case, we give a strong edge-coloring $\phi$ for $G$ with at most $2\Delta(G) + 1$ colors.

**Case A.** $\deg_G(v) = 3$. There are three possibilities to consider.

**Case A.1.** $u = y_3$. Obtain the reduction $G'$ of $G$ by adding two new edges $vx_1$ and $vy_1$ to the induced subgraph of $G$ on the vertex set $V(G) \setminus \{v_1, v_2\}$, as indicated in Figure 4. Clearly, $\Delta(G') = \Delta(G) \geq 4$ and $|C'| < |C|$.

Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = 1$ and $\phi(v_2y_1) = 2$ (see Figure 4). We find admissible colors $w_1$, $w_2$, and $w_3$, one by one. The colors that can not be assigned to $v_1v_1$ are from $\{1, 2, t_1, t_2\}$ and the labels used by edges incident to $u$. Therefore, there are at most $\Delta(G) + 4$ forbidden colors for $v_1v_1$. Since $\Delta(G) \geq 4$, there exists an admissible color for $v_1v_1$. Color $v_1v_1$ by such an admissible color $w_1$.

Next we color $v_2v_2$ which has the forbidden colors in $\{1, 2, w_1, s\}$ and the labels used for edges incident to $u$. Similarly, we can find an admissible color for $v_2v_2$. Finally, the forbidden colors for $v_1v_2$ are in $\{1, 2, w_1, w_2, r_1, r_2, s, t_1, t_2\}$. If $s \in \{t_1, t_2\}$, then there is an admissible color for $v_1v_2$. Otherwise, we re-color $v_1v_1$ by $s$, creating an admissible color for $v_1v_2$. 
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Case A.1. $u$ is adjacent to $y_1$, and $\Delta(G) \geq 5$. Obtain the reduction $G'$ in the same way as in Case A.1, as indicated in Figure 5. Clearly, $\Delta(G') = \Delta(G) \geq 4$ and $|C'| < |C|$.

Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = 1$ and $\phi(v_2y_1) = 2$ (see Figure 5). We find admissible colors $w_1$, $w_2$, and $w_3$, one by one. By the same argument as in Case A.1, one can easily show that there exists an admissible color $w_1$. Color $vv_1$ by such an admissible color.

Next we color $vv_2$ which has the forbidden colors in $\{1, 2, w_1, s_1, s_2\}$ and the labels used for edges incident to $u$. Since $\Delta(G) \geq 5$, we can find an admissible color $w_2$. Finally, the forbidden colors for $v_1v_2$ are in $\{1, 2, w_1, w_2, r, s_1, s_2, t_1, t_2\}$. Thus, there exists an admissible color $w_3$.

Case A.2. $u$ is adjacent to $y_3$, and $\Delta(G) \geq 5$. Obtain the reduction $G'$ in the same way as in Case A.1, as indicated in Figure 5. Clearly, $\Delta(G') = \Delta(G) \geq 4$ and $|C'| < |C|$.

Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = 1$ and $\phi(v_2y_1) = 2$ (see Figure 5). We find admissible colors $w_1$, $w_2$, and $w_3$, one by one. By the same argument as in Case A.1, one can easily show that there exists an admissible color $w_1$. Color $vv_1$ by such an admissible color.

Next we color $vv_2$ which has the forbidden colors in $\{1, 2, w_1, s_1, s_2\}$ and the labels used for edges incident to $u$. Since $\Delta(G) \geq 5$, we can find an admissible color $w_2$. Finally, the forbidden colors for $v_1v_2$ are in $\{1, 2, w_1, w_2, r, s_1, s_2, t_1, t_2\}$. Thus, there exists an admissible color $w_3$.

Case A.3. $u$ is adjacent to $y_3$, and $\Delta(G) = 4$. Then $\text{deg}_G(y_3)$ is either 3 or 4. Obtain the reduction $G'$ from $G$ with partial labels to some vertices as indicated in Figure 6(a) and 6(b), respectively. Clearly, $\Delta(G') \leq \Delta(G)$ and $|C'| < |C|$. Assume that $\text{deg}_G(y_3) = 3$. Then $\Delta(G') = \Delta(G) = 4$. We find
admissible colors $w_1$, $w_2$, and $w_3$, one after another. For $v_1v_2$, the forbidden colors are in $\{1, 2, 3, r_1, t_1, t_2\}$. Hence there is an admissible color $w_1$ for $v_1v_2$. Next, the forbidden colors for $y_1y_2$ are in $\{1, 2, 3, w_1, r_2, s_1, s_2\}$. We can color $y_1y_2$ by an admissible color $w_2$. Finally, the forbidden colors for $v_2y_1$ are in $\{1, 2, 3, w_1, w_2, r_1, r_2\}$. Again, there exists an admissible color $w_3$ for $v_2y_1$.

