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# ASYMPTOTIC SHARPNESS OF BOUNDS ON HYPERTREES

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**Abstract**

The hypertree can be defined in many different ways. Katona and Szabó introduced a new, natural definition of hypertrees in uniform hypergraphs and investigated bounds on the number of edges of the hypertrees. They showed that a $k$-uniform hypertree on $n$ vertices has at most $\binom{n}{k-1}$ edges and they conjectured that the upper bound is asymptotically sharp. Recently, Szabó verified that the conjecture holds by recursively constructing an infinite sequence of $k$-uniform hypertrees and making complicated analyses for it. In this note we give a short proof of the conjecture by directly constructing a sequence of $k$-uniform $k$-hypertrees.

**Keywords:** hypertree, semicycle in hypergraph, chain in hypergraph.

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## 1. Introduction

Paths, cycles and trees are among the most fundamental objects in graph theory. As we have known, trees have a number of interesting structural properties, and trees are the most common objects in all of graph theory. These concepts have been generalized to hypergraphs in a lot of different ways [1, 3, 4].

Recently, Katona and Szabó [2] generalized the notion of trees to uniform hypergraphs and discussed lower and upper bounds on the number of edges of such hypertrees. They showed that a $k$-uniform hypertree on $n$ vertices has at most $\binom{n}{k-1}$ edges and they posed some conjectures for bounds on the number of edges in the hypertrees.

We now recall definitions of hypertrees for $k$-uniform hypergraphs given in [2]. Let $\mathcal{F} = (V, \mathcal{E})$ be a $k$-uniform hypergraph (with no multiple edges).

The hypergraph $\mathcal{F}$ is a *chain* if there exists a sequence $v_1, v_2, \ldots, v_l$ of its vertices such that every vertex appears at least once (possibly more times), $v_1 \neq v_l$ and $\mathcal{E}$ consists of $l-k+1$ distinct edges of the form $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$, $1 \leq i \leq l-k+1$. The length of the chain is $l-k+1$, i.e., the number of its edges.

The hypergrah $\mathcal{F}$ is a *semicycle* if there exists a sequence $v_1, v_2, \ldots, v_l$ of its vertices such that every vertex appears at least once (possibly more times), $v_1 = v_l$ and for all $1 \leq i \leq l-k+1$, $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ are distinct edges of $\mathcal{F}$. The length of the semicycle $\mathcal{F}$ is $l-k+1$, the number of its edges. From the definition it follows that every semicycle has at least 3 edges.

A $k$-uniform hypergraph $\mathcal{H}$ is *chain-connected* if every pair of its vertices is connected by a chain. A $k$-uniform hypergraph $\mathcal{H}$ is *semicycle-free* if it contains no semicycle as a subhypergraph. A *hypertree* is a $k$-uniform hypergraph $\mathcal{H}$ ($k \geq 2$) such that $\mathcal{H}$ is chain-connected and semicycle-free. A hypertree is called an *$l$-hypertree* if every chain in it is of length at most $l$.

Katona and Szabó [2] investigated lower and upper bounds on the number of edges of hypertrees. They obtained the following results on the upper bounds.

**Theorem 1** (Katona, Szabó [2]). *If $\mathcal{H}$ is a semicycle-free $k$-uniform hypergraph on $n$ vertices, then $|\mathcal{E}(\mathcal{H})| \leq \binom{n}{k-1}$, and this bound is asymptotically sharp for $k = 3$.*

**Theorem 2** (Katona, Szabó [2]). *Let $1 \leq l \leq k$ and $\mathcal{H}$ be a $k$-uniform $l$-hypertree on $n$ vertices. Then $|\mathcal{E}(\mathcal{H})| \leq \frac{1}{k-l+1}\binom{n}{k-1}$. This bound is asymptotically sharp in the case $l = 2, k = 3$.*

**Conjecture 3** (Katona, Szabó [2]). *The upper bound in Theorem 1 can be reached by a sequence of $k$-hypertrees.*

Recently, Szabó [5] proved the above conjecture by recursively constructing a sequence of $k$-hypertrees. However, the construction is intricate and technical.

