Discussiones Mathematicae
Graph Theory 37 (2017) 797–809
doi:10.7151/dmgt.1964

A DEGREE CONDITION IMPLYING ORE-TYPE CONDITION FOR EVEN $[2, b]$-FACTORS IN GRAPHS

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Abstract

For a graph G and even integers $b \geqslant a \geqslant 2$, a spanning subgraph F of G such that $a \leqslant \deg_F(x) \leqslant b$ and $\deg_F(x)$ is even for all $x \in V(F)$ is called an even $[a, b]$-factor of G. In this paper, we show that a 2-edge-connected graph G of order n has an even $[2, b]$-factor if $\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\left\{\frac{2n}{2+b}, 3\right\}$ for any nonadjacent vertices x and y of G. Moreover, we show that for $b \geqslant 3a$ and $a > 2$, there exists an infinite family of 2-edge-connected graphs G of order n with $\delta(G) \geqslant a$ such that G satisfies the condition $\deg_G(x) + \deg_G(y) > \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G, but has no even $[a, b]$-factors. In particular, the infinite family of graphs gives a counterexample to the conjecture of Matsuda on the existence of an even $[a, b]$-factor.

Keywords: $[a, b]$-factor, even factor, 2-edge-connected, minimum degree.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

In this paper, we consider only finite undirected graphs with no loops and no multiple edges. For a graph G, we let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For a vertex x of G, $\deg_G(x)$ denotes the *degree* of x in G. We let $\delta(G)$ denote the *minimum degree* of G. For two integers a and b with $1 \leqslant a \leqslant b$, a spanning subgraph F of G such that $a \leqslant \deg_F(x) \leqslant b$ for all $x \in V(F)$ is called an $[a,b]$-*factor* of G. A $[k,k]$-factor is usually called a k-*factor*. An $[a,b]$-factor F is said to be a *parity* $[a,b]$-*factor* if $\deg_F(x) \equiv a \equiv b \pmod 2$ for all $x \in V(F)$. In particular, a parity $[a,b]$-factor is an *even* $[a,b]$-*factor* if $a \equiv b \equiv 0 \pmod 2$.

We first introduce some known results on degree conditions for the existence of an even $[2,b]$-factor.

Theorem 1 (Kouider and Vestergaard [1])**.** *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $\delta(G) \geqslant \max\{\frac{2n}{2+b}, 3\}$, then G has an even $[2,b]$-factor.*

Theorem 2 (Matsuda [4])**.** *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $\deg_G(x) + \deg_G(y) \geqslant \max\{\frac{4n}{2+b}, 5\}$ for any nonadjacent vertices x and y of G, then G has an even $[2,b]$-factor.*

In this paper, we prove the following theorem, which implies Theorems 1 and 2.

Theorem 3. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\left\{\frac{2n}{2+b}, 3\right\} \tag{1}$$

for any nonadjacent vertices x and y of G, then G has an even $[2,b]$-factor.

Let x and y be nonadjacent vertices of G. Then $\delta(G) \geqslant \max\{\frac{2n}{2+b}, 3\}$ implies $\deg_G(x)+\deg_G(y) \geqslant \max\{\frac{4n}{2+b}, 5\}$, and $\deg_G(x)+\deg_G(y) \geqslant \max\{\frac{4n}{2+b}, 5\}$ implies $\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\{\frac{2n}{2+b}, 3\}$. Hence Theorem 3 implies Theorems 1 and 2.

Additionally, we here show that Theorem 3 is stronger than Theorem 2. In order to show that, we construct an infinite family of graphs as follows: For a positive integer t and an even integer $b \geqslant 4$, we define the graph G_0 obtained from $\frac{b}{2}$ cliques $K_t^1, K_t^2, \ldots, K_t^{\frac{b}{2}}$ of order t and one vertex v_0 by joining a vertex v_0 to two vertices of K_t^i for each $1 \leqslant i \leqslant \frac{b}{2}$ (a clique means a complete graph), and let $\mathcal{G}_0 = \{G_0(b,t) \mid t \in \mathbb{Z}^+, b \in 2\mathbb{Z}^+, t > \frac{b^2+b-6}{b-2}\}$. For each $G_0 \in \mathcal{G}_0$, it is easily seen that G_0 is 2-edge-connected, and that the order of G_0 is $n = \frac{b}{2}t + 1$. By the definition of G_0, we have $n > \frac{b^3+b^2-4b-4}{2(b-2)}$. Hence it follows that if $b \geqslant 4$, then