Figure 6. Case A.3.

Assume $\deg_G(y_3) = 4$. Note, even if $\Delta(G') = 3$ or $T'$ is a star (or double star), we can still find a strong edge coloring for $G'$ by up to 9 colors. The forbidden colors for $y_1y_3$ are in $\{1, 2, 3\}$ and labels used on edges incident to $u$. Thus there are at most $\Delta(G) + 3$ forbidden colors. We color $y_1y_3$ by an admissible color $w_1$. Next, the forbidden colors for $v_1v_2$ are $\{1, 2, 3, 4, w_1, t_1, t_2\}$. Because $2\Delta(G) + 1 \geq 9$, we can find an admissible color $w_2$ for $v_1v_2$. The forbidden colors for $y_2z$ are in $\{1, 2, 3, 4, w_1, r, s_1, s_2\}$. Again, there is an admissible color $w_3$ for $y_2z$. Finally, the forbidden colors for $v_2y_1$ are from $\{1, 2, 3, 4, w_1, w_2, w_3, r\}$. So there is an admissible color $w_4$ for $v_2y_1$.

Case B. $\deg_G(v) \geq 4$. We consider two cases separately.

Case B.1. $\Delta(G) = 4$. Then $\deg_G(v) = 4$. There are two subcases.

Subcase B.1.1. $\deg_G(u) = 3$. Obtain the reduction $G'$ of $G$ by adding two new edges $vx_1$ and $vy_1$ to the induced subgraph of $G$ on the vertex set $V(G) \setminus \{v_1, v_2, v_3\}$ as depicted in Figure 7.

Since we assumed earlier that $\deg_G(u_{i-1}) \geq \deg_G(u_1) = \deg_G(v) = 4$, we have $\Delta(G') = \Delta(G) = 4$, and $|C'| < |C|$ holds. We fix colors on some edges as shown in Figure 7. Note that in Figure 7(a) we assign $\phi(y_1y_2) = \phi(vv_2) = 3$ but in Figure 7(b) we assign $\phi(y_1y_3) = \phi(vv_2) = 3$ and $\phi(y_1y_2) = s$. We find admissible colors $w_1$, $w_2$, $w_3$, and $w_4$. 
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For the subcase depicted in Figure 7(a), the forbidden colors for $vv_1$ are in \{1, 2, 3, $t_1$, $t_2$\} and the three colors used in the neighborhood of $u$. Thus, there are at most 8 forbidden colors, implying there is an admissible color $w_1$ for $vv_1$. Next, the forbidden colors for $vv_3$ are in \{1, 2, 3, $w_1$\} and the three colors used in the neighborhood of $u$. There is an admissible color $w_2$ for $vv_3$. The forbidden colors for $v_1v_2$ are in \{1, 2, 3, $w_1$, $w_2$, $r_1$, $t_1$, $t_2$\}, so there is an admissible color $w_3$ for $v_1v_2$. Finally, the forbidden colors for $v_2v_3$ are in \{1, 2, 3, $w_1$, $w_2$, $w_3$, $r_1$, $r_2$\}. Therefore, there is an admissible color $w_4$ for $v_2v_3$.

For the subcase depicted in Figure 7(b), the arguments are the same as in Figure 7(a) except for $v_2v_3$, which has forbidden colors from \{1, 2, 3, $w_1$, $r_2$\} and the three colors used in the neighborhood of $u$. So there is an admissible color $w_2$ for $v_2v_3$.

Subcase B.1.2. $\deg_G(u) = 4$. We distinguish several cases. In each case $\Delta(G') \leq \Delta(G)$ and $|C'| < |C|$ hold.

