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# PROPER CONNECTION OF DIRECT PRODUCTS

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**Abstract**

The proper connection number of a graph is the least integer $k$ for which the graph has an edge coloring with $k$ colors, with the property that any two vertices are joined by a properly colored path. We prove that given two connected non-bipartite graphs, one of which is (vertex) 2-connected, the proper connection number of their direct product is 2.

**Keywords:** direct product of graphs, proper connection of graphs.

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An *edge coloring* of a graph is an assignment of colors to its edges. A *proper edge coloring* is an edge coloring for which adjacent edges never have the same color. The *proper connection number* of a graph is the least integer $k$ for which the graph has an edge coloring with $k$ colors, with the property that any two vertices are joined by a properly colored path. The proper connection number of a graph $G$ is denoted $\mathrm{pc}(G)$. This invariant has been studied in [2, 6] and is a natural extension of the rainbow connection number of a graph [3, 4, 5]. (The *rainbow connection number* of $G$ is the minimum number of colors needed to edge-color $G$ in such a way that any two vertices are joined by a path for which no two edges are colored the same.)

The rainbow connection number of graph powers and graph products is investigated in [1]. (See [7] for a survey of graph products.) A recent paper [8] determines the proper connection number of three of the four standard graph products. For the Cartesian product, the authors show $\mathrm{pc}(G\Box H) = 2$ for non-trivial connected graphs $G$ and $H$. For the strong product $\mathrm{pc}(G\boxtimes H)$ is either 1 or 2 depending on whether or not $G$ and $H$ are both complete. A similar result holds for the lexicographic product, where $\mathrm{pc}(G \circ H)$ is 1 or 2, depending on whether or not the product is complete. However, the proper connection number of a direct product $G \times H$ is not known. We prove here that if $G$ and $H$ are connected non-bipartite graphs and one is 2-connected, then the proper connection number of their direct product is 2.

Recall that the *direct product* of $G$ and $H$ is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edges $\{(g, h)(g', h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}$. Figure 1 shows an example. Here neither factor is 2-connected, and the proper connection number of the product exceeds 2 because in any edge 2-coloring a pair of the vertices $a, b, c$ is joined only by a monochromatic path. Thus the assumption of 2-connectivity in our result cannot be relaxed.

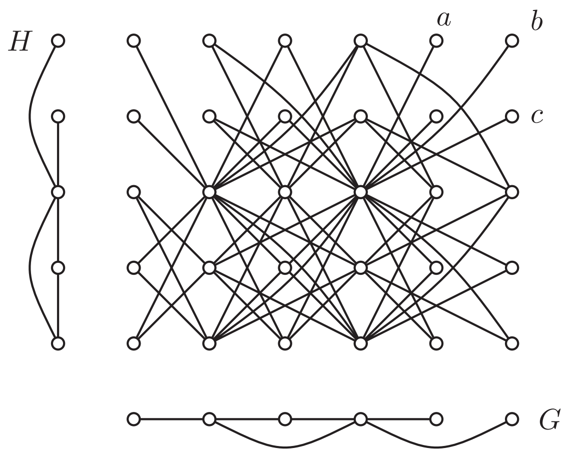


Figure 1. A direct product with $\mathrm{pc}(G \times H) > 2$.

Our results involve simple undirected finite graphs without loops, though our proofs use orientations. Denote the vertices of an $n$-cycle $C_n$ as $0, 1, 2, \ldots, n-1$; its edges are $i(i+1)$, with addition modulo $n$.

Given two cycles $C_m$ and $C_n$, we define the *standard edge 2-coloring* of the product $C_m \times C_n$ to be the assignment of two colors, *bold* and *dashed*, to the edges of $C_m \times C_n$ such that any edge of form $(i, j)(i+1, j+1)$ is colored bold, and any edge of form $(i, j)(i+1, j-1)$ is colored dashed (with arithmetic modulo $m$ and $n$ on respective arguments). This is illustrated in Figure 3, for odd cycles $m = 2p+1$ and $n = 2q+1$, where we regard the product as embedded on a torus. The left-most column of vertices is identified with the right-most column, and the top row of vertices is identified with the bottom row.