$$\deg_{G_0}(x) + \deg_{G_0}(v_0) = |V(K_t^1)| - 1 + b = t - 1 + b = \frac{2n-2}{b} - 1 + b < \frac{4n}{2+b},$$

and

$$\max\{\deg_{G_0}(x), \deg_{G_0}(v_0)\} = |V(K_t^1)| - 1 = t - 1 = \frac{2n-2}{b} - 1 > \frac{2n}{2+b}$$

for any vertex $x \in \left(\bigcup_{1 \leqslant i \leqslant \frac{b}{2}} V(K_t^i)\right) \setminus N_{G_0}(v_0)$. Thus Theorem 3 guarantees the existence of an even $[2, b]$-factor in G_0, but Theorem 2 does not. Consequently, Theorem 3 is stronger than Theorem 2.

In order to prove Theorem 3, we actually prove the following two theorems, which are obtained from Theorem 3 by dividing it into two cases on the order n of a graph G.

Theorem 4. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $n \geqslant b+3$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b} \tag{2}$$

for any nonadjacent vertices x and y of G, then G has an even $[2, b]$-factor.

Theorem 5. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $n \leqslant b+2$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant 3 \tag{3}$$

for any nonadjacent vertices x and y of G, then G has an even $[2, b]$-factor.

Combining these, we can obtain Theorem 3.

In the rest of this section, we discuss extending "an even $[2, b]$-factor" in Theorem 3 to "an even $[a, b]$-factor" briefly. In 2005, Matsuda [4] posed the following conjecture as a natural generalization of Theorem 2.

Conjecture 6 (Matsuda [4]). *Let $2 \leqslant a \leqslant b$ be even integers, and let G be a 2-edge-connected graph of order $n \geqslant 2a + b + \frac{a^2-3a}{b} - 2$. If $\delta(G) \geqslant a$ and $\deg_G(x) + \deg_G(y) \geqslant \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G, then G has an even $[a, b]$-factor.*

In 2004, Kouider and Vestergaard constructed an infinite family of k-connected graphs G^* of order n with $\delta(G^*) \geqslant \frac{an}{a+b}$ having no even $[a, b]$-factors such that $b > 3a^2$, $k \leqslant a - 1$ and k is odd (see Example 3 in [2]). If n is sufficiently large and $k \geqslant 3$, then the graph G^* satisfies the hypothesis of Conjecture 6. Thus G^* is a kind of counterexamples in the case where $b > 3a^2$. Nevertheless, Conjecture 6 was open when $b \leqslant 3a^2$.

In this paper, we also prove that Conjecture 6 does not hold even when $3a \leqslant b \leqslant 3a^2$. Furthermore, we prove that the similar degree condition to (1) (i.e., $\max\{\deg_G(x), \deg_G(y)\} > \frac{an}{a+b}$) does not guarantee the existence of an even $[a, b]$-factor even when the difference of a and b is not so large.

Proposition 7. *Let $4 \leqslant a \leqslant b$ be even integers. Then the following assertions hold:*

(i) *For $b \geqslant 3a$, there exists an infinite family of 2-edge-connected graphs G of order n with $\delta(G) \geqslant a$ such that G satisfies $\deg_G(x) + \deg_G(y) > \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G, but has no $[a, b]$-factors.*

(ii) *For $b > a$, there exists an infinite family of 2-edge-connected graphs G of order n with $\delta(G) \geqslant a$ such that G satisfies $\max\{\deg_G(x), \deg_G(y)\} > \frac{an}{a+b}$ for any nonadjacent vertices x and y of G, but has no $[a, b]$-factors.*

Although Conjecture 6 is not true in the case where $b \geqslant 3a$ by Proposition 7, the case where $3a > b \geqslant a$ is still open.

The organization of the paper is as follows. In Section 2, Proposition 7 is described in detail. We introduce preliminaries used in our proofs of Theorems 4 and 5 in Section 3, and we show the sharpness of Theorems 4 and 5 in Section 4. In Section 5, we prove Theorems 4 and 5.

2. Construction of Graphs Without Even $[a, b]$-Factors

In this section, we mention in more detail on Proposition 7. We here construct an infinite family, which gives a new counterexample to the conjecture of Matsuda.