1. $u = y_3$, $u$ is adjacent to neither $x_1$ nor $x_3$, and $|\{\psi(uw), \psi(uz)\} \cap \{\psi(x_1x_2), \psi(x_1x_3)\}| \leq 1$, where $z$ is the fourth neighbor of $u$, as shown in Figure 8(a). Without loss of generality, assume that $\psi(uz) \notin \{\psi(x_1x_2), \psi(x_1x_3)\}$. Let $\phi(v_1v_2) = \psi(uz) = 3$ and $\phi(v_2v_3) = \psi(uw) = 4$, as indicated in Figure 8(a). Note, $t_1$, $t_2 \neq 3$. The forbidden colors for $v_1v_1$ are in \{1, 2, 3, 4, $w_1$, $t_1$, $t_2$\}. So there is an admissible color for $w_1$. Next, the forbidden colors for $w_2$ are in \{1, 2, 3, 4, $w_1$, $s$\}. Again, there is an admissible color for $w_2$. The forbidden colors for $w_3$ are in \{1, 2, 3, 4, $w_1$, $w_2$\}, so there is an admissible color for $w_3$.

2. $u = y_3$, $u$ is adjacent to neither $x_1$ nor $x_3$, and $\{\psi(uw), \psi(uz)\} = \{\psi(x_1x_2), \psi(x_1x_3)\},$ where $z$ is the fourth neighbor of $u$. Without loss of generality, we assume that $\psi(x_1x_2) = \psi(uw) = 5$ and $\psi(x_1x_3) = \psi(uz) = 7$. Let $\psi(uw) = 3$, $\phi(v_1v_2) = \psi(yu_1) = 4$, $\phi(v_2v_3) = 5$, and $\phi(v_3v_3) = \psi(y_1y_2) = 6$, as indicated in Figure 8(b). Clearly, the remaining edges $v_1v_1$ and $v_2v_3$ can be colored by any two colors not in the set \{1, 2, 3, \ldots, 7\}.
Figure 8. Subcase B.1.2.
(3) \( u = y_3 \) and \( u = x_3 \) (that is, \( u \) is adjacent to both \( y_1 \) and \( x_1 \)). Let \( \phi(v_1v_2) = \psi(uy_1) = 3 \), \( \phi(v_2v_3) = \psi(uw) = 4 \) and \( \phi(vv_2) = \psi(y_1y_2) = 5 \) as indicated in Figure 8(c). We find admissible colors \( w_1 \) and \( w_2 \). The forbidden colors for \( vv_1 \) are in \{1, 2, 3, 4, 5, 6, 7, \( t_1 \)\}. Hence, there is an admissible color \( w_1 \) for \( vv_1 \). Then the forbidden colors for \( vv_3 \) are in \{1, 2, 3, 4, 5, 6, 7, \( w_1 \)\}. Thus, there is an admissible color \( w_2 \) for \( vv_3 \).

(4) \( u \) is adjacent to \( y_3 \), \( u = x_3 \), and \( \deg_G(y_3) = 3 \). (Symmetrically, \( u \) is adjacent to \( x_3 \), \( u = y_3 \), and \( \deg_G(x_3) = 3 \).) Take \( P = y_1, y_3, u, w, u_4, \ldots, u_l \) as a longest path, and such a graph was discussed in Subcase A.3 (see Figure 6(b), where the positions of \( y_3 \) and \( v \) are switched).

(5) \( u \) is adjacent to \( y_3 \), \( u = x_3 \), and \( \deg_G(y_3) = 4 \). Let \( z \) be the fourth neighbor of \( y_3 \). (Symmetrically, \( u \) is adjacent to \( x_3 \), \( u = y_3 \), and \( \deg_G(x_3) = 4 \).) The reduction \( G' \) and partial labels are shown in Figure 8(d). The forbidden colors for \( vv_2 \) are in \{1, 2, 3, 4, 5, 6, 7\}. Hence, there is an admissible color \( w_1 \) for \( vv_2 \). The forbidden colors for \( y_2z \) are in \{1, 2, 3, 4, 5, 7, \( s_1, s_2 \)\}. Thus, there is an admissible color \( w_2 \) for \( y_2z \). The forbidden colors for \( y_3z \) are from \{1, 2, 3, 4, 5, 6, 7, \( w_2 \)\}, leaving an admissible color \( w_3 \) for \( y_1y_3 \).

(6) \( u \) is adjacent to both \( x_3 \) and \( y_3 \), and \( \deg_G(x_3) = 3 \) or \( \deg_G(y_3) = 3 \). Say \( \deg_G(x_3) = 3 \) (the other case is symmetric). Then take \( P = x_1, x_3, u, w, u_4, \ldots, u_l \) as a longest path, and such case has been discussed in Case A (see Figure 6).