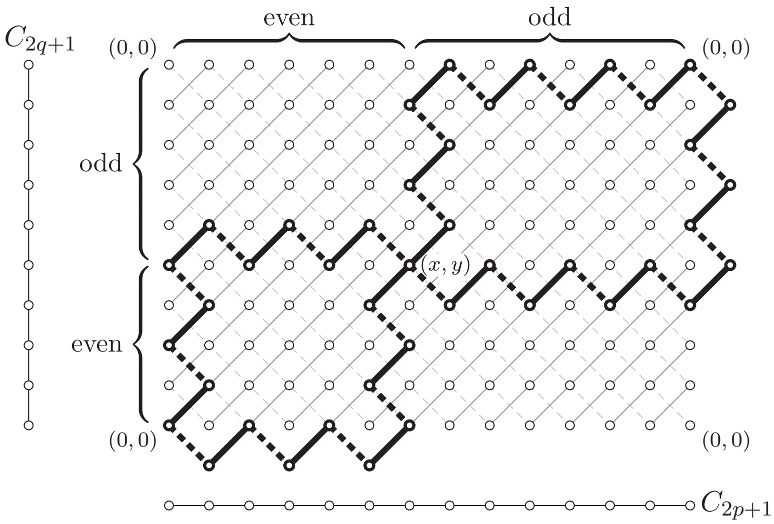


Figure 2. The four paths when $x$ and $y$ are both even. For clarity the product is shown embedded in a torus.

**Lemma 1.** *The proper connection number of the direct product of two odd cycles is* 2*. Further, in the standard edge* 2*-coloring any two vertices are joined by four types of properly colored paths, namely those that*

- *begin in* bold *and end in* dashed*,*
- *begin in* dashed *and end in* bold*,*
- *begin in* bold *and end in* bold*,*
- *begin in* dashed *and end in* dashed*.*

***Proof.*** Let the cycles be $C_{2p+1}$ and $C_{2q+1}$. Give $C_{2p+1} \times C_{2q+1}$ the standard edge 2-coloring. We now show that any two vertices in the product are joined by paths of the prescribed types. By symmetry we can assume one vertex is $(0, 0)$. Say the other is $(x, y)$. We break into cases, depending on the parity of $x$ and $y$.

First assume $x$ and $y$ are both even. The following paths have the prescribed types. (These paths are illustrated in Figure 2.)

$$(0,0)\ (1,1)\ (0,2)\ (1,3)\ \ldots\ (1,y-1)\ (0,y)\ \mid\ (1,y+1)(2,y)\ (3,y+1)\ (4,y)\ldots(x-1,y+1)(x,y)$$
$$(0,0)(1,-1)(2,0)(3,-1)\ \ldots(x-1,-1)(x,0)\ \mid\ (x-1,1)\ (x,2)(x-1,3)(x,4)\ldots(x-1,y-1)(x,y)$$
$$(0,0)(-1,-1)(-2,0)(-3,-1)\ldots(x+1,0)(x,-1)\ \mid\ (x+1,-2)(x,-3)(x+1,-4)\ \ldots(x+1,y+1)(x,y)$$
$$(0,0)\ (1,-1)\ (0,-2)\ (1,-3)\ \ldots\ (0,y+1)\ (1,y)\ \mid\ (0,y-1)\ (-1,y)\ (-2,y-1)\ \ldots(x+1,y-1)(x,y)$$

For clarity an artificial separating bar | shows where the pattern switches from alternating back and forth along an edge in one factor to alternating in the other.

The case in which $x$ and $y$ are both odd is similar, though we will not write the four paths explicitly. The construction is illustrated in Figure 3. The case where $x$ and $y$ have opposite parity is shown in Figure 4. ■

