UPPER BOUNDS FOR THE STRONG CHROMATIC INDEX OF HALIN GRAPHS

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Abstract

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The strong chromatic index of a graph $G$, denoted by $\chi'_s(G)$, is the minimum number of vertex induced matchings needed to partition the edge set of $G$. Let $T$ be a tree without vertices of degree 2 and have at least one vertex of degree greater than 2. We construct a Halin graph $G$ by drawing $T$ on the plane and then drawing a cycle $C$ connecting all its leaves in such a way that $C$ forms the boundary of the unbounded face. We call $T$ the characteristic tree of $G$. Let $G$ denote a Halin graph with maximum degree $\Delta$ and characteristic tree $T$. We prove that $\chi'_s(G) \leq 2\Delta + 1$ when $\Delta \geq 4$. In addition, we show that if $\Delta = 4$ and $G$ is not a wheel, then $\chi'_s(G) \leq \chi'_s(T) + 2$. A similar result for $\Delta = 3$ was established by Lih and Liu [21].

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1. Introduction

Let $G$ be a simple graph. The distance between two edges $e$ and $e'$ in $G$ is the minimum $k$ for which there is a sequence $e = e_0, e_1, \ldots, e_k = e'$ of distinct edges such that for $1 \leq i \leq k$, $e_{i-1}$ and $e_i$ share an end vertex. A strong edge-coloring of a graph is a function that assigns to each edge a color such that any two edges with distance at most two must receive different colors. A strong $k$-edge-coloring is a strong edge-coloring using $k$ colors. The strong chromatic index of a graph $G$, denoted by $\chi'_s(G)$, is the minimum $k$ such that $G$ admits a strong $k$-edge-coloring. The pre-image of each color in a strong edge-coloring is an induced matching. Thus, the strong chromatic index is also the minimum number of vertex induced matchings needed to partition the edge set of $G$.

Denote the maximum degree of a graph $G$ by $\Delta(G)$ (or, simply by $\Delta$ when $G$ is clear in the context). A trivial upper bound is that $\chi'_s(G) \leq 2\Delta(G)^2 - 2\Delta(G) + 1$. Fouquet and Jolivet [13] established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta(G)^2 - 2\Delta(G)$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [24]. The following conjecture was posed by Erdős and Nešetřil [10, 11].

Conjecture 1. For any graph $G$ of maximum degree $\Delta$,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even}; \\ \frac{3}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd}. \end{cases}$$

For graphs with maximum degree $\Delta(G) = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [18], independently. For $\Delta(G) = 4$, while Conjecture 1 asserts that $\chi'_s(G) \leq 20$, Horák [17] obtained $\chi'_s(G) \leq 23$ and Cranston [8] proved $\chi'_s(G) \leq 22$. For general graphs $G$ with maximum degree $\Delta$, Molloy and Reed [22] showed that $\chi'_s(G) \leq 1.998\Delta^2$. Most recently, this bound has been improved by Bruhn and Joos [4] to $1.93\Delta^2$.

Strong edge-coloring for planar graphs has been investigated by many authors. Fouquet and Jolivet [13, 14] first studied strong edge-coloring for cubic planar graphs. Let $G$ be a planar graph with maximum degree $\Delta$ and girth $g$. Faudree et al. [12] proved that $\chi'_s(G) \leq 4\Delta + 4$. Bensmail et al. [2] established the bound $\chi'_s(G) \leq 3\Delta + 1$ for $g \geq 6$. Hudák et al. [19] showed $\chi'_s(G) \leq 3\Delta$ if $g \geq 7$, and the bound is sharp for some subcubic (that is, $\Delta \leq 3$) planar graphs. Furthermore, Hocquard et al. [16] showed that $\chi'_s(G) \leq 9$ for subcubic planar graphs $G$ which do not contain cycles of lengths 4 or 5. DeOrsey et al. [9] recently reduced this bound to $\chi'_s(G) \leq 5$ if $g \geq 30$. For planar graphs with large girth, Borodin and Ivanova [3] established a rather tight bound $\chi'_s(G) \leq 2\Delta - 1$ if $g \geq 40(\Delta/2) + 1$; Chang et al. [7] further confirmed that the bound also holds if $g \geq 10\Delta + 46$. Clearly, the bound $\chi'_s(G) \leq 2\Delta - 1$ becomes sharp when $G$ contains two adjacent vertices of maximum degree $\Delta$. 
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By definition, a trivial lower bound of $\chi'_s(G)$ for a graph $G$ would be $\sigma(G)$, where

$$\sigma(G) := \max\{\deg_G(u) + \deg_G(v) - 1 \mid uv \in E(G)\}.$$  

If $G$ has no edges, then define $\sigma(G) = 0$. It is known and easy to verify that for a tree $T$, we have $\chi'_s(T) = \sigma(T)$. Wu and Lin [25] proved that if $\sigma(G) \leq 4$ and $G$ is not isomorphic to the graph of the 5-cycle with a chord connecting two non-adjacent vertices, then $\chi'_s(G) \leq \chi'_s(T) + 2$.

Let $G$ be different from a wheel. Then $\chi'_s(G) \leq \chi'_s(T) + 2$.