Construction of the family $\mathcal{G}^*$. For an integer t and even integers a and b such that $t \geqslant a + 2$ and $b \geqslant a \geqslant 4$, we construct a graph $G^*(a, b, t)$ as follows: Recall that a clique means a complete graph. Let C^0, C_t^1, C_t^2 be three disjoint cliques of order 2, t and t, respectively. Let $V(C^0) = \{x, y\}$, and let $u_1, u_2, \ldots, u_{a-1}$ (resp., $v_1, v_2, \ldots, v_{a-1}$) be distinct $a - 1$ vertices of C_t^1 (resp., C_t^2). We define the graph $G^*(a, b, t)$ obtained from C^0, C_t^1 and C_t^2 by adding xv_1, xu_i, yu_1 and yv_i for $2 \leqslant i \leqslant a - 1$ (see Figure 1), and let $\mathcal{G}^* = \{G^*(a, b, t) \mid t \geqslant a + 2, b \geqslant a \geqslant 4\}$.

For each $G^* \in \mathcal{G}^*$, it is easy to check the following:

(i) $\delta(G^*) = a$ $(= \deg_{G^*}(x) = \deg_{G^*}(y))$,

(ii) the order of G^* is $n \geqslant 2a + b + \frac{a^2-3a}{b} - 2$ if t is large enough,

(iii) G^* is 2-edge-connected from $a \geqslant 4$.

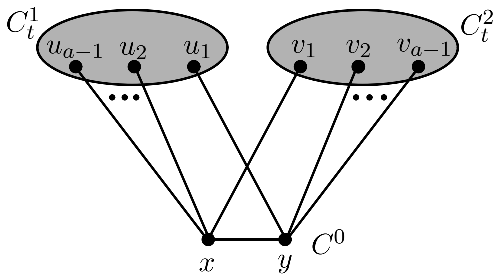

Figure 1. The graph $G^*(a, b, t)$.

Lemma 8. *Let $4 \leqslant a \leqslant b$ be even integers. Then the following assertions hold:*

(i) *For $b \geqslant 3a$, every graph $G^* \in \mathcal{G}^*$ satisfies $\deg_{G^*}(x) + \deg_{G^*}(y) > \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G^*.*

(ii) *For $b > a$, every graph $G^* \in \mathcal{G}^*$ satisfies $\max\{\deg_{G^*}(x), \deg_{G^*}(y)\} > \frac{an}{a+b}$ for any nonadjacent vertices x and y of G^*.*

Proof. (i) Let $G^* \in \mathcal{G}^*$. By the construction of G^*, the following two facts hold:

(F1) Vertices having the minimum degree are only x and y, and $G^*[\{x, y\}]$ is a clique;

(F2) Vertices having the second smallest degree belong to $V(C_t^1) \setminus N_{G^*}(V(C^0))$ or to $V(C_t^2) \setminus N_{G^*}(V(C^0))$, each of which is nonadjacent to x and y.

In view of (F1) and (F2), it suffices to check the degree condition only for two vertices $w \in V(C_t^1) \setminus N_{G^*}(V(C^0))$ and $z \in V(C^0)$. By $b \geqslant 3a$ and $a \geqslant 4$, we obtain

$$\deg_{G^*}(w) + \deg_{G^*}(z) = t - 1 + a \geqslant t + 3 = \frac{n}{2} + 2 > \frac{2an}{a+b}.$$

(ii) Let $G^* \in \mathcal{G}^*$. Similarly to the proof of (i), it suffices to check the degree condition only for two vertices $w \in V(C_t^1) \setminus N_{G^*}(V(C^0))$ and $z \in V(C^0)$. By $b > a \geqslant 4$, we get

$$\max\{\deg_{G^*}(w), \deg_{G^*}(z)\} = t - 1 = \frac{n}{2} - 2 > \frac{an}{a+b}.$$

■

Lemma 9. *Every graph $G^* \in \mathcal{G}^*$ has no even $[a, b]$-factors.*

Proof. Suppose that $G^* \in \mathcal{G}^*$ has an even $[a, b]$-factor F. Since $\deg_{G^*}(x) = a = \deg_{G^*}(y)$, we obtain $\deg_F(x) = a = \deg_F(y)$. Also, since $|V(C_t^1) \cap N_F(V(C^0))| = a - 1$ is odd, $F[V(C_t^1)]$ is a graph having odd number of vertices with odd degree. This is a contradiction. ■

By Lemmas 8, 9 and the construction of $\mathcal{G}^*$, Proposition 7 can be proved if t is large enough.

3. Preliminaries

In this section, we give notation and lemmas used in our proofs of Theorems 4 and 5.