(7) \( u \) is adjacent to both \( x_3 \) and \( y_3 \), and \( \deg_G(x_3) = \deg_G(y_3) = 4 \). The reduction \( G' \) and partial labels are indicated in Figure 8(e). Since \( \deg_G(u_{l-1}) \geq \deg_G(v) = 4 \), we have \( \Delta(G') = \Delta(G) \). The forbidden colors for \( y_2z_1 \) are from \{1, 2, 3, 5, 6, 7, \( s_1, s_2 \)\}. Hence, there is an admissible color \( w_1 \) for \( y_2z_1 \). The forbidden colors for \( y_2y_3 \) are in \{1, 2, 3, 4, 5, 6, 7, \( w_1 \)\}. Thus, there is an admissible color \( w_2 \) for \( y_2y_3 \). The forbidden colors for \( vv_2 \) are from \{1, 2, 3, 4, 5, 6, 7\}. So there is an admissible color \( w_3 \) for \( vv_2 \).

(8) \( u \) is adjacent to \( y_3 \), but not \( x_1 \) or \( x_3 \). Then \( u \) must have another neighbor, say \( z \), besides \( y_3 \), that is a leaf or distance one away from the adjoining cycle \( C \). The position of \( z \) will be similar to the one in Figure 8(b) (where \( z \) might be on the cycle). We then consider the longest path \( P' = y_1, y_3, u, \ldots, u_l \), which falls in one of the cases discussed earlier.

**Case B.2.** \( \Delta(G) \geq 5 \). Obtain the reduction \( G' \) by adding two new edges \( vx_1 \) and \( vy_1 \) to the induced subgraph of \( G \) on the vertex set \( V(G) \setminus \{v_1, v_2, \ldots, v_k\} \), \( k \geq 3 \), as shown in Figure 9. Since \( \deg_G(u_{l-1}) \geq \deg_G(v) \), we have \( \Delta(G') = \Delta(G) \), and \( |C'| < |C| \) holds. Without loss of generality, let \( \phi(v_1x_1) = \psi(vx_1) = 1 \) and \( \phi(v_1y_1) = \psi(vy_1) = 2 \).

For \( u = \psi(y_3) \) (or \( u \) is adjacent to \( y_3 \), respectively), let \( \phi(vv_2) = \psi(y_1y_2) = 3 \) (\( \phi(vv_2) = \psi(y_1y_3) = 3 \), respectively) as indicated in Figure 9(a) (Figure 9(b), respectively). If \( \deg_G(v) = 4 \), then the coloring scheme is the same as the ones used in Subcase B.1.1.
Thus we assume \( \deg_G(v) \geq 5 \). We proceed to color the remaining edges, \( vv_1, vv_3, \ldots, vv_k \) and \( v_jv_{j+1} \), for \( j = 1, 2, \ldots, k - 1 \).

![Figure 9. Case B.2.](image)

For \( u = y_3 \) (see Figure 9(a)), the forbidden colors for \( vv_1 \) are \( \{1, 2, 3, t_1, t_2\} \) and colors used in the neighborhood of \( u \). So there are at most \( \Delta(G) + 5 \leq 2\Delta(G) \) forbidden colors. Hence, there exists an admissible color for \( vv_1 \). Next we color \( vv_k \), which has forbidden colors \( \{1, 2, 3, \phi(vv_1)\} \) and the labels used for edges incident to \( u \). Again, there is an admissible color for \( vv_k \). For \( i = 3, 4, \ldots, k - 1 \), we color \( vv_i \) one after another. By direct calculation, the number of forbidden colors for \( vv_i \) is at most \( \deg_G(u) + \deg_G(v) \). Hence, we can color all \( vv_i \) by admissible colors.

Next we color \( v_1v_2 \), which has forbidden colors \( \{1, t_1, t_2\} \) and colors used in the neighborhood of \( v \). Hence there is an admissible color for \( v_1v_2 \). Next we sequentially color \( v_jv_{j+1} \) for \( j = 2, 3, \ldots, k - 2 \). Using the assumption that \( \Delta(G) \geq 5 \), one can easily verify that there exists an admissible color at each step. Finally, the forbidden colors for \( v_{k-1}v_k \) are \( \{2, s, \phi(v_{k-2}v_{k-1}), \phi(v_{k-3}v_{k-2})\} \) and the labels used in the neighborhood of \( v \). Thus we can find an admissible color for \( v_{k-1}v_k \).

