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# CONSTRUCTION OF COSPECTRAL INTEGRAL REGULAR GRAPHS

Ravindra B. Bapat [1]

*Indian Statistical Institute*
*Delhi Centre, 7 S.J.S.S. Marg*
*New Delhi 110 016, India*

**e-mail:** rbb@isid.ac.in

and

Masoud Karimi [2]

*School of Mathematical Sciences*
*Anhui University, Hefei, China*
*and*
*Department of Mathematics, Bojnourd Branch*
*Islamic Azad University, Bojnourd, Iran*
*P.O. Box 94176-94686*

**e-mail:** karimimth@yahoo.com

## Abstract

Graphs $G$ and $H$ are called cospectral if they have the same characteristic polynomial. If eigenvalues are integral, then corresponding graphs are called integral graph. In this article we introduce a construction to produce pairs of cospectral integral regular graphs. Generalizing the construction of $G_4(a,b)$ and $G_5(a,b)$ due to Wang and Sun, we define graphs $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ and show that they are cospectral integral regular when $G$ is an integral $q$-regular graph of order $m$ and $H$ is an integral $q$-regular graph of order $(b-2)m$ for some integer $b \geq 3$.

**Keywords:** eigenvalue, cospectral graphs, adjacency matrix, integral graphs.

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[2] Corresponding author.

## 1. Introduction

We consider simple graphs, that is, graphs without loops or parallel edges. For basic notions in graph theory we refer to [12], whereas for preliminaries on graphs and matrices, see [3]. By the eigenvalues of a graph $G$, we mean the eigenvalues of its adjacency matrix $A(G)$. Graphs $G$ and $H$ are said to be cospectral if they have the same eigenvalues, counting multiplicities, or equivalently, they have the same characteristic polynomial. A graph with only integer eigenvalues is termed an integral graph. There is considerable literature on construction of cospectral graphs, see [7, 9]. Some graph operations, which when applied on integral graphs produce integral graphs, are described in [1, 2, 6]. Other results on integral graphs can be found in [11]. For all other facts or terminology on graph spectra, see [6].

The complete join of graphs $G$ and $H$, denoted by $G \vee H$, is a graph with $V(G \vee H) := V(G) \cup V(H)$ and

$$E(G \vee H) := E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}.$$

The complement of graph $G$ is denoted by $\overline{G}$, and $G + H$ denotes $\overline{\overline{G} \vee \overline{H}}$.

Through the paper $I_b$ and $J_{a\times b}$, respectively, denote the identity matrix of order $b$, and the matrix with all ones of order $a \times b$. In block matrices, the symbol 0 will be used to denote zero matrix; also we denote by $\mathbf{1}$ and $\mathbf{0}$ vectors of all ones and all zeros, respectively. The dimension of these matrices and vectors will be mentioned explicitly, unless they can be determined by the context. If $A$ and $B$ are matrices of order $m \times n$ and $p \times q$, respectively, then the Kronecker product of $A$ and $B$, denoted $A \otimes B$, is the $mp \times nq$ block matrix $[a_{ij}B]$. For a given vector $v$, by $v'$ we mean the transposition of $v$.

In Section 2 of this paper we shall give some lemmas on computing characteristics polynomial of some types of block matrices.

Bussemaker and Cvetković [5] introduced connected integral cubic graphs. In 1978, Schwenk, independently, obtained these graphs and denotes these thirteen integral graphs by $G_1, G_2, \ldots, G_{13}$; see [8]. The graphs $G_4$ and $G_5$ are a pair of non-isomorphic connected cospectral integral cubic graphs on 20 vertices.

Section 3 of the paper is motivated by Wang and Sun [10] who constructed graphs $G_4(a, b)$ and $G_5(a, b)$ based on $G_4$ and $G_5$. They showed that for any positive integer $a$, $G_4(a, a+2)$ and $G_5(a, a+2)$ form a pair of integral cospectral $(a+2)$-graphs and concluded that there exist infinitely many pairs of cospectral integral graphs. We shall give a generalization of $G_4(a, b)$ and $G_5(a, b)$. In fact two new families of regular graphs will be constructed such that they are cospectral and integral. For the sake of completeness and comparing the results, we recall the adjacency matrices of $G_4(a, b)$ and $G_5(a, b)$. The adjacency matrix of $G_4(a, b)$ is $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$, where

$$(1)\quad A_0 = \begin{pmatrix} 0_{a\times a} & J_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ J_{b\times a} & 0_{b\times b} & I_b & 0_{b\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & B \\ 0_{b\times a} & 0_{b\times b} & B & 0_{b\times b} \end{pmatrix}, \ A_1 = \begin{pmatrix} 0_{a\times a} & 0_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & I_b \end{pmatrix}$$

and

$$(2)\qquad B = \begin{pmatrix} 1 & J_{1\times(b-2)} & 0 \\ J_{(b-2)\times 1} & J_{(b-2)\times(b-2)} - I_{b-2} & J_{(b-2)\times 1} \\ 0 & J_{1\times(b-2)} & 1 \end{pmatrix}.$$