Our notation is standard possibly except the following. Let G be a graph. For a vertex x of G, $N_G(x)$ denotes the set of vertices adjacent to x in G; $\deg_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we let $N_G(A)$ denote the union of $N_G(x)$ as x ranges over A. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, $e_G(A, B)$ denotes the number of those edges of G which join a vertex in A and a vertex in B. For $A \subseteq V(G)$, the subgraph of G induced by A is denoted by $G[A]$, and $G-A$ denotes the subgraph $G[V(G)-A]$. A vertex set A is called *independent* if $G[A]$ has no edges.

In our proofs of Theorems 4 and 5, we depend on the following lemma, which is a special case of the parity (g, f)-factor theorem of Lovász [3] (for this necessary and sufficient criterion, an alternative proof was given by Tutte [5]).

Lemma 10 (Lovász [3])**.** *Let $b \geqslant 2$ be an even integer, and let G be a graph. Then G has an even $[2,b]$-factor if and only if*

$$\begin{aligned}\theta_G(S,T) &:= b|S| + \sum_{y\in T}(\deg_{G-S}(y) - 2) - h_G(S,T) \\ &= b|S| + \sum_{y\in T}(\deg_G(y) - 2) - e_G(S,T) - h_G(S,T) \geqslant 0\end{aligned}$$

for all disjoint subsets S and T of $V(G)$, where $h_G(S,T)$ is the number of components C of $G-S-T$ such that $e_G(V(C),T) \equiv 1 \pmod 2$, and such a component C is briefly called an odd component of $G-S-T$.

In addition to the above lemma, we use the following two lemmas in our proofs. Since they are well-known, we omit the proofs (see [4] in detail).

Lemma 11. *Let G be a graph, and let S and T be disjoint subsets of $V(G)$. Then the following assertion hold:*

$$\theta_G(S,T) \equiv 0 \pmod 2.$$

Lemma 12. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph. Let S and T be disjoint subsets of $V(G)$ for which $\theta_G(S,T) \leqslant -2$. Then the following assertions holds:*

(i) $2|T| \geqslant b|S| + 2$,
(ii) $|T| \geqslant 2$.

For a graph G satisfying the hypothesis in Theorem 3, we show the following lemma.

Lemma 13. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n such that $\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\{\frac{2n}{2+b}, 3\}$ for any nonadjacent vertices x and y of G. Assume that there exist disjoint subsets S and T of $V(G)$ satisfying $\theta_G(S,T) \leqslant -2$. Choose such subsets S and T so that $|T|$ is as small as possible. Then the following assertions hold:*

(i) *T is an independent set of G,*
(ii) $\sum_{y\in T} \deg_G(y) \geqslant 3|T| - 1$.

Proof. To prove (i), let $T' = T - \{v\}$ for any $v \in T$. Then $T' \neq \emptyset$ by Lemma 12(ii). By the choice of T and Lemma 11, we have $\theta_G(S,T') \geqslant 0$ and $\theta_G(S,T) \leqslant -2$. Thus, by subtracting these inequalities, $2 \leqslant \theta_G(S,T') - \theta_G(S,T) \leqslant -\deg_{G-S}(v) + 2 + h_G(S,T) - h_G(S,T')$, which implies $\deg_{G-S}(v) \leqslant h_G(S,T) - h_G(S,T')$. This inequality together with $e_G(v, V(G) - S - T) \geqslant h_G(S,T) - h_G(S,T')$ yields $\deg_{G[T]}(v) = \deg_{G-S}(v) - e_G(\{v\}, V(G) - S - T) \leqslant 0$, which means that T is an independent subset of $V(G)$. Thus, (i) holds.

Suppose that there exist two vertices $x, y \in T$ satisfying $\deg_G(x) = \deg_G(y) = 2$. Then, in the case where $n \geqslant b+3$, by (i) and the condition of Theorem 4, we have $2 = \max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b}$, which contradicts $n \geqslant b+3$. In the case where $n \leqslant b+2$, by (i) and the condition of Theorem 5, we have $2 = \max\{\deg_G(x), \deg_G(y)\} \geqslant 3$, a contradiction. In either case, we obtain a contradiction. Hence T has at most one vertex t with $\deg_G(t) = 2$. Consequently, we have $\sum_{y \in T} \deg_G(y) \geqslant 3(|T| - 1) + 2 = 3|T| - 1$. Thus, (ii) holds. ■

4. Sharpness of Theorems 4 and 5

In this section, we discuss the sharpness of Theorems 4 and 5. In Theorems 4 and 5, the degree conditions (2) and (3) are best possible. Moreover, the hypothesis "2-edge-connected" cannot be dropped. For Theorem 4, the lower bound of the order (i.e., "$b+3$") is sharp. Although our result is a generalization of Theorem 2, the examples in [4] are applicable to Theorems 4 and 5 as they stand. Here we include them for the convenience of the reader.