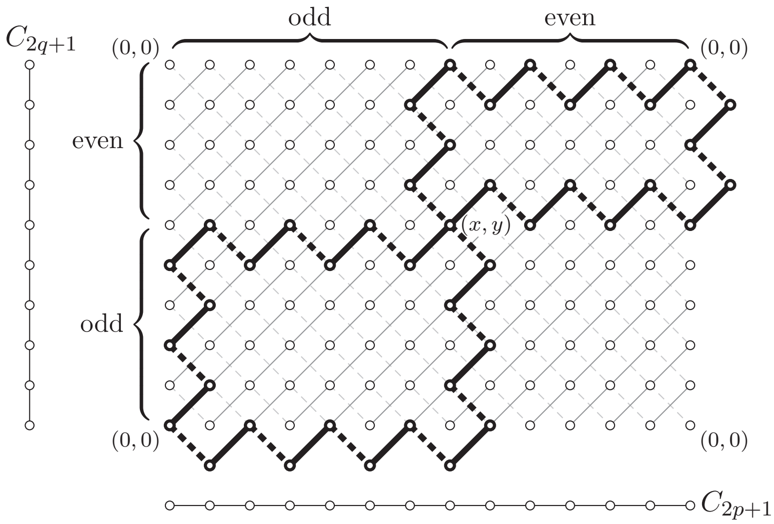


Figure 3. The four paths when $x$ and $y$ are both odd.

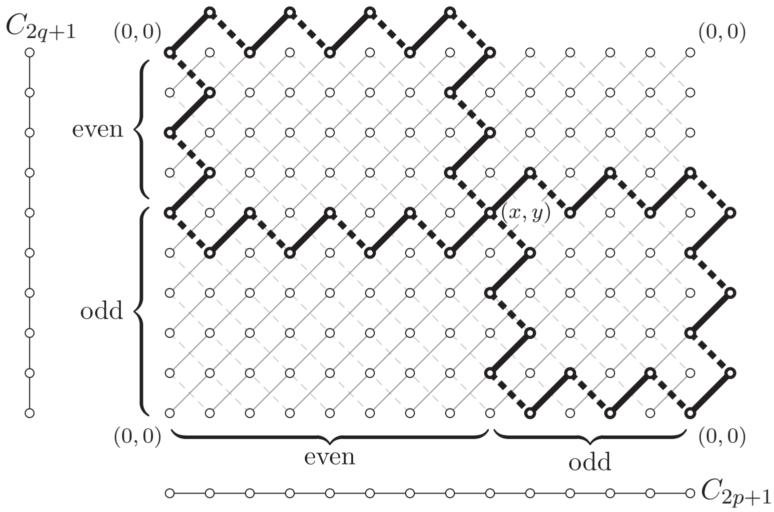


Figure 4. The four paths when $x$ and $y$ have opposite parity.

The proof of our main result will use ear decompositions. Recall that an *ear decomposition* of a graph is an edge-disjoint sequence $C, P_1, P_2, P_3, \ldots, P_k$, where $C$ is a cycle in the graph, each $P_i$ is a path whose internal vertices have degree 2 in $C \cup P_1 \cup P_2 \cup \cdots \cup P_i$, and any edge of the graph belongs to a unique member of the sequence. A theorem of Whitney [9, 10] holds that a graph is 2-connected if and only if it has an ear decomposition, and, moreover, an ear decomposition may begin with any cycle of the graph.

**Theorem 2.** *If $G$ and $H$ are connected non-bipartite graphs, and one of them is (vertex) 2-connected, then* $\mathrm{pc}(G \times H) = 2$.

***Proof.*** Let $G$ and $H$ be as stated, with $H$ 2-connected.

First we argue that it suffices to assume that $G$ has a particularly simple structure. Let $K$ be a connected spanning subgraph of $G$ that has only one cycle, $B$, which is an odd cycle (as in Figure 5). Then $K \times H$ is a connected spanning non-complete subgraph of $G \times H$, so $1 < \mathrm{pc}(G \times H) \leq \mathrm{pc}(K \times H)$. Thus it suffices to prove the proposition for $K \times H$ instead of $G \times H$. Equivalently, there is no loss of generality in assuming that $G$ has only one cycle $B$, which is odd. We assume this henceforward.

Next we define an edge 2-coloring of $G \times H$. (We will eventually show that under this coloring, any two vertices of $G \times H$ are joined by a properly colored path.) Our coloring will be defined in terms of certain orientations of $G$ and $H$.

Give $G$ an orientation for which $B$ is a directed cycle and all other edges are oriented toward it, as shown in Figure 5.