The adjacency matrix of $G_5(a,b)$ is $\begin{pmatrix} M_0 & M_1 \\ M_1 & M_0 \end{pmatrix}$, where

$$(3)\quad M_0 = \begin{pmatrix} 0_{a\times a} & J_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ J_{b\times a} & 0_{b\times b} & I_b & I_b \\ 0_{b\times a} & I_b & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & 0_{b\times b} \end{pmatrix}, \ M_1 = \begin{pmatrix} 0_{a\times a} & 0_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & B & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & B \end{pmatrix}$$

and $B$ is the same as in (2).

## 2. Some Lemmas

A square matrix is said to be regular if all its row sums and column sums are equal. The common value of the row and column sum is called the regularity of the matrix. Clearly, in this case the regularity is an eigenvalue with the all ones vector as an eigenvector.

The next result is known when $A$ and $B$ are adjacency matrices of graphs (see [6], Theorem 2.8, p. 57). We present a more general statement for completeness.

**Theorem 1.** *Let $A$ and $B$ be symmetric, regular matrices of orders $p, s$ and regularity $q, w$ respectively. If $q, \mu_2, \dots, \mu_p$ and $w, \lambda_2, \dots, \lambda_s$, are respectively the eigenvalues of $A$ and $B$, and $\alpha$, $\beta$ are real scalars, then the eigenvalues of the matrix*

$$T = \begin{pmatrix} A & \beta J_{p\times s} \\ \alpha J_{s\times p} & B \end{pmatrix}$$

*are $\mu_2, \dots, \mu_p, \lambda_2, \dots, \lambda_s$ and $\frac{q+w\pm\sqrt{(q+w)^2+4(\alpha\beta ps - wq)}}{2}$.*

The proof of the next result is easy and is omitted.

**Lemma 2.** *Suppose that $X$ and $Y$ are square matrices of the same order. Let*

$$T = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}. \tag{4}$$

*Then the eigenvalues of $T$ are the eigenvalues of $X - Y$ and the eigenvalues of $X + Y$.*

**Lemma 3.** *Suppose that $r, x$ and $y$ are real scalars. Then*

$$\begin{aligned} \det(A - \lambda I) &= (-1)^{a-b}\lambda^{a-1}(\lambda - r)^{b-1}(\lambda^2 - r\lambda - 2)^{b-1} \\ &\times \left[\lambda^4 - 2\lambda^3 r + \left(-b^2 + (-xya + 2)b - 2 + r^2\right)\lambda^2 \right. \\ &\left. + (2 + b^2 + (xya - 2)b)r\lambda + bxya(b-1)^2\right], \end{aligned}$$

*where*

$$A = \begin{pmatrix} I_a & yJ_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ xJ_{b\times a} & rI_b & I_b & 0_{b\times b} \\ 0_{b\times a} & I_b & I_b & B \\ 0_{b\times a} & 0_{b\times b} & B & rI_b \end{pmatrix}$$

*and $B$ is the matrix in* (2).

***Proof.*** By interchanging the second and third columns and then the second and third rows of $A - \lambda I$ we obtain

$$\det(A - \lambda I) = \det \left(\begin{array}{cc|cc} -\lambda I_a & 0_{a\times b} & yJ_{a\times b} & 0_{a\times b} \\ 0_{b\times a} & -\lambda I_b & I_b & B \\ \hline xJ_{b\times a} & I_b & (r-\lambda)I_b & 0_{b\times b} \\ 0_{b\times a} & B & 0_{b\times b} & (r-\lambda)I_b \end{array}\right).$$