Example 1. The degree condition (2) is best possible in the sense that we cannot replace $\frac{2n}{2+b}$ with $\frac{2n-2}{2+b}$ (noting that $\frac{2n-1}{2+b}$ cannot be an integer, and thus an integer $a > \frac{2n}{2+b}$ if and only if $a > \frac{2n-1}{2+b}$). To check it, we construct an infinite family of 2-edge-connected graphs G_1 of order sufficiently large n without even $[2,b]$-factors such that the degree condition of G_1 is a little smaller than $\frac{2n}{2+b}$ as follows: For a positive integer t and an even integer $b \geqslant 2$, let K_{2t} (resp., $(bt+1)K_1$) be a clique of order $2t$ (resp., $bt+1$ cliques of order 1). We define the graph $G_1(b,t)$ obtained by joining K_{2t} and $(bt+1)K_1$, and let $\mathcal{G}_1 = \{G_1(b,t) \mid t \in \mathbb{Z}^+, b \geqslant 2 \text{ is even}\}$. For each $G_1 \in \mathcal{G}_1$, the order of G_1 is $n = (2+b)t + 1$ and G_1 is 2-edge-connected. Also, it follows that

$$\frac{2n}{2+b} > \max\{\deg_{G_1}(x), \deg_{G_1}(y)\} = 2t = \frac{2n}{2+b} - \frac{2}{2+b} > \frac{2n}{2+b} - 1$$

for any nonadjacent vertices $x, y \in V((bt+1)K_1)$. However, G_1 has no $[2,b]$-factors as $b|V(K_{2t})| < 2|V((bt+1)K_1)|$.

Example 2. The condition "2-edge-connected" in Theorem 4 cannot be deleted for $b \geqslant 6$. To check it, we construct an infinite family of connected graphs G_2

of order sufficiently large n without even $[2,b]$-factors such that G_2 satisfies the condition $\max\{\deg_{G_2}(x), \deg_{G_2}(y)\} \geqslant \frac{2n}{2+b}$ for any nonadjacent vertices $x, y \in V(G_2)$, but is not 2-edge-connected as follows: For a positive integer t and an even integer $b \geqslant 6$, we define the graph $G_2(t)$ obtained from two cliques K_t^1, K_t^2 and one vertex v_0 by joining a vertex v_0 to a vertex of K_t^1 and to a vertex of K_t^2, and let $\mathcal{G}_2 = \{G_2(t) \mid t \in \mathbb{Z}^+\}$. For each $G_2 \in \mathcal{G}_2$, G_2 is not 2-edge-connected. Also, the order of G_2 is $n = 2t+1$, and it follows that $\max\{\deg_{G_2}(u), \deg_{G_2}(v_0)\} = \deg_{G_2}(u) = t-1 = \frac{n-3}{2} \geqslant \frac{2n}{2+b}$ for any vertex $u \in (V(K_t^1) \setminus N_{G_2}(v_0)) \cup (V(K_t^2) \setminus N_{G_2}(v_0))$ for $b \geqslant 6$. However, G_2 has no even $[2,b]$-factors. In fact, putting $S = \emptyset$ and $T = \{v_0\}$ in Lemma 10, we can check that both K_t^1 and K_t^2 are odd components of $G - S - T$, and thus $\theta_{G_2}(\emptyset, \{v_0\}) = \deg_{G_2}(v_0) - 2 - 2 = -2 < 0$.