For the case that \( u \) is adjacent to \( y_3 \), the argument is the same except for the edge \( vv_k \), which has forbidden colors from \( \{1, 2, 3, s, \phi(vv_1)\} \) and the labels used by the edges incident to \( u \). As \( \Delta(G) \geq 5 \), we can find an admissible color for \( vv_k \). This completes the proof of Theorem 4.

3. **Proof of Theorem 5**

Let \( G = T \cup C \) be a Halin graph with \( \Delta(G) = 4 \), and let \( G \) be different from a wheel. By Theorem 4, if \( \chi'_s(T) = 7 \), then \( \chi'_s(G) \leq \chi'_s(T) + 2 \). So Theorem 5
holds. Thus we assume $\chi'_s(T) = 6$. That is, every vertex of degree 4 is adjacent to vertices of degree 3 only. Similarly to the previous section, we proceed by induction on $|C|$, the length of $C$. If $|C| = 4$, then $G = W_4$ which contradicts the assumption. If $|C| = 5$, then $T = D_{3,4}$ is a double star. The result follows by Lemma 6. If $|C| = 6$, the only three possible graphs are in Figure 2(a), 2(b), and 2(c). So the result follows.

Similarly to the proof of Theorem 4, we consider a reduction $G' = T' \cup C'$ of $G$ with characteristic tree $T'$ and adjoint cycle $C'$. If $\Delta(G') = 4$ and $G'$ is not a wheel, then $\chi'_s(G') \leq \chi'_s(T') + 2 \leq \chi'_s(T) + 2$ follows by the induction hypothesis, since $|C'| < |C|$. If $G' = W_4$ or if $G'$ is a cubic Halin graph different from $N_{e_2}$, then $\chi'_s(G') \leq 8 = \chi'_s(T) + 2$ by Theorem 3, Lemma 6, and Lemma 7. Finally, the case when $G' = N_{e_2}$ is considered at the end of the proof.

Assume $|C| \geq 7$. Let $P = u_0, u_1, \ldots, u_l$ be a longest path in $T$, where $l$ is the length of $P$. The result holds if $T$ is a double star by Lemma 6 (note that $b \geq 4$). Thus, we assume $l \geq 4$. Without loss of generality, we also assume that $\deg_G(u_1) \leq \deg_G(u_{l-1})$.

Case A. There exists a longest path $P$ with both non-leaf ends of degree 4. That is, $\deg_G(u_1) = \deg_G(u_{l-1}) = 4$. Then $\deg_G(u_2) = 3$. Consider the following two cases.

Case A.1. In $T$, $u_2$ has exactly one neighbor that is a leaf.

The reduction $G'$ along with proposed colors for some edges are depicted in Figure 10. We now find admissible colors $w_1, w_2, w_3, w_4,$ and $w_5$. First we can find an admissible color $w_1$ for $u_1u_2$ that is different from 1, 2 and the colors used in the neighborhood of $u_3$. Next, we can find an admissible color $w_2$ for $v_1v_2$ that is not in $\{1, 2, 3, 4, 5, w_1\}$. Finally, we find three pairwise distinct admissible colors $w_3, w_4, w_5$, which are not in $\{1, 2, 3, w_1, w_2\}$.
Case A.2. In $T$, none of the neighbors of $u_2$ is a leaf.

Without loss of generality, we assume that the colors assigned by $\psi$ to the edges incident to $u_3$ are 3, 4, 5, and 6 (if $u_3$ has degree 3, then we only use colors 3, 4, and 5, and ignore the respective edge labeled by 6 in Figure 11). Consider two possibilities. For the graph depicted in each Figure 11(a) and 11(b) we obtain the reduction $G'$ and complete the labeling $\phi$ by using only eight colors, respectively.

Figure 11. Case A.2.

Case B. Every longest path $P$ has $\text{deg}_G(u_1) = 3$. That is, at least one non-leaf end has degree 3.

Case B.1. $\text{deg}_G(u_2) = 3$.

Subcase B.1.1. In $T$, $u_2$ has exactly one neighbor that is a leaf. The reduction $G'$ along with proposed colors for some edges are depicted in Figure 12. Note if $u_3$ has degree 3, we simply ignore the edge labeled by $t_3$ in Figure 12. We color $u_1u_2$ by a color $w_1$ not from $\{1, 2, 3, t_1, t_2, t_3\}$. Next, color $v_1v_2$ by a color $w_2$.