In this note we give a shorter proof of the conjecture by directly constructing a sequence of $k$-hypertrees.

We will prove the main result below in next section.

**Theorem 4.** *For $k \geq 3$, there exists an infinite sequence of $k$-hypertrees where the number of edges is asymptotically $\binom{n}{k-1}$.*

## 2. Proof of Theorem 4

Let $\mathcal{H} = (V, \mathcal{E})$ be an arbitrary $k$-uniform $k$-hypertree and let $V = \{v_1, v_2, \dots, v_n\}$. Now let us define a new $k$-uniform hypergraph $\mathcal{H}' = (V \cup V', \mathcal{E} \cup \mathcal{E}')$, where $V' = \{1, 2, \dots, k-1\}^n$, i.e., the set of $n$-dimensional vectors over $\{1, 2, \dots, k-1\}$, and $\mathcal{E}' = \{\{v_i, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\} \mid v_i \in V,\ \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1} \in V'$, where the $i$th coordinate of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}$ is the smallest coordinate where all the coordinates are distinct$\}$.

By the definition of $\mathcal{E}'$, if $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$, then all of $1, 2, \dots, k-1$ appear in the $i$th column of the $(k-1) \times n$ matrix

$$M = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{k-1} \end{pmatrix},$$

where every $\mathbf{u}_i$ is regarded as a row vector, but at least one of $1, 2, \dots, k-1$ do not appear in the $i'$th column of the matrix $M$ for each $i' < i$.

We first prove that $\mathcal{H}'$ is a $k$-uniform $k$-hypertree.

**Lemma 5.** *$\mathcal{H}'$ is a $k$-uniform $k$-hypertree.*

***Proof.*** To prove that $\mathcal{H}'$ is a $k$-uniform $k$-hypertree, we need to verify that $\mathcal{H}'$ satisfies the following three properties.

(i) $\mathcal{H}'$ is chain-connected. Clearly, any two vertices of $V$ are chain-connected, since $\mathcal{H}$ is a hypertree and all of its edges are edges of $\mathcal{H}'$. For any $\mathbf{u}_1, \mathbf{u}_2 \in V'$, let $i$ denote the position of the first coordinate where they differ. Then we consider the vertices $\mathbf{u}_3, \dots, \mathbf{u}_{k-1} \in V'$ each of which the first $i-1$ coordinates are the same as the first $i-1$ coordinates of $\mathbf{u}_1, \mathbf{u}_2$ but the $i$th coordinates of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}$ differ from each other. By the definition of $\mathcal{E}'$, we see that $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$. This implies that $\mathbf{u}_1, \mathbf{u}_2$ are connected by a chain of length one in $\mathcal{H}'$. For any $\mathbf{u}_1 \in V'$ and $v_i \in V$, let $\mathbf{u}_2, \dots, \mathbf{u}_{k-1}$ be $k-2$ vertices in $V'$ such that the first $i-1$ coordinates of each $\mathbf{u}_i$ $(2 \leq i \leq k-1)$ are the same as the first $i-1$ coordinates of $\mathbf{u}_1$, but the $i$th coordinates of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}$ differ from each other. By the definition of $\mathcal{E}'$, $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$. So $\mathbf{u}_1$ and $v_i$ are connected by a chain of length one.

(ii) $\mathcal{H}'$ is semicycle-free. Suppose, to the contrary, that $\mathcal{H}'$ contains a semicycle $C$. By the definition, we have $|e \cap e'| \leq 1$ for all $e \in \mathcal{E}$, $e' \in \mathcal{E}'$. This implies that all edges in $C$ belong to either $\mathcal{E}$ or $\mathcal{E}'$ since $k \geq 3$. If all edges in $C$ lie in $\mathcal{E}$, then $C$ is also a semicycle of $\mathcal{H}$, which contradicts that $\mathcal{H}$ is semicycle-free. Therefore, all edges in $C$ lie in $\mathcal{E}'$.