A Halin graph is a plane graph $G$ constructed as follows. Let $T$ be a tree with at least 4 vertices, called the characteristic tree of $G$. All vertices of $T$ are either of degree 1, called leaves, or of degree at least 3. We draw $T$ on the plane. Let $C$ be a cycle, called the adjoint cycle of $G$, connecting all leaves of $T$ in such a way that $C$ forms the boundary of the unbounded face. We usually write $G = T \cup C$ to reveal the characteristic tree and the adjoint cycle. For $n \geq 3$, the wheel $W_n$ with $n + 1$ vertices is a particular Halin graph whose characteristic tree is the complete bipartite graph $K_{1,n}$ (called a star). A graph is said to be cubic if the degree of every vertex is 3. For $h \geq 1$, a cubic Halin graph $Ne_h$, called a necklace, was introduced in [23]. Its characteristic tree $T$ consists of the path $v_0, v_1, \ldots, v_h$, $v_{h+1}$ and leaves $v'_1, v'_2, \ldots, v'_h$ such that the unique neighbor of $v'_i$ in $T$ is $v_i$ for $1 \leq i \leq h$ and vertices $v_0, v'_1, \ldots, v'_h, v_{h+1}$ are connected in this order to form the adjoint cycle $C_{h+2}$.

Lai, Lih and Tsai [20] proved the following result.

**Theorem 2** [20]. If a Halin graph $G = T \cup C$ is different from a certain necklace $Ne_2$ and any wheel $W_n$, $n \not\equiv 0$ (mod 3), then $\chi'_s(G) \leq \chi'_s(T) + 3$.

For cubic Halin graphs, Lih and Liu improved the above bound as follows.

**Theorem 3** [21]. A cubic Halin graph $G$ different from $Ne_2$ or $Ne_4$ satisfies $\chi'_s(G) \leq 7$.

The exact values of $\chi'_s(G)$ for special families of cubic Halin graphs were determined by Shiu and Tam [24] and by Chang and Liu [6].

For a Halin graph $G = T \cup C$ with maximum degree $\Delta$, since $\chi'_s(T) \leq 2\Delta - 1$, the bound in Theorem 2 implies that $\chi'_s(G) \leq 2\Delta + 2$. We improve this bound and establish a result similar to Theorem 3 for Halin graphs of maximum degree 4.

**Theorem 4.** Let $G$ be a Halin graph with maximum degree $\Delta \geq 4$. Then $\chi'_s(G) \leq 2\Delta + 1$.

**Theorem 5.** Let $G = T \cup C$ be a Halin graph with maximum degree $\Delta = 4$, and let $G$ be different from a wheel. Then $\chi'_s(G) \leq \chi'_s(T) + 2$.
Both bounds in Theorems 4 and 5 are sharp. Consider the graph $G$ in Figure 1. A strong edge-coloring of $G$ must use at least 7 colors on the edges incident to $u$ or $v$. Let these colors be $\{1, 2, \ldots, 7\}$. Next, since the edges $w_1$ and $w_2$ must use colors different from $\{1, 2, \ldots, 7\}$, at least 8 colors are needed. Assume that we only have 8 colors. Then $w_1$ and $w_2$ must be colored by the same new color, say color 8. This implies that the four edges $e_1, e_2, e_3, e_4$ shown in Figure 1 only have three admissible colors, from the set $\{5, 6, 7\}$, which is a contradiction as these edges must receive different colors. Hence $\chi_s'(G) \geq 9$. By coloring $e_1, e_2, e_3, e_4$ with colors 5, 6, 7, 9 and the last edge with color 4, it follows that $\chi_s'(G) = 9$. This example shows that both bounds in Theorems 4 and 5 are sharp.

![Figure 1](image.png)

Figure 1. An example showing sharp bounds of Theorems 4 and 5.

2. PROOF OF THEOREM 4

A double star is a tree with exactly two non-leaf vertices. Denote by $D_{a,b}$ a double star, where $a, b$ are the degrees of the two non-leaf vertices and $a \leq b$. Prior to the proof of Theorem 4, we quote several known results as follows.

**Lemma 6** [20]. Let $G = T \cup C$ be a Halin graph. If $T = D_{a,b}$ is a double star with $a \leq b$, then

\[
\chi_s'(G) = \begin{cases} 
\chi_s'(T) + 4 & \text{if } a = b = 3; \\
\chi_s'(T) + 2 & \text{if } a = 3 \text{ and } b \geq 4; \\
\chi_s'(T) + 1 & \text{if } a \geq 4.
\end{cases}
\]

If $T = K_{1,k}$ (that is, $G$ is a wheel $W_k$), then

\[
\chi_s'(W_k) = \begin{cases} 
k + 3 & \text{if } k \equiv 0 \pmod{3}; \\
k + 5 & \text{if } k = 5; \\
k + 4 & \text{otherwise}.
\end{cases}
\]
Lemma 7 [23]. Suppose \( h \geq 1 \). Then
\[
\chi'_s(Ne_h) = \begin{cases} 
6 & \text{if } h \text{ is odd;} \\
7 & \text{if } h \geq 6 \text{ and } h \text{ is even;} \\
8 & \text{if } h = 4; \\
9 & \text{if } h = 2.
\end{cases}
\]

Proof of Theorem 4. Let \( G = T \cup C \) be a Halin graph with \( \Delta(G) \geq 4 \). If \( T \) is a star or a double star, by Lemma 6, the conclusion of Theorem 4 follows. Assume that \( T \) is neither a star nor a double star. We proceed by induction on \(|C|\), the length of \( C \). The shortest length of \( C \) is 6. Three possible graphs along with their corresponding strong edge-colorings satisfying the desired upper bounds are shown in Figure 2. So the result follows.