Example 3. The lower bound of order $n \geqslant b+3$ in Theorem 4 is sharp for $b \geqslant 4$. To check it, we construct an infinite family of 2-edge-connected graphs G_3 of order $n = b+2$ without even $[2,b]$-factors such that G_3 satisfies the condition $\max\{\deg_{G_3}(x), \deg_{G_3}(y)\} \geqslant \frac{2n}{2+b}$ for any nonadjacent vertices x and y of G_3 as follows: For an even integer $b \geqslant 4$, we define the graph $G_3(b)$ obtained from two vertices v_1, v_2 and a path P_b of order b by joining each v_i to two endvertices of P_b, and let $\mathcal{G}_3 = \{G_3(b) \mid b \geqslant 4 \text{ is even}\}$. For each $G_3 \in \mathcal{G}_3$, G_3 is 2-edge-connected. Also, the order of G_3 is $n = b+2$, and it follows that $\max\{\deg_{G_3}(v_1), \deg_{G_3}(v_2)\} = 2 = \frac{2n}{2+b}$. However, it is clear that G_3 has no $[2,b]$-factors. Note that G_3 also shows that the degree condition (3) in Theorem 5 is best possible in the sense that we cannot replace 3 with 2.

5. Proof of Theorems 4 and 5

In this section, we prove Theorems 4 and 5. Suppose that a graph G satisfies the hypothesis of Theorems 4 or 5. By Lemmas 10 and 11, it suffices to show that there exist no disjoint subsets S and T of $V(G)$ for which

$$\theta_G(S,T) \leqslant -2. \tag{4}$$

5.1. Proof of Theorem 4

Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order $n \geqslant b+3$ such that $\max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b}$ for any nonadjacent vertices x and y of G. By way of contradiction, suppose that G does not have an even $[2,b]$-factor. Then by Lemmas 10 and 11, there exist disjoint subsets S and T of G satisfying (4). We choose such S and T so that $|T|$ is as small as possible.

Let $t_1, t_2, \ldots, t_{|T|}$ be the vertices of T. Note that $|T| \geqslant 2$ by Lemma 12(ii). Without loss of generality, we may assume that $\deg_G(t_1) \leqslant \deg_G(t_2) \leqslant \cdots \leqslant$

$\deg_G(t_{|T|})$. By Lemma 13(i), $T = \{t_1, t_2, \ldots, t_{|T|}\}$ is an independent set of G. Consequently, by the condition of Theorem 4, we have

$$\max\{\deg_G(t_1), \deg_G(t_i)\} = \deg_G(t_i) \geqslant \frac{2n}{2+b}$$

for each $2 \leqslant i \leqslant |T|$. By this inequality, we obtain

$$(5) \qquad \sum_{y \in T} \deg_G(y) = \sum_{y \in T \setminus \{t_1\}} \deg_G(y) + \deg_G(t_1) \geqslant (|T|-1)\frac{2n}{2+b} + \deg_G(t_1).$$

We divide the proof into two cases on the cardinality of $|T|$.

Case 1. $|T| \geqslant b+1$.

Claim 14. $|S| \leqslant \frac{2n}{2+b} - 1$.

Proof. Suppose that $|S| > \frac{2n}{2+b} - 1$, i.e., $2n-(2+b)|S| < 2+b$. Since the both sides of this inequality are even, $2n-(2+b)|S| \leqslant b$ holds. By $n \geqslant |S|+|T|+h_G(S,T)$, this implies

$$\begin{aligned} 2|T| - b|S| &\leqslant 2(n - |S| - h_G(S,T)) - b|S| \\ &= 2n - (2+b)|S| - 2h_G(S,T) \leqslant b - 2h_G(S,T). \end{aligned}$$

Thus, it follows from (4) and $2|T| - b|S| \leqslant b - 2h_G(S,T)$ that

$$\begin{aligned} \sum_{y \in T} \deg_{G-S}(y) &\leqslant 2|T| - b|S| + h_G(S,T) - 2 \\ &\leqslant b - 2h_G(S,T) + h_G(S,T) - 2 \leqslant b-2. \end{aligned}$$

Since $|T| \geqslant b+1$, there exist at least two vertices x and y of T such that $\deg_{G-S}(x) = \deg_{G-S}(y) = 0$. Therefore by the condition of Theorem 4, we have

$$(6) \qquad |S| \geqslant \max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b}.$$

On the other hand, by Lemma 12(i) and $n \geqslant |S|+|T|+h_G(S,T)$, we have $2(n-|S|-h_G(S,T)) \geqslant 2|T| \geqslant b|S|+2$, which implies $|S| \leqslant \frac{2(n-h_G(S,T)-1))}{2+b} < \frac{2n}{2+b}$. This contradicts (6). □