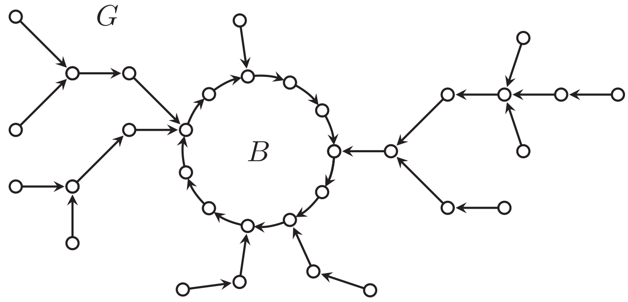


Figure 5. Orientation of the graph $G$.

We next construct an orientation of $H$ that has neither sources nor sinks. Give $H$ an ear decomposition $C, P_1, P_2, P_3, \ldots, P_k$ for which $C$ is an odd cycle. Orient the edges of $C$ so that it is a directed cycle, and orient the edges of each $P_i$ so that it is a directed path, as in Figure 6. (Each $P_i$ has two such orientations; choose one arbitrarily.) By construction this orientation has neither sources nor sinks.

Now we define our edge 2-coloring of $G \times H$. Color an edge $(g, h)(g', h')$ bold if $gg'$ is directed from $g$ to $g'$ in the orientation of $G$ and $hh'$ is directed from $h$ to $h'$ in the orientation of $H$. (Or, symmetrically, if $gg'$ is directed from $g'$ to $g$ and $hh'$ from $h'$ to $h$.) Color $(g, h)(g', h')$ dashed if $gg'$ is directed from $g$ to $g'$ but $hh'$ is directed from $h'$ to $h$.

In essence, $(g, h)(g', h')$ is colored bold if $gg'$ and $hh'$ are oriented the same (both left to right, or both right to left), and $(g, h)(g', h')$ is colored dashed if $gg'$ and $hh'$ are oriented oppositely.

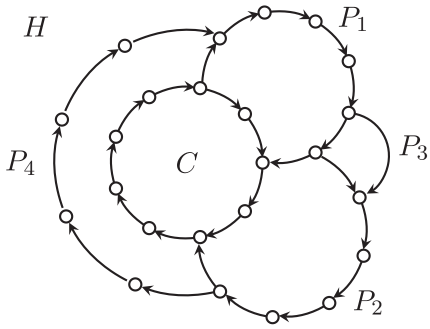


Figure 6. Orientation of the ear decomposition of $H$.

Notice that under this coloring the subgraph $B \times C$ has the standard edge 2-coloring for the product of two cycles. Lemma 1 says that any two vertices of $B \times C$ are joined by properly colored paths that begin and end with edges of any color we desire. We claim that this same property holds for $B \times H$.

**Claim.** *Consider the subgraph $B \times H \subseteq G \times H$. With the 2-coloring inherited from $G \times H$, the graph $B \times H$ has the property that any two of its vertices can be joined by paths that begin and end with all possible combinations of the two colors* (*as in Lemma* 1).

To prove the claim, take two vertices $(b_0, h_0)$ and $(b'_0, h'_0)$ of $B \times H$. We now produce properly colored paths that join them and meet the requirements of the proposition. If it happens that both $(b_0, h_0)$ and $(b'_0, h'_0)$ belong to $B \times C$, then the claim follows from Lemma 1 because the 2-coloring of $G \times H$ restricts to the standard edge 2-coloring of the product of cycles $B \times C$. Otherwise, at least one of $h_0$ and $h'_0$ is not a vertex of $C$ (though possibly $h_0 = h'_0$). Because $H$ is 2-connected, $H - E(C)$ has two paths $P : h_0h_1, h_2 \cdots h_k$ and $P' : h'_0h'_1, h'_2 \cdots h'_\ell$ that are vertex-disjoint (except possibly $h_0 = h'_0$), and whose terminal vertices $h_k$ and $h'_\ell$ belong to $C$, but for which no internal vertices belong to $C$. (Possibly one of these paths is trivial if $h_0$ or $h'_0$ already belongs to $C$.)