![Figure 2. All Halin graphs with \(|C| = 6\) and \(\Delta(G) = 4\).](image)

Assume \(|C| \geq 7\). Let \( P = u_0, u_1, \ldots, u_l \) be a longest path in \( T \) with length \( l \). As \( T \) is neither a star nor a double star, so \( l \geq 4 \). Without loss of generality, we assume \( \deg_G(u_{l-1}) \geq \deg_G(u_1) \).

Denote \( u_1 = v, u_2 = u, u_3 = w \), and label the \( k \geq 2 \) leaf neighbors of \( v \) as \( v_1, v_2, \ldots, v_k \). Since \( P \) is a longest path in \( T \), it is easy to see that \( v_1, v_2, \ldots, v_k \) must be on the adjoint cycle \( C \). Let \( x_1, x_2, y_1, y_2 \) be vertices on \( C \), where \( x_1 \) is adjacent to \( v_1 \) and \( x_2 \); \( y_1 \) is adjacent to \( v_k \) and \( y_2 \). Let \( x_3 \) and \( y_3 \) be vertices not on \( C \), where \( x_1x_3 \) and \( y_1y_3 \) are edges in \( T \) (see Figure 3).

Since \( G \) is a Halin graph and \( u \) is a vertex of degree at least 3, there exists a path \( P' \) in \( T \) from \( u \) to \( x_1 \) or from \( u \) to \( y_1 \) with \( P \cap P' = \{u\} \). Without loss of generality, we shall assume that \( P' \) is from \( u \) to \( y_1 \). By our assumption that \( P \) is a longest path, it must be that \(|P'| \leq 2 \). Thus, either \( u = y_3 \) or \( u \) is adjacent to \( y_3 \).

In the following, we denote by \( G' = T' \cup C' \) the Halin graph obtained by adding some new edges to an induced subgraph of \( G \) such that \(|C'| < |C|\) and \( \Delta(G') \leq \Delta(G) \). If \( \Delta(G') \geq 4 \), then \( \chi'_s(G') \leq 2\Delta(G) + 1 \) holds because \( T' \) is a star or double star (see the beginning of the proof) or by the inductive hypothesis as \(|C'| < |C|\). If \( \Delta(G') = 3 \), then \( \chi'_s(G') \leq 9 \leq 2\Delta(G) + 1 \) by Theorem 2,
Lemma 6, and because $\Delta(G) \geq 4$. In the following case analysis these steps will be repeatedly used, while may not be mentioned explicitly all the time.

![Figure 3. The neighborhood around one end of the longest path $P$.](image)

We call $G'$ a reduction of $G$. Depending on various situations, different types of $G'$ are created. In the corresponding figures, the dashed lines represent new edges added in $G'$, and dark vertices represent the vertices that are temporarily deleted from $G$.

Let $\psi$ be a strong edge-coloring of $G'$ using the minimum number of colors. A strong edge-coloring $\phi$ of $G$ is obtained as follows. We color the edges that are in both $G$ and $G'$ by the same colors used in $\psi$, i.e., let $\phi(e) = \psi(e)$ for every $e \in E(G) \cap E(G')$. For edges in $e \in E(G) \setminus E(G')$, we develop different coloring schemes for different cases, and in each case, we give a strong edge-coloring $\phi$ for $G$ with at most $2\Delta(G) + 1$ colors.

**Case A.** $\deg_G(v) = 3$. There are three possibilities to consider.

**Case A.1.** $u = y_3$. Obtain the reduction $G'$ of $G$ by adding two new edges $vx_1$ and $vy_1$ to the induced subgraph of $G$ on the vertex set $V(G) \setminus \{v_1, v_2\}$, as indicated in Figure 4. Clearly, $\Delta(G') = \Delta(G) \geq 4$ and $|C'| < |C|$. Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = 1$ and $\phi(v_2y_1) = 2$ (see Figure 4). We find admissible colors $w_1, w_2,$ and $w_3$, one by one. The colors that can not be assigned to $vv_1$ are from $\{1, 2, t_1, t_2\}$ and the labels used by edges incident to $u$. Therefore, there are at most $\Delta(G) + 4$ forbidden colors for $vv_1$. Since $\Delta(G) \geq 4$, there exists an admissible color for $vv_1$. Color $vv_1$ by such an admissible color $w_1$.

Next we color $vv_2$ which has the forbidden colors in $\{1, 2, w_1, s\}$ and the labels used for edges incident to $u$. Similarly, we can find an admissible color for $vv_2$. Finally, the forbidden colors for $v_1v_2$ are in $\{1, 2, w_1, w_2, r_1, r_2, s, t_1, t_2\}$. If $s \in \{t_1, t_2\}$, then there is an admissible color for $v_1v_2$. Otherwise, we re-color $vv_1$ by $s$, creating an admissible color for $v_1v_2$. 


Case A.2. $u$ is adjacent to $y_3$, and $\Delta(G) \geq 5$. Obtain the reduction $G'$ in the same way as in Case A.1, as indicated in Figure 5. Clearly, $\Delta(G') = \Delta(G) \geq 4$ and $|C'| < |C|$.

Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(vx_1) = 1$ and $\phi(vy_1) = 2$ (see Figure 5). We find admissible colors $w_1$, $w_2$, and $w_3$, one by one. By the same argument as in Case A.1, one can easily show that there exists an admissible color $w_1$. Color $vv_1$ by such an admissible color.