Without loss of generality, let $e_1 = \{v_1, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ be an edge in $C$. Then, by definition, the first coordinates of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ are the first coordinates that are different from each other. We may assume that $i$ is the first coordinate of $\mathbf{u}_i$ for $1 \leq i \leq k-1$. Clearly, for any $1 < j \leq n$, $\{v_j, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ does not belong to $\mathcal{E}'$. Let $e_1$ and $e_2$ be two consecutive edges in $C$. Then, by the definition of the semicycle, $|e_1 \cap e_2| = k-1$. This implies that $v_1$ must be in $e_2$, and so each edge of $C$ contains the vertex $v_1$.

If we write down the vertices of the semicycle in a sequence, denoting the vertices from $V$ by $v_i$ and those from $V'$ by $\mathbf{u}_j$, there are $k$ possible sequences as follows:

(1) $v_1$,$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$: only one edge, which obviously cannot be a semicycle.

(2) $\mathbf{u}_1, v_1, \mathbf{u}_2, u_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$: only two edges. This sequence cannot be a semicycle because a semicycle must have at least three edges.

(3) $\mathbf{u}_1, \mathbf{u}_2, v_1, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k, \mathbf{u}_{k+1}$: there are three edges. By the definition of a semicycle, the first and the last vertices of the sequence must be the same. Because $\{\mathbf{u}_1, \mathbf{u}_2, v_1, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}\}$ is an edge of $\mathcal{E}'$, the first coordinate of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ differ from each other. We may assume $\{1, 2, \ldots, k-1\}$ are respectively the first coordinate of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$. Besides, $\mathbf{u}_2, v_i, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}$, $\mathbf{u}_k$ is also an edge of $\mathcal{E}'$. The first coordinate of $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$ differ from each other. So the first coordinate of $\mathbf{u}_k$ must be 1, which is the same with the first coordinate of $\mathbf{u}_1$. Similarly, for the edge $v_i, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k, \mathbf{u}_{k+1}$, we may get that the first coordinate of $\mathbf{u}_{k+1}$ must be 2. Obviously, $\mathbf{u}_1$ and $\mathbf{u}_{k+1}$ differ in the first coordinate. As $\mathbf{u}_1$ and $\mathbf{u}_{k+1}$ are not the same vertices, this sequence cannot be a semicycle.

$\vdots$

(k) $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}, v_1, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{2k-2}$. We assume that $\{1, 2, \ldots, k-1\}$ are respectively the first coordinate of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$. According to the chain-connected properties, the $k$ edges in this sequence all contain the vertex $v_1$. So the first coordinates of the vertices in every edge except $v_1$ differ from each other. For $k \leq j \leq 2k-2$, the first coordinate of $\mathbf{u}_j$ are the same as $\mathbf{u}_{j-k+1}$. So the first coordinate of $\mathbf{u}_{2k-2}$ is $k-1$. As $\mathbf{u}_1$ and $\mathbf{u}_{2k-2}$ are not the same vertices, this sequence cannot be a semicycle.

Without loss of generality, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{i-1}, v_1, \mathbf{u}_i, \ldots, \mathbf{u}_t$ be the sequence of vertices in $C$ such that $\{v_1, \mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}\}$, $i = 1, 2, \ldots, t-(k-2)$ are all the edges of $C$. Note that every semicycle has at least 3 edges. Then $t \geq k+1$ and the first coordinates of $\mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}$ differ from each other. By the definition of the semicycle, it can be verified that $t \leq 2k-2$ and $\mathbf{u}_t = \mathbf{u}_1$. So the length of $C$ is at most $k$. Hence, the first coordinate of $\mathbf{u}_k$ is the same as the first coordinate of $\mathbf{u}_1$, so the first coordinate of $\mathbf{u}_k$ is also 1. In fact, it is easy to see that the first coordinate of $\mathbf{u}_j$ is the same as that of $\mathbf{u}_{j-k+1}$ for each $j$, $k \leq j \leq t \leq 2k-2$. Thus the first coordinate of $\mathbf{u}_t$ is $t-k+1$. Obviously, $t-k+1 \neq 1$ as $t \leq 2k-2$. This contradicts the fact that $\mathbf{u}_1 = \mathbf{u}_t$.