Note that neither $P$ nor $P'$ is necessarily a directed path in the orientation of $H$. In traversing them we may go with the orientation and also against it. But we can find a walk $W : b_0b_1b_2 \cdots b_k$ in $B$ for which the path $(b_0, h_0)(b_1, h_1)(b_2, h_2) \cdots (b_k, h_k)$ in $B \times H$ is properly colored, and begins with an edge that is either solid or bold. If we want $(b_0, h_0)(b_1, h_1)$ to be solid, we select $b_1$ so that $b_0b_1$ has the same orientation as $h_0h_1$, and if we want it dashed we go the other way on $B$, selecting $b_1$ so $b_0b_1$ is oriented opposite to $h_0h_1$. Moving on to $(b_1, h_1)(b_2, h_2)$ we can make this edge either solid or dashed with a judicious choice of $b_2$. Continuing this process, we get a path $Q : (b_0, h_0)(b_1, h_1)(b_2, h_2) \cdots (b_k, h_k)$ in $B \times H$ that is properly colored, and we are free to choose the color of the first edge.

Likewise there is a path $Q' : (b'_0, h'_0)(b'_1, h'_1)(b'_2, h'_2) \cdots (b'_\ell, h'_\ell)$ in $B \times H$ that is properly colored, and again we are free to choose the color of the first edge. By construction $Q$ and $Q'$ are vertex-disjoint, and they terminate in $B \times C$. With the exception of their terminal vertices $(b_k, h_k)$ and $(b'_\ell, h'_\ell)$, no other vertex belongs to $B \times C$. Lemma 1 guarantees a path $R$ in $B \times C$ from $(b_k, h_k)$ to $(b'_\ell, h'_\ell)$ for which $Q \cup R \cup Q'$ is properly colored. The combinations of these paths yield the desired set of four paths. This completes the claim.

To finish the proof we take two arbitrary vertices $(g_0, h_0)$ and $(g'_0, h'_0)$ of $G \times H$, and produce a properly colored path joining them.

Now, $G - E(B)$ has directed (possibly trivial) paths $P : g_0 g_1 g_2 \cdots g_k$ and $P' : g'_0 g'_1 g'_2 \cdots g'_\ell$ that terminate at vertices of $B$. Our plan is to use them to construct two disjoint properly colored paths in $(G - E(B)) \times H$, joining $(g_0, h_0)$ and $(g'_0, h'_0)$ to distinct vertices of $B \times H$, and then use the above claim to join these endpoints with an appropriate properly colored path in $B \times H$.

*Case* 1. Suppose $g_0$ and $g'_0$ are in different components of $G - E(B)$, so $P$ and $P'$ do not meet. Choose arbitrary edges $h_0 h_1$ and $h'_0 h'_1$ of $H$. In $(G - E(B)) \times H$ we have vertex-disjoint properly colored paths

$$Q : (g_0, h_0)(g_1, h_1)(g_2, h_0)(g_3, h_1)(g_4, h_0) \cdots (g_k, h_*),$$
$$Q' : (g'_0, h'_0)(g'_1, h'_1)(g'_2, h'_0)(g'_3, h'_1)(g'_4, h'_0) \cdots (g'_\ell, h'_*),$$

where $h_* = h_0$ or $h_* = h_1$ (depending on the parity of $k$), and $h'_* = h'_0$ or $h'_* = h'_1$. By the above claim, $B \times H$ has a path $R$ joining $(g_k, h_*)$ to $(g'_\ell, h'_*)$, for which the path $Q \cup R \cup Q'$ is properly colored.

*Case* 2. Suppose $g_0$ and $g'_0$ are in the same component of $G - E(B)$. Now, $P$ and $P'$ terminate at the same vertex $g_k = g'_\ell$ of $B$, and they merge at some vertex $g_{k-a} = g'_{\ell-a}$. That is, $a$ is the largest non-negative integer for which $g_{k-i} = g'_{\ell-i}$ for $a \geq i \geq 0$. (Possibly $a = 0$, in which case $P$ and $P'$ meet only at $g_k = g'_\ell$. At the other extreme, $P \subseteq P'$ if $a = k$, and $P' \subseteq P$ if $a = \ell$.)