Next we color $vv_2$ which has the forbidden colors in $\{1, 2, w_1, s_1, s_2\}$ and the labels used for edges incident to $u$. Since $\Delta(G) \geq 5$, we can find an admissible color $w_2$. Finally, the forbidden colors for $v_1v_2$ are in $\{1, 2, w_1, w_2, r, s_1, s_2, t_1, t_2\}$. Thus, there exists an admissible color $w_3$.

Case A.3. $u$ is adjacent to $y_3$, and $\Delta(G) = 4$. Then $\deg_G(y_3)$ is either 3 or 4. Obtain the reduction $G'$ from $G$ with partial labels to some vertices as indicated in Figure 6(a) and 6(b), respectively. Clearly, $\Delta(G') \leq \Delta(G)$ and $|C'| < |C|$. Assume that $\deg_G(y_3) = 3$. Then $\Delta(G') = \Delta(G) = 4$. We find
admissible colors $w_1$, $w_2$, and $w_3$, one after another. For $v_1 v_2$, the forbidden colors are in $\{1, 2, 3, r_1, t_1, t_2\}$. Hence there is an admissible color $w_1$ for $v_1 v_2$. Next, the forbidden colors for $y_1 y_2$ are in $\{1, 2, 3, w_1, w_2, s_1, s_2\}$. We can color $y_1 y_2$ by an admissible color $w_2$. Finally, the forbidden colors for $v_2 y_1$ are in $\{1, 2, 3, w_1, w_2, r_1, r_2\}$. Again, there exists an admissible color $w_3$ for $v_2 y_1$.

![Diagram](image)

Figure 6. Case A.3.

Assume $\deg_G(y_3) = 4$. Note, even if $\Delta(G') = 3$ or $T'$ is a star (or double star), we can still find a strong edge coloring for $G'$ by up to 9 colors. The forbidden colors for $y_1 y_3$ are in $\{1, 2, 3\}$ and labels used on edges incident to $u$. Thus there are at most $\Delta(G) + 3$ forbidden colors. We color $y_1 y_3$ by an admissible color $w_1$. Next, the forbidden colors for $v_1 v_2$ are $\{1, 2, 3, 4, w_1, t_1, t_2\}$. Because $2\Delta(G) + 1 \geq 9$, we can find an admissible color $w_2$ for $v_1 v_2$. The forbidden colors for $y_2 z$ are in $\{1, 2, 3, 4, w_1, r, s_1, s_2\}$. Again, there is an admissible color $w_3$ for $y_2 z$. Finally, the forbidden colors for $v_2 y_1$ are from $\{1, 2, 3, 4, w_1, w_2, w_3, r\}$. So there is an admissible color $w_4$ for $v_2 y_1$.

**Case B. $\deg_G(v) \geq 4$.** We consider two cases separately.

**Case B.1. $\Delta(G) = 4$.** Then $\deg_G(v) = 4$. There are two subcases.

**Subcase B.1.1. $\deg_G(u) = 3$.** Obtain the reduction $G'$ of $G$ by adding two new edges $v x_1$ and $v y_1$ to the induced subgraph of $G$ on the vertex set $V(G) \setminus \{v_1, v_2, v_3\}$ as depicted in Figure 7.

Since we assumed earlier that $\deg_G(u_{l-1}) \geq \deg_G(u_1) = \deg_G(v) = 4$, we have $\Delta(G') = \Delta(G) = 4$, and $|C'| < |C|$ holds. We fix colors on some edges as shown in Figure 7. Note that in Figure 7(a) we assign $\phi(y_1 y_2) = \phi(v v_2) = 3$ but in Figure 7(b) we assign $\phi(y_1 y_3) = \phi(v v_2) = 3$ and $\phi(y_1 y_2) = s$. We find admissible colors $w_1$, $w_2$, $w_3$, and $w_4$. 

For the subcase depicted in Figure 7(a), the forbidden colors for \(v_1v_1\) are in \(\{1, 2, 3, t_1, t_2\}\) and the three colors used in the neighborhood of \(u\). Thus, there are at most 8 forbidden colors, implying there is an admissible color \(w_1\) for \(v_1v_1\). Next, the forbidden colors for \(v_2v_2\) are in \(\{1, 2, 3, w_1\}\) and the three colors used in the neighborhood of \(u\). There is an admissible color \(w_2\) for \(v_2v_2\). The forbidden colors for \(v_1v_2\) are in \(\{1, 2, 3, w_1, w_2, r_1, t_1, t_2\}\), so there is an admissible color \(w_3\) for \(v_1v_2\). Finally, the forbidden colors for \(v_2v_3\) are in \(\{1, 2, 3, w_1, w_2, w_3, r_1, r_2\}\). Therefore, there is an admissible color \(w_4\) for \(v_2v_3\).

For the subcase depicted in Figure 7(b), the arguments are the same as in Figure 7(a) except for \(v_3v_3\), which has forbidden colors from \(\{1, 2, 3, w_1, r_2\}\) and the three colors used in the neighborhood of \(u\). So there is an admissible color \(w_2\) for \(v_3v_3\).

**Subcase B.1.2.** \(\deg_G(u) = 4\). We distinguish several cases. In each case \(\Delta(G') \leq \Delta(G)\) and \(|C'| < |C|\) hold.