Figure 12. Subcase B.1.1.
not from \( \{1, 2, 3, 4, 5, w_1\} \). Finally, color \( u_1v_1 \) by an admissible color \( w_3 \) not in \( \{1, 2, 3, w_1, w_2\} \).

Subcase B.1.2. In \( T \), none of the neighbors of \( u_2 \) is a leaf. Then \( u_2 \) has two neighbors, denoted as \( u_1 \) and \( v_4 \), that are distance one away from the adjoining cycle \( C \). First consider the case that \( v_4 \) has degree 4. Then by our assumption of Case B, the degree of the other non-leaf end of the path \( P \) must have degree 3. We consider the reverse order of \( P \), denoted as \( P^* \), as our longest path. That is, \( P^* = u_l, u_{l-1}, u_{l-2}, \ldots, u_1, u_0 \), where \( \deg_G(u_{l-1}) = 3 \). If \( P^* \) falls again in Subcase B.1.2, \( \deg_G(u_{l-2}) = 3 \) and none of the neighbors of \( v_{l-2} \) is a leaf, then by the assumption of Case B, every non-leaf neighbor of \( v_{l-2} \) that is distance two away from the adjoining cycle \( C \) must be degree 3 (for otherwise, there is a longest path with both non-leaf ends of degree 4, which was discussed in Case A).

Therefore, we only need to consider the case that \( \deg_G(v_4) = 3 \), which is shown in Figure 13, where the reduction \( G' \) and partial labels are indicated.

![Figure 13. The second possibility of Subcase B.1.2.](image)

We shall find colors for the remaining edges. First, color \( v_3v_4 \) and \( v_1v_2 \) by two admissible colors \( w_1 \) and \( w_2 \) different from \( \{1, 2, 3, 4, 5\} \). Next, color \( v_2v_4 \) and \( v_1v_2 \) by two admissible colors \( w_3 \) and \( w_4 \) not from \( \{1, 2, 3, w_1, w_2\} \), and assign \( u_1v_1 \) the color \( w_5 = w_1 \). Finally, color \( u_0u_1 \) by an admissible color \( w_6 \) different from \( \{1, 2, 3, w_4, w_5, t_1, t_2\} \). Since we have 8 colors, this can be accomplished.

Case B.2. \( \deg_G(u_2) = 4 \). Then \( \deg_G(u_3) = 3 \).

Subcase B.2.1. In \( T \), \( u_2 \) has exactly two neighbors that are leaves. Consider possible situations depicted in Figure 14. Figure 14(a) shows the situation that the two leaves are adjacent on \( C \). We color \( v_2v_3 \) by a color \( w_1 \) not from the set \( \{1, 2, 3, 4, 5, s_1, s_2\} \). Next, color \( u_2v_2 \) and \( u_1u_2 \) by two colors \( w_2 \) and \( w_3 \) not in \( \{1, 2, 3, 4, 5, w_1\} \).
Figure 14. Five possibilities of Subcase B.2.1.
Now assume that the two leaves are not adjacent on $C$. The length of a longest path from $u_3$ to the adjoint cycle $C$ on one side of $v_1$ is at most three, as $P$ is a longest path. Suppose the length is one. Then there is only one possibility which is shown in Figure 14(b). Color $u_2v_4$ by a color $w_1$ not in $\{1, 2, 3, 4, 5, t_1, t_2\}$. Color $u_2v_2$ by a color $w_2$ not in $\{1, 2, 3, 4, 5, 6, w_1\}$. Finally, color $u_1u_2$ by a color $w_3$ not in $\{1, 2, 3, 4, 5, w_1, w_2\}$.

If there is a path of length two from $u_3$ to the adjoint cycle $C$, then there are two possibilities as shown in Figure 14(c) and Figure 14(d). Assume that the colors used in the neighborhood of $u_4$ are from the set $\{3, 4, 5, 8\}$. We directly color the remaining edges as depicted on those two figures.