By (4), (5), Claim 14, $e_G(S,T) \leqslant |S||T|$, $h_G(S,T) \leqslant n - |S| - |T|$ and $b+1-|T| \leqslant 0$ (by the assumption of Case 1), we obtain

$$\begin{aligned} -2 &\geqslant \theta_G(S,T) \\ &\geqslant b|S| + (|T|-1) \cdot \frac{2n}{2+b} + \deg_G(t_1) - |S||T| - 2|T| - (n - |S| - |T|) \end{aligned}$$

$$
\begin{aligned}
&= (b+1-|T|)|S| + \left(\frac{2n}{2+b} - 1\right)|T| + \deg_G(t_1) - \frac{2n}{2+b} - n \\
&\geqslant (b+1-|T|)\left(\frac{2n}{2+b} - 1\right) + \left(\frac{2n}{2+b} - 1\right)|T| + \deg_G(t_1) - \frac{2n}{2+b} - n \\
&= b\left(\frac{2n}{2+b} - 1\right) + \deg_G(t_1) - 1 - n,
\end{aligned}
$$

which implies $\deg_G(t_1) \leqslant \frac{(2-b)n}{2+b} + b - 1$. If $b \geqslant 4$, then by $n > b+2$,

$$\deg_G(t_1) \leqslant \frac{n}{2+b}(2-b) + b - 1 < 2 - b + b - 1 = 1,$$

i.e., $\deg_G(t_1) = 0$, which means that t_1 is an isolated vertex. If $b = 2$, then $\deg_G(t_1) \leqslant 1$ holds. In either case, we get a contradiction because G is 2-edge-connected.

Case 2. $|T| \leqslant b$. By Lemma 12(i), we have $|S| < \frac{2|T|}{b} \leqslant 2$, which means that $|S| = 0$ or $|S| = 1$.

Let h_1 (resp., h_2) be the number of odd components C of $G - S - T$ such that $e_G(V(C), T) = 1$ (the number of odd components C of $G - S - T$ such that $e_G(V(C), T) \neq 1$, i.e., $e_G(V(C), T) \geqslant 3$). Then $h_G(S, T) = h_1 + h_2$.

Claim 15. $|S| = 1$.

Proof. Suppose that $|S| = 0$, i.e., $S = \emptyset$. Since G is 2-edge-connected, we obtain $h_1 = 0$. Then $h_G(\emptyset, T) = h_2$ holds. Hence it follows from (4) and $\sum_{y \in T} \deg_G(y) \geqslant 3h_2$ that

$$
\begin{aligned}
-2 \geqslant \theta_G(\emptyset, T) &= \sum_{y \in T} \deg_G(y) - 2|T| - h_G(\emptyset, T) \\
&\geqslant 3h_2 - 2|T| - h_2 = 2h_2 - 2|T|,
\end{aligned}
$$

implying $|T| \geqslant h_2 + 1$. By this inequality, (4) and Lemma 13(ii), we have

$$
\begin{aligned}
-2 \geqslant \theta_G(\emptyset, T) &= \sum_{y \in T} \deg_G(y) - 2|T| - h_G(\emptyset, T) \\
&\geqslant (3|T| - 1) - 2|T| - h_2 = |T| - h_2 - 1 \geqslant 0.
\end{aligned}
$$

This is a contradiction. □

Since $\sum_{y \in T} \deg_{G-S}(y) \geqslant h_1 + 3h_2$ and $h_G(S, T) = h_1 + h_2$, it follows from Claim 15 and (4) that

$$-2 \geqslant \theta_G(S, T) \geqslant b + (h_1 + 3h_2) - 2|T| - (h_1 + h_2) = 2h_2 - 2|T| + b,$$

that is,

$$|T| \geqslant h_2 + \frac{b+2}{2}. \tag{7}$$

Claim 16. $h_1 \geqslant \frac{b+4}{2}$.

Proof. By (4), (7), Lemma 13(ii), Claim 15, $e_G(S,T) \leqslant |S||T| \leqslant b$ and $h_G(S,T) = h_1 + h_2$,we obtain

$$\begin{aligned}-2 \geqslant \theta_G(S,T) &\geqslant b + (3|T| - 1) - b - 2|T| - (h_1 + h_2)\\ &\geqslant |T| - h_1 - h_2 - 1 \geqslant \frac{b+2}{2} - h_1 - 1,\end{aligned}$$

which implies $h_1 \geqslant \frac{b+4}{2}$, as desired. □

For each $1 \leqslant i \leqslant h_1$, let C'_i be the odd components of $G - S - T$ such that $e_G(V(C'_i, T) = 1$. Without loss of generality, we may assume that $|C'_1| \leqslant |C'_2| \leqslant \cdots \leqslant |C'_{h_1}|$. Note that there exist at least two components C'_1 and C'_2 by Claim 16. For two vertices $u_1 \in V(C'_1)$ and $u_2 \in V(C'_2)$, it follows from the definition of C'_i, Claim 15 and the condition of Theorem 4 that