1. \(u = y_1\), \(u\) is adjacent to neither \(x_1\) nor \(x_3\), and \(|\{\psi(uw), \psi(uz)\} \cap \{\psi(x_1x_2), \psi(x_1x_3)\}| \leq 1\), where \(z\) is the fourth neighbor of \(u\), as shown in Figure 8(a). Without loss of generality, assume that \(\psi(uz) \notin \{\psi(x_1x_2), \psi(x_1x_3)\}\). Let \(\phi(v_1v_2) = \psi(ux) = 3\) and \(\phi(v_2v_3) = \psi(uw) = 4\), as indicated in Figure 8(a). Note, \(t_1, t_2 \neq 3\). The forbidden colors for \(v_1v_1\) are in \(\{1, 2, 3, 4, 5, 6, t_1, t_2\}\). So there is an admissible color for \(w_1\). Next, the forbidden colors for \(w_2\) are in \(\{1, 2, 3, 4, 5, 6, w_1, s\}\). Again, there is an admissible color for \(w_2\). The forbidden colors for \(w_3\) are in \(\{1, 2, 3, 4, 5, 6, w_1, w_2\}\), so there is an admissible color for \(w_3\).

2. \(u = y_2\), \(u\) is adjacent to neither \(x_1\) nor \(x_3\), and \(\{\psi(uw), \psi(uz)\} = \{\psi(x_1x_2), \psi(x_1x_3)\}\), where \(z\) is the fourth neighbor of \(u\). Without loss of generality, we assume that \(\psi(x_1x_2) = \psi(uw) = 5\) and \(\psi(x_1x_3) = \psi(uz) = 7\). Let \(\psi(uw) = 3\), \(\phi(v_1v_2) = \psi(y_1y_1) = 4\), \(\phi(v_2v_3) = 5\), and \(\phi(uw) = \psi(y_1y_2) = 6\), as indicated in Figure 8(b). Clearly, the remaining edges \(v_1v_1\) and \(v_2v_2\) can be colored by any two colors not in the set \(\{1, 2, 3, \ldots, 7\}\).
(a) Condition (1).

(b) Condition (2).

(c) Condition (3).

(d) Condition (5).

(e) Condition (7).

Figure 8. Subcase B.1.2.
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(3) \( u = y_3 \) and \( u = x_3 \) (that is, \( u \) is adjacent to both \( y_1 \) and \( x_1 \)). Let \( \phi(v_1v_2) = \psi(uw_1) = 3 \), \( \phi(v_2v_3) = \psi(uw) = 4 \) and \( \phi(v_3v_2) = \psi(y_1y_2) = 5 \) as indicated in Figure 8(c). We find admissible colors \( w_1 \) and \( w_2 \). The forbidden colors for \( vv_1 \) are in \( \{1, 2, 3, 4, 5, 6, 7, t_1\} \). Hence, there is an admissible color \( w_1 \) for \( vv_1 \). Then the forbidden colors for \( vv_3 \) are in \( \{1, 2, 3, 4, 5, 6, 7, w_1\} \). Thus, there is an admissible color \( w_2 \) for \( vv_3 \).

(4) \( u \) is adjacent to \( y_3 \), \( u = x_3 \), and \( \deg_G(y_3) = 3 \). (Symmetrically, \( u \) is adjacent to \( x_3 \), \( u = y_3 \), and \( \deg_G(x_3) = 3 \).) Take \( P = y_1, y_3, u, w, u_4, \ldots, u_l \) as a longest path, and such a graph was discussed in Subcase A.3 (see Figure 6(b), where the positions of \( y_3 \) and \( v \) are switched).

(5) \( u \) is adjacent to \( y_3 \), \( u = x_3 \), and \( \deg_G(y_3) = 4 \). Let \( z \) be the fourth neighbor of \( y_3 \). (Symmetrically, \( u \) is adjacent to \( x_3 \), \( u = y_3 \), and \( \deg_G(x_3) = 4 \).) The reduction \( G' \) and partial labels are shown in Figure 8(d). The forbidden colors for \( vv_2 \) are in \( \{1, 2, 3, 4, 5, 6, 7\} \). Hence, there is an admissible color \( w_1 \) for \( vv_2 \). The forbidden colors for \( y_2z \) are in \( \{1, 2, 3, 4, 5, 7, s_1, s_2\} \). Thus, there is an admissible color \( w_2 \) for \( y_2z \). The forbidden colors for \( y_1y_3 \) are from \( \{1, 2, 3, 4, 5, 6, 7, w_2\} \), leaving an admissible color \( w_3 \) for \( y_1y_3 \).

(6) \( u \) is adjacent to both \( x_3 \) and \( y_3 \), and \( \deg_G(x_3) = 3 \) or \( \deg_G(y_3) = 3 \). Say \( \deg_G(x_3) = 3 \) (the other case is symmetric). Then take \( P = x_1, x_3, u, w, u_4, \ldots, u_l \) as a longest path, and such case has been discussed in Case A (see Figure 6).

(7) \( u \) is adjacent to both \( x_3 \) and \( y_3 \), and \( \deg_G(x_3) = \deg_G(y_3) = 4 \). The reduction \( G' \) and partial labels are indicated in Figure 8(e). Since \( \deg_G(u_{l-1}) \geq \deg_G(v) = 4 \), we have \( \Delta(G') = \Delta(G) \). The forbidden colors for \( y_2z_1 \) are from \( \{1, 2, 3, 4, 5, 6, 7, s_1, s_2\} \). Hence, there is an admissible color \( w_1 \) for \( y_2z_1 \). The forbidden colors for \( y_2y_3 \) are in \( \{1, 2, 3, 4, 5, 6, 7, w_1\} \). Thus, there is an admissible color \( w_2 \) for \( y_2y_3 \). The forbidden colors for \( vv_2 \) are from \( \{1, 2, 3, 4, 5, 6, 7\} \). So there is an admissible color \( w_3 \) for \( vv_2 \).