Assume that there is a path of length three from $u_3$ to the adjoint cycle $C$ which intersects $P$ only at $u_3$. Let $u_3, v_2, v_1, v_0$ be such a path from $u_3$ to $C$. Then there is another longest path in $T$, $P' = u_1, u_{l-1}, \ldots, u_3, v_2, v_1, v_0$. Assume $\deg_G(v_1) = 4$. By our assumption that every longest path has at least one non-leaf end of degree 3, it must be that $\deg_G(u_{l-1}) = 3$. We then consider $P^*$, the reverse ordering of $P$, namely, $P^* = u_1, u_{l-1}, \ldots, u_3, v_2, v_1, v_0$. Observe that the same situation will not occur to $P^*$, since if $\deg_G(u_{l-2}) = 4$, $\deg_G(u_{l-3}) = 3$, there is a path of length three from $u_{l-3}$ to $C$ (denoted as $u_{l-3}, v_2', v_1', v_0'$), and $\deg_G(v_1') = 4$, then we obtain a longest path $v_0', v_1', v_2', u_{l-3}, \ldots, u_0$ with both non-leaf ends of degree 4, which has been discussed in Case A.

Thus, assume $\deg_G(v_1) = 3$. By symmetry of considering $P$ and $P'$, the only possibility is drawn in Figure 14(e), in which an extended strong edge-coloring is shown using 8 colors.

**Figure 15.** Two possibilities of Subcase B.2.2.

**Subcase B.2.2.** In $T$, $u_2$ has exactly one neighbor that is a leaf. There are two possible situations as shown in Figure 15. In Figure 15(a), a strong edge-coloring is given on the extended edges of $G'$. In Figure 15(b), we color the edges by the following sequence: Color the two edges labeled as $w_1$ by an admissible color not from $\{1, 2, 3, t_1, t_2\}$. Color the two edges labeled as $w_2$ by an admissible color not
from \{1, 2, 3, w_1, s_1, s_2\}. Color the edge labeled as $w_3$ by an admissible color not from \{1, 2, 3, 4, w_1, w_3\}. Finally, color the remaining two edges labeled as $w_4$ and $w_5$ by two different admissible colors not from \{1, 2, 3, w_1, w_2, w_3\}.

**Subcase B.2.3.** In $T$, none of the neighbors of $u_2$ is a leaf. The reduction $G'$ and the completion of $\phi$ using eight colors are demonstrated in Figure 16. This completes all cases.

\[\text{Figure 16. Subcase B.2.3.}\]

We now discuss the situation that the reduction graph $G'$ is $Ne_2$. Notice that this does not occur in Case A. For Subcase B.1.1, if $G' = Ne_2$, then $G$ is a cubic graph, contradicting our assumption that $\chi_s'(T) = 6$. Similarly, for the second possibility in Subcase B.1.2, $G'$ is not $Ne_2$.

These leave a total of fourteen possible situations from the first possibility (Figure 11(b)) of Subcase B.1.2, as well as Subcases B.2.1, B.2.2 and B.2.3, when the reduction graph $G'$ is $Ne_2$. These fourteen situations are depicted in Figure 17, where a strong edge coloring using at most eight colors is given in each situation. This completes the proof of Theorem 5.

For a Halin graph $G = T \cup C$ with maximum degree at most 4 and $G$ is not a wheel, $Ne_2$, nor $Ne_4$, it has been shown that $\chi_s'(G) \leq \chi_s'(T) + 2$, and the bound is sharp (cf. [21] and Theorem 5). We propose

**Conjecture 8.** If $G = T \cup C$ is a Halin graph other than a wheel, $Ne_2$, or $Ne_4$, then $\chi_s'(G) \leq \chi_s'(T) + 2$.

If the answer to Conjecture 8 is affirmative, then the bound is sharp for infinitely many graphs besides the ones mentioned in Lemmas 6 and 7. Let $a, b, c$ be positive integers, $b \geq 4$. A tree $T$ is a **triple star**, denoted by $T = S_{a,b,c}$, if it has exactly three non-leaf vertices which have degrees $a, b, c$ (in this order on a longest path), respectively. We draw $T$ on the plane by fixing a longest path of
Figure 17. Fourteen special graphs.
length four horizontally, \(u_0 - u - v - w - w_0\) (where \(u, v, w\) are non-leaf vertices), and draw at least one pendant edge of \(v\) towards each of the up and down sides of the path. For instance, Figure 1 shows \(T = S_{3,4,4}\). Let \(k \geq 4\) be a positive integer. Similar to the argument for Figure 1, one can show that if \(T = S_{3,k,3}\), then \(\chi'_s(G) = \chi'_s(T) + 2\).

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