$$\begin{aligned}\frac{2n}{2+b} &\leqslant \max\{\deg_G(u_1), \deg_G(u_2)\}\\ &\leqslant \max\{|C'_1| - 1 + e_G(u_1, S \cup T), |C'_2| - 1 + e_G(u_2, S \cup T)\}\\ &\leqslant \max\{|C'_1| + 1, |C'_2| + 1\} = |C'_2| + 1,\end{aligned}$$

that is, $|C'_2| \geqslant \frac{2n}{2+b} - 1$. Hence, we have

$$\sum_{i=1}^{h_1} |C'_i| \geqslant |C'_1| + (h_1 - 1)\left(\frac{2n}{2+b} - 1\right).$$

It follows from this inequality, (7) and Claim 16 that

$$\begin{aligned}n &\geqslant |S| + |T| + |C'_1| + (h_1 - 1)\left(\frac{2n}{2+b} - 1\right)\\ &\geqslant 1 + h_2 + \frac{b+2}{2} + |C'_1| + \frac{b+2}{2}\left(\frac{2n}{2+b} - 1\right) > n,\end{aligned}$$

which is a contradiction. Consequently, this completes the proof of Theorem 4.

5.2. Proof of Theorem 5

Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order $n \leqslant b + 2$ such that $\max\{\deg_G(x), \deg_G(y)\} \geqslant 3$ for any nonadjacent vertices x and y of G. By way of contradiction, suppose that G does not have an even $[2, b]$-factor. Then by Lemmas 10 and 11, there exist disjoint subsets S and T of G satisfying (4). We choose such S and T so that $|T|$ is as small as possible.

By Lemma 12(i), $|T| \geqslant \frac{b|S|}{2} + 1$. If $|S| \geqslant 2$, then we obtain $n \geqslant |S| + |T| \geqslant |S| + \left(\frac{b|S|}{2} + 1\right) \geqslant b + 3$, which contradicts that $n \leqslant b + 2$. Hence we have that $|S| = 0$ or $|S| = 1$.

Claim 17. $|S| = 1$.

Proof. Suppose that $|S| = 0$, i.e., $S = \emptyset$. Since G is 2-edge-connected, all of the odd components C of $G - T$ satisfy $e_G(V(C), T) \geqslant 3$. By (4),

$$\begin{aligned} -2 \geqslant \theta_G(\emptyset, T) &= \sum_{y \in T} \deg_G(x) - 2|T| - h_G(\emptyset, T) \\ &\geqslant 3h_G(\emptyset, T) - 2|T| - h_G(\emptyset, T) = 2h_G(\emptyset, T) - 2|T|, \end{aligned}$$

implying

$$|T| \geqslant h_G(\emptyset, T) + 1. \tag{8}$$

Then it follows from (4), (8) and Lemma 13(ii) that

$$\begin{aligned} -2 \geqslant \theta_G(\emptyset, T) &= \sum_{y \in T} \deg_G(y) - 2|T| - h_G(\emptyset, T) \\ &\geqslant (3|T| - 1) - 2|T| - h_G(\emptyset, T) = |T| - h_G(\emptyset, T) - 1 \geqslant 0. \end{aligned}$$

This is a contradiction. □

By (4), Lemma 13(ii), Claim 17 and $e_G(S, T) \leqslant |T|$, we have

$$\begin{aligned} h_G(S, T) &\geqslant b + \sum_{y \in T} \deg_G(y) - e_G(S, T) - 2|T| + 2 \\ &\geqslant b + (3|T| - 1) - |T| - 2|T| + 2 = b + 1. \end{aligned}$$

Therefore by the above inequality and Lemma 12(ii), we obtain $n \geqslant |S| + |T| + h_G(S, T) \geqslant 1 + 2 + (b + 1) \geqslant b + 4$, which contradicts the assumption that $n \leqslant b + 2$. This completes the proof of Theorem 5.

Acknowledgment

The authors would like to thank Professor Yoshimi Egawa, Doctor Michitaka Furuya and the anonymous referees for their helpful comments and suggestions which made this paper improved.

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Received 1 February 2016
Revised 17 June 2016
Accepted 15 July 2016