(8) \( u \) is adjacent to \( y_3 \), but not \( x_1 \) nor \( x_3 \). Then \( u \) must have another neighbor, say \( z \), besides \( y_3 \), that is a leaf or distance one away from the adjoining cycle \( C \). The position of \( z \) will be similar to the one in Figure 8(b) (where \( z \) might be on the cycle). We then consider the longest path \( P' = y_1, y_3, u, \ldots, u_l \), which falls in one of the cases discussed earlier.

Case B.2. \( \Delta(G) \geq 5 \). Obtain the reduction \( G' \) by adding two new edges \( vx_1 \) and \( vy_1 \) to the induced subgraph of \( G \) on the vertex set \( V(G) \setminus \{v_1, v_2, \ldots, v_k\} \), \( k \geq 3 \), as shown in Figure 9. Since \( \deg_G(u_{l-1}) \geq \deg_G(v) \), we have \( \Delta(G) = \Delta(G') \), and \( |C'| < |C| \) holds. Without loss of generality, let \( \phi(v_1x_1) = \psi(vx_1) = 1 \) and \( \phi(v_ky_1) = \psi(vy_1) = 2 \).

For \( u = y_3 \) (or \( u \) is adjacent to \( y_3 \), respectively), let \( \phi(vv_2) = \psi(y_1y_2) = 3 \) (\( \phi(vv_2) = \psi(y_1y_3) = 3 \), respectively) as indicated in Figure 9(a) (Figure 9(b), respectively). If \( \deg_G(v) = 4 \), then the coloring scheme is the same as the ones used in Subcase B.1.1.
Thus we assume \( \deg_G(v) \geq 5 \). We proceed to color the remaining edges, \( vv_1, vv_3, \ldots, vv_k \) and \( v_jv_{j+1} \), for \( j = 1, 2, \ldots, k-1 \).

For \( u = y_3 \) (see Figure 9(a)), the forbidden colors for \( vv_1 \) are \( \{1, 2, 3, t_1, t_2\} \) and colors used in the neighborhood of \( u \). So there are at most \( \Delta(G)+5 \leq 2\Delta(G) \) forbidden colors. Hence, there exists an admissible color for \( vv_1 \). Next we color \( vv_k \), which has forbidden colors \( \{1, 2, 3, \phi(vv_1)\} \) and the labels used for edges incident to \( u \). Again, there is an admissible color for \( vv_k \). For \( i = 3, 4, \ldots, k-1 \), we color \( vv_i \) one after another. By direct calculation, the number of forbidden colors for \( vv_i \) is at most \( \deg_G(u)+\deg_G(v) \). Hence, we can color all \( vv_i \) by admissible colors.

Next we color \( v_1v_2 \), which has forbidden colors \( \{1, t_1, t_2\} \) and colors used in the neighborhood of \( v \). Hence there is an admissible color for \( v_1v_2 \). Next we sequentially color \( v_jv_{j+1} \) for \( j = 2, 3, \ldots, k-2 \). Using the assumption that \( \Delta(G) \geq 5 \), one can easily verify that there exists an admissible color at each step. Finally, the forbidden colors for \( v_{k-1}v_k \) are \( \{2, s, \phi(v_{k-2}v_{k-1}), \phi(v_{k-3}v_{k-2})\} \) and the labels used in the neighborhood of \( v \). Thus we can find an admissible color for \( v_{k-1}v_k \).

For the case that \( u \) is adjacent to \( y_3 \), the argument is the same except for the edge \( vv_k \), which has forbidden colors from \( \{1, 2, 3, s, \phi(vv_1)\} \) and the labels used by the edges incident to \( u \). As \( \Delta(G) \geq 5 \), we can find an admissible color for \( vv_k \). This completes the proof of Theorem 4.

3. Proof of Theorem 5

Let \( G = T \cup C \) be a Halin graph with \( \Delta(G) = 4 \), and let \( G \) be different from a wheel. By Theorem 4, if \( \chi'_s(T) = 7 \), then \( \chi'_s(G) \leq \chi'_s(T) + 2 \). So Theorem 5
holds. Thus we assume $\chi'(T) = 6$. That is, every vertex of degree 4 is adjacent to vertices of degree 3 only. Similarly to the previous section, we proceed by induction on $|C|$, the length of $C$. If $|C| = 4$, then $G = W_4$ which contradicts the assumption. If $|C| = 5$, then $T = D_{3,4}$ is a double star. The result follows by Lemma 6. If $|C| = 6$, the only three possible graphs are in Figure 2(a), 2(b), and 2(c). So the result follows.

Similarly to the proof of Theorem 4, we consider a reduction $G' = T' \cup C'$ of $G$ with characteristic tree $T'$ and adjoint cycle $C'$. If $\Delta(G') = 4$ and $G'$ is not a wheel, then $\chi'(G') \leq \chi'_s(T') + 2 \leq \chi'_s(T) + 2$ follows by the induction hypothesis, since $|C'| < |C|$. If $G' = W_4$ or if $G'$ is a cubic Halin graph different from $N_{e_2}$, then $\chi'_s(G') \leq 8 = \chi'_s(T) + 2$ by Theorem 3, Lemma 6, and Lemma 7. Finally, the case when $G' = N_{e_2}$ is considered at the end of the proof.