First suppose $k - a$ and $\ell - a$ have opposite parity (and w.l.o.g., suppose it is $k - a$ that is even). Choose $h_0 h_1 \in E(H)$ and $h'_0 h'_1 \in E(H)$ with $h_0 \neq h'_1$ and $h_1 \neq h'_0$. Form the following properly colored paths in $(G - E(B)) \times H$:

$$Q : (g_0, h_0)(g_1, h_1)(g_2, h_0)(g_3, h_1)(g_4, h_0) \cdots (g_{k-a}, h_0)(g_{k-a+1}, h_1)(g_{k-a+2}, h_0) \cdots (g_k, h_*),$$
$$Q' : (g'_0, h'_0)(g'_1, h'_1)(g'_2, h'_0)(g'_3, h'_1)(g'_4, h'_0) \cdots (g'_{\ell-a}, h'_1)(g'_{\ell-a+1}, h'_0)(g'_{\ell-a+2}, h'_1) \cdots (g'_\ell, h'_*).$$

Notice $h_* \neq h'_*$, and these paths are disjoint and end in $B \times H$. By our claim, $B \times H$ has a path $R$ joining $(g_k, h_*)$ to $(g'_\ell, h'_*)$, for which the path $Q \cup R \cup Q'$ is properly colored.

Next suppose $k - a$ and $\ell - a$ are both even. Choose $h_0 h_1 \in E(H)$ and $h'_0 h'_1 \in E(H)$ with $h_1 \neq h'_1$, and also so that their orientations are opposite (i.e.,

$h_0h_1$ is directed from $h_0$ to $h_1$, and $h'_0h'_1$ is directed from $h'_1$ to $h'_0$, or vice versa). This is possible because $H$ is 2-connected and its orientation has neither sources nor sinks. We have paths

$$Q : (g_0, h_0)(g_1, h_1)(g_2, h_0)(g_3, h_1)(g_4, h_0)(g_5, h_1) \cdots (g_{k-a+1}, h_1)(g_{k-a}, h_0),$$
$$Q' : (g'_0, h'_0)(g'_1, h'_1)(g'_2, h'_0)(g'_3, h'_1)(g'_4, h'_0)(g_5, h'_1) \cdots (g_{\ell-a+1}, h'_1)(g'_{\ell-a}, h'_0).$$

The first begins with a bold edge and ends with a dashed edge. The second begins dashed and ends bold. If it happens that $h_0 = h'_0$, then $Q$ and $Q'$ intersect only at their last vertex, so $Q \cup Q'$ is a properly colored path from $(g_0, h_0)$ to $(g'_0, h'_0)$. If $h_0 \neq h'_0$ then the paths may be continued as indicated until reaching $B \times H$. Then, by our claim, $B \times H$ has a path $R$ for which $Q \cup R \cup Q'$ is a properly colored path joining $(g_0, h_0)$ to $(g'_0, h'_0)$.

Finally suppose $k - a$ and $\ell - a$ are both odd. Let $h_0h_1$ and $h'_0h'_1$ be as in the previous paragraph. We have properly colored paths

$$Q : (g_0, h_0)(g_1, h_1)(g_2, h_0)(g_3, h_1)(g_4, h_0)(g_5, h_1) \cdots (g_{k-a}, h_1)(g_{k-a+1}, h_0),$$
$$Q' : (g'_0, h'_0)(g'_1, h'_1)(g'_2, h'_0)(g'_3, h'_1)(g'_4, h'_0)(g'_5, h'_1) \cdots (g'_{\ell-a}, h'_1)(g'_{\ell-a+1}, h'_0).$$

Now, $Q$ begins bold and ends dashed, and $Q'$ begins dashed and ends bold. If $h_0 = h'_0$, then $Q$ and $Q'$ meet only at their last vertex, so $Q \cup Q'$ is a properly colored path from $(g_0, h_0)$ to $(g'_0, h'_0)$. If $h_0 \neq h'_0$ then the paths may be continued as indicated until reaching $B \times H$. Then $B \times H$ has a path $R$ for which $Q \cup R \cup Q'$ is a properly colored path joining $(g_0, h_0)$ to $(g'_0, h'_0)$. ■

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