(iii) $\mathcal{H}'$ is a $k$-hypertree. For any $e \in \mathcal{E}$, $e' \in \mathcal{E}'$, since $|e \cap e'| \leq 1$ and $k \geq 3$, all chains in $\mathcal{H}'$ belong to either $\mathcal{E}$ or $\mathcal{E}'$. Let $P$ be a chain in $\mathcal{H}'$. If $P$ belongs to $\mathcal{E}$, $P$ is also a chain in $\mathcal{H}$. Since $\mathcal{H}$ is $k$-hypertree, every chain in it is of length at most $k$, so $P$ is of length at most $k$ in $\mathcal{H}'$. If $P$ belongs to $\mathcal{E}'$, as we noted in the proof in (ii), $P$ contains at most $2k-1$ vertices. This implies that $P$ is of length at most $k$ in $\mathcal{H}'$. ■

Return to the proof of Theorem 3.

By the construction of $\mathcal{H}'$, we have $|V \cup V'| = n + (k-1)^n$. Now we count the number of edges of $\mathcal{H}'$. For each $v_i \in V$, let $\mathcal{E}'_i = \{\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \mid \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1} \in V'\}$. Then $\mathcal{E}' = \bigcup_{i=1}^n \mathcal{E}'_i$. By the construction of $\mathcal{E}'_i$, it is easy to see that

$$|\mathcal{E}'_i| = ((k-1)^{k-1} - (k-1)!)^{i-1}((k-1)^{k-1})^{n-i}.$$

Hence,

$$|\mathcal{E}'| = x^{n-1} + yx^{n-2} + y^2x^{n-3} + \cdots + y^{n-1},$$

where $x = (k-1)^{k-1} - (k-1)!$, $y = (k-1)^{k-1}$. Therefore,

$$|\mathcal{E}(\mathcal{H}')| = |\mathcal{E} \cup \mathcal{E}'| \geq |\mathcal{E}'| = \frac{y^n - x^n}{y - x} = \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!}.$$

We count the limit of the ratio $|\mathcal{E}(\mathcal{H}')| \Big/ \binom{|V(\mathcal{H}')|}{k-1}$.

$$\begin{aligned}\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} &\geq \frac{\frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!}}{\binom{n+(k-1)^n}{k-1}} \\ &= \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!} \\ &\quad \cdot \frac{(k-1)![n + (k-1)^n - (k-1)]!}{[n + (k-1)^n]!}\end{aligned}$$

$$\geq \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{[n + (k-1)^n]^{k-1}}$$

$$= \frac{1 - \left[\frac{(k-1)^{k-1} - (k-1)!}{(k-1)^{k-1}}\right]^n}{\left[\frac{n+(k-1)^n}{(k-1)^n}\right]^{k-1}} = \frac{1 - \left[1 - \frac{(k-1)!}{(k-1)^{k-1}}\right]^n}{\left[\frac{n}{(k-1)^n} + 1\right]^{k-1}} \to 1 (n \to \infty).$$

On the other hand, by Theorem 2, we have

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \leq 1.$$

So, when $n \to \infty$, we obtain

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \to 1.$$

Thus, if $\{\mathcal{H}_i\}_{i=1}^{\infty}$ is a sequence of $k$-uniform $k$-hypertrees on $n$ $(n \geq k)$ vertices such that $\lim_{n\to\infty} |V(\mathcal{H}_i)| = \infty$, then,

$$|\mathcal{E}(\mathcal{H}'_i)| \sim \binom{|V(\mathcal{H}'_i)|}{k-1}.$$

■

Now let us review the construction given in [5]. In [5], the author constructed a $k$-hypertree $H_i^k = (V_{2^i,k}, E_{2^i,k})$, where $|V_{2^i}| = 2^i + F(2^i, k-1)$, $|E_{2^i,k}| = \binom{2^i}{k-1} + |D_{n,k}|$, and $D_{n,k}$ is the set of edges of a hypertree $F_{n,k} = (U_{n,k}, D_{n,k})$. It is proved that $|E_{2^i,k}|$ is asymptotically $\binom{|V_{2^i}|}{k-1}$. The construction of $H_i^k$ and counting its number of edges are intricate and technical. This note provides an elegant construction of the desired $k$-hypertree by using vectors and matrices, and the proof is easy.


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