Assume $|C| \geq 7$. Let $P = u_0, u_1, \ldots, u_l$ be a longest path in $T$, where $l$ is the length of $P$. The result holds if $T$ is a double star by Lemma 6 (note that $b \geq 4$). Thus, we assume $l \geq 4$. Without loss of generality, we also assume that $\deg_G(u_1) \leq \deg_G(u_{l-1})$.

**Case A.** There exists a longest path $P$ with both non-leaf ends of degree 4. That is, $\deg_G(u_1) = \deg_G(u_{l-1}) = 4$. Then $\deg_G(u_2) = 3$. Consider the following two cases.

**Case A.1.** In $T$, $u_2$ has exactly one neighbor that is a leaf.

The reduction $G'$ along with proposed colors for some edges are depicted in Figure 10. We now find admissible colors $w_1, w_2, w_3, w_4, w_5$. First we can find an admissible color $w_1$ for $u_1u_2$ that is different from 1, 2, and the colors used in the neighborhood of $u_3$. Next, we can find an admissible color $w_2$ for $v_1v_2$ that is not in $\{1, 2, 3, 4, 5, w_1\}$. Finally, we find three pairwise distinct admissible colors $w_3, w_4, w_5$, which are not in $\{1, 2, 3, w_1, w_2\}$. 

![Figure 10](image-url)
Case A.2. In $T$, none of the neighbors of $u_2$ is a leaf.

Without loss of generality, we assume that the colors assigned by $\psi$ to the edges incident to $u_3$ are 3, 4, 5, and 6 (if $u_3$ has degree 3, then we only use colors 3, 4, and 5, and ignore the respective edge labeled by 6 in Figure 11). Consider two possibilities. For the graph depicted in each Figure 11(a) and 11(b) we obtain the reduction $G'$ and complete the labeling $\phi$ by using only eight colors, respectively.

Figure 11. Case A.2.

Case B. Every longest path $P$ has $\deg_G(u_1) = 3$. That is, at least one non-leaf end has degree 3.

Case B.1. $\deg_G(u_2) = 3$.

Subcase B.1.1. In $T$, $u_2$ has exactly one neighbor that is a leaf. The reduction $G'$ along with proposed colors for some edges are depicted in Figure 12. Note if $u_3$ has degree 3, we simply ignore the edge labeled by $t_3$ in Figure 12. We color $u_1u_2$ by a color $w_1$ not from \{1, 2, 3, $t_1$, $t_2$, $t_3$\}. Next, color $v_1v_2$ by a color $w_2$.
not from \{1, 2, 3, 4, 5, w_1\}. Finally, color \(u_1v_1\) by an admissible color \(w_3\) not in \{1, 2, 3, w_1, w_2\}.

**Subcase B.1.2.** In \(T\), none of the neighbors of \(u_2\) is a leaf. Then \(u_2\) has two neighbors, denoted as \(u_1\) and \(v_2\), that are distance one away from the adjoining cycle \(C\). First consider the case that \(v_4\) has degree 4. Then by our assumption of Case B, the degree of the other non-leaf end of the path \(P\) must have degree 3. We consider the reverse order of \(P\), denoted as \(P^*\), as our longest path. That is, \(P^* = u_l, u_l-1, u_l-2, \ldots, u_1, u_0\), where \(\deg_G(u_{l-1}) = 3\). If \(P^*\) falls again in Subcase B.1.2, \(\deg_G(u_{l-2}) = 3\) and none of the neighbors of \(u_{l-2}\) is a leaf, then by the assumption of Case B, every non-leaf neighbor of \(v_{l-2}\) that is distance two away from the adjoining cycle \(C\) must be degree 3 (for otherwise, there is a longest path with both non-leaf ends of degree 4, which was discussed in Case A).

Therefore, we only need to consider the case that \(\deg_G(v_4) = 3\), which is shown in Figure 13, where the reduction \(G'\) and partial labels are indicated.

![Figure 13. The second possibility of Subcase B.1.2.](image)

We shall find colors for the remaining edges. First, color \(v_3v_4\) and \(v_1v_2\) by two admissible colors \(w_1\) and \(w_2\) different from \{1, 2, 3, 4, 5\}. Next, color \(v_2v_4\) and \(v_1v_2\) by two admissible colors \(w_3\) and \(w_4\) not from \{1, 2, 3, w_1, w_2\}, and assign \(u_1v_1\) the color \(w_5 = w_1\). Finally, color \(u_0u_1\) by an admissible color \(w_6\) different from \{1, 2, 3, w_4, w_5, t_1, t_2\}. Since we have 8 colors, this can be accomplished.

**Case B.2.** \(\deg_G(u_2) = 4\). Then \(\deg_G(u_3) = 3\).

**Subcase B.2.1.** In \(T\), \(u_2\) has exactly two neighbors that are leaves. Consider possible situations depicted in Figure 14. Figure 14(a) shows the situation that the two leaves are adjacent on \(C\). We color \(v_2v_3\) by a color \(w_1\) not from the set \{1, 2, 3, 4, 5, s_1, s_2\}. Next, color \(u_2v_2\) and \(u_1u_2\) by two colors \(w_2\) and \(w_3\) not in \{1, 2, 3, 4, 5, w_1\}.  

...
Figure 14. Five possibilities of Subcase B.2.1.
Now assume that the two leaves are not adjacent on $C$. The length of a longest path from $v_3$ to the adjoint cycle $C$ on one side of $v_1$ is at most three, as $P$ is a longest path. Suppose the length is one. Then there is only one possibility which is shown in Figure 14(b). Color $u_2v_4$ by a color $w_1$ not in $\{1, 2, 3, 4, 5, t_1, t_2\}$. Color $u_2v_2$ by a color $w_2$ not in $\{1, 2, 3, 4, 5, 6, w_1\}$. Finally, color $u_1u_2$ by a color $w_3$ not in $\{1, 2, 3, 4, 5, w_1, w_2\}.

If there is a path of length two from $u_3$ to the adjoint cycle $C$, then there are two possibilities as shown in Figure 14(c) and Figure 14(d). Assume that the colors used in the neighborhood of $u_4$ are from the set $\{3, 4, 5, 8\}$. We directly color the remaining edges as depicted on those two figures.

Assume that there is a path of length three from $u_3$ to the adjoint cycle $C$ which intersects $P$ only at $u_3$. Let $u_3, v_2, v_1, v_0$ be such a path from $u_3$ to $C$. Then there is another longest path in $T$, $P' = u_1, u_{l-1}, \ldots, u_3, v_2, v_1, v_0$. Assume $\deg_G(v_1) = 4$. By our assumption that every longest path has at least one non-leaf end of degree 3, it must be that $\deg_G(u_{l-1}) = 3$. We then consider $P^*$, the reverse ordering of $P$, namely, $P^* = u_1, u_{l-1}, \ldots, u_3, v_2, v_1, v_0$. Observe that the same situation will not occur to $P^*$, since if $\deg_G(u_{l-2}) = 4$, $\deg_G(u_{l-3}) = 3$, there is a path of length three from $u_{l-3}$ to $C$ (denoted as $u_{l-3}, v'_2, v'_1, v'_0$), and $\deg_G(v'_1) = 4$, then we obtain a longest path $v'_0, v'_1, v'_2, u_{l-3}, \ldots, u_0$ with both non-leaf ends of degree 4, which has been discussed in Case A.

Thus, assume $\deg_G(v_1) = 3$. By symmetry of considering $P$ and $P'$, the only possibility is drawn in Figure 14(e), in which an extended strong edge-coloring is shown using 8 colors.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure15.png}
\caption{Two possibilities of Subcase B.2.2.}
\end{figure}

**Subcase B.2.2.** In $T$, $u_2$ has exactly one neighbor that is a leaf. There are two possible situations as shown in Figure 15. In Figure 15(a), a strong edge-coloring is given on the extended edges of $G'$. In Figure 15(b), we color the edges by the following sequence: Color the two edges labeled as $w_1$ by an admissible color not from $\{1, 2, 3, t_1, t_2\}$. Color the two edges labeled as $w_2$ by an admissible color not
from \(\{1, 2, 3, w_1, s_1, s_2\}\). Color the edge labeled as \(w_3\) by an admissible color not from \(\{1, 2, 3, 4, 5, w_1, w_2\}\). Finally, color the remaining two edges labeled as \(w_4\) and \(w_5\) by two different admissible colors not from \(\{1, 2, 3, w_1, w_2, w_3\}\).

Subcase B.2.3. In \(T\), none of the neighbors of \(u_2\) is a leaf. The reduction \(G'\) and the completion of \(\phi\) using eight colors are demonstrated in Figure 16. This completes all cases.

We now discuss the situation that the reduction graph \(G'\) is \(Ne_2\). Notice that this does not occur in Case A. For Subcase B.1.1, if \(G' = Ne_2\), then \(G\) is a cubic graph, contradicting our assumption that \(\chi'(T) = 6\). Similarly, for the second possibility in Subcase B.1.2, \(G'\) is not \(Ne_2\).

These leave a total of fourteen possible situations from the first possibility (Figure 11(b)) of Subcase B.1.2, as well as Subcases B.2.1, B.2.2 and B.2.3, when the reduction graph \(G'\) is \(Ne_2\). These fourteen situations are depicted in Figure 17, where a strong edge coloring using at most eight colors is given in each situation. This completes the proof of Theorem 5.

For a Halin graph \(G = T \cup C\) with maximum degree at most 4 and \(G\) is not a wheel, \(Ne_2\), nor \(Ne_4\), it has been shown that \(\chi'_s(G) \leq \chi'_s(T) + 2\), and the bound is sharp (cf. [21] and Theorem 5). We propose

**Conjecture 8.** If \(G = T \cup C\) is a Halin graph other than a wheel, \(Ne_2\), or \(Ne_4\), then \(\chi'_s(G) \leq \chi'_s(T) + 2\).

If the answer to Conjecture 8 is affirmative, then the bound is sharp for infinitely many graphs besides the ones mentioned in Lemmas 6 and 7. Let \(a, b, c\) be positive integers, \(b \geq 4\). A tree \(T\) is a *triple star*, denoted by \(T = S_{a,b,c}\), if it has exactly three non-leaf vertices which have degrees \(a, b,\) and \(c\) (in this order on a longest path), respectively. We draw \(T\) on the plane by fixing a longest path of...
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Figure 17. Fourteen special graphs.
length four horizontally, $u_0 - u - v - w - w_0$ (where $u,v, w$ are non-leaf vertices), and draw at least one pendant edge of $v$ towards each of the up and down sides of the path. For instance, Figure 1 shows $T = S_{3,4,4}$. Let $k \geq 4$ be a positive integer. Similar to the argument for Figure 1, one can show that if $T = S_{3,k,3}$, then $\chi_s'(G) = \chi_s'(T) + 2$.

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