First, assume that $\lambda \neq 0$ and put $k := -\lambda^2 + r\lambda + 1$. Applying the Schur complement formula for computing determinant shows

$$\begin{aligned} &\det(A - \lambda I) \\ &= (-\lambda)^{a+b} \times \det\left(\begin{pmatrix} (r-\lambda)I_b & 0_{b\times b} \\ 0_{b\times b} & (r-\lambda)I_b \end{pmatrix} + \tfrac{1}{\lambda}\begin{pmatrix} xJ_{b\times a} & I_b \\ 0_{b\times a} & B \end{pmatrix}\begin{pmatrix} yJ_{a\times b} & 0_{a\times b} \\ I_b & B \end{pmatrix}\right) \\ &= (-\lambda)^{a+b}\det\left((r-\lambda)I_{2b} + \frac{1}{\lambda}\begin{pmatrix} xyaJ_b + I_b & B \\ B & B^2 \end{pmatrix}\right) \\ &= (-\lambda)^{a-b}\det\begin{pmatrix} xyaJ_b + kI_b & B \\ B & (b-2)J_b + kI_b \end{pmatrix} \\ &= (-\lambda)^{a-b}\det\left((xyaJ_b + kI_b)((b-2)J_b + kI_b) - B^2\right) \\ &= (-\lambda)^{a-b}\det\left(((xya + b - 2)k + (b-2)(xyab - 1))J_b + (k^2 - 1)I_b\right) \\ &= (-\lambda)^{a-b}(k^2 - 1)^{b-1}(k^2 - 1 + b(xya + b - 2)k + b(b-2)(xyab - 1)) \\ &= (-\lambda)^{a-b}(k-1)^{b-1}(k+1)^{b-1}\left[k^2 - 1 + b(xya + b - 2)k + b(b-2)(xyab - 1)\right]. \end{aligned}$$

By replacing $k = -\lambda^2 + r\lambda + 1$ and then simplifying the above relation we can see that

$$\begin{aligned}\det(A - \lambda I) &= (-1)^{a-b}\lambda^{a-1}(\lambda - r)^{b-1}(\lambda^2 - r\lambda - 2)^{b-1} \\ &\quad \times \left[\lambda^4 - 2\lambda^3 r + (-b^2 + (-xya + 2)b - 2 + r^2)\lambda^2 \right. \\ &\quad \left. + (2 + b^2 + (xya - 2)b)r\lambda + bxya(b-1)^2\right].\end{aligned}$$

To complete the proof, we need to consider the case $\lambda = 0$. To this end, note that $\lambda = 0$ is an eigenvalue of $A$. By assuming $\lambda \neq 0$ we could compute $3b + 1$ nonzero eigenvalues, and so the multiplicity of $\lambda = 0$ must be $a - 1$. ■

**Lemma 4.** *Suppose that $x, y$ and $r$ are scalars and $B$ is the matrix in (2). Then*

$$\det(rB + xJ + yI) = (y + r)(y - r)^{b-2}(bx + y + rb - r).$$

***Proof.*** Since

$$B = \begin{pmatrix} 1 & J_{1\times(b-2)} & 0 \\ J_{(b-2)\times 1} & J_{(b-2)\times(b-2)} - I_{b-2} & J_{(b-2)\times 1} \\ 0 & J_{1\times(b-2)} & 1 \end{pmatrix},$$

then

$$B + xJ + yI = \left(\begin{array}{c|ccccc|c} 1+x+y & x+1 & \dots & \dots & \dots & x+1 & x \\ \hline x+1 & x+y & 1+x & \dots & \dots & 1+x & x+1 \\ \vdots & 1+x & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & x+1 & \vdots \\ x+1 & 1+x & \dots & \dots & 1+x & x+y & x+1 \\ \hline x & x+1 & \dots & \dots & \dots & x+1 & 1+x+y \end{array}\right).$$

Subtract the first column from the other columns, then add the last row to the first row and finally add rows $2, \dots, b-2$ to the first row to get

$$\left(\begin{array}{c|ccccc|c} b(x+1)+y-1 & 0 & \dots & \dots & \dots & 0 & 0 \\ \hline x+1 & y-1 & 0 & \dots & \dots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 & \vdots \\ x+1 & 0 & \dots & \dots & 0 & y-1 & 0 \\ \hline x & 1 & \dots & \dots & \dots & 1 & 1+y \end{array}\right).$$

In this way we have a triangular matrix, and so

$$\det(B + xJ + yI) = (y + 1)(y - 1)^{b-2}(bx + y + b - 1).$$

Now

$$\begin{aligned}\det(rB + xJ + yI) &= r^b \det\left(B + \frac{x}{r}J + \frac{y}{r}I\right)\\ &= r^b\left(\frac{y}{r} + 1\right)\left(\frac{y}{r} - 1\right)^{b-2}\left(b\frac{x}{r} + \frac{y}{r} + b - 1\right)\\ &= (y + r)(y - r)^{b-2}(bx + y + rb - r)\end{aligned}$$

and the proof is complete. ■

**Lemma 5.** *If* $T = \begin{pmatrix} xI_a & \beta J_{a\times b} \\ \alpha J_{b\times a} & xI_b \end{pmatrix}$ *is invertible, then*

$$T^{-1} = \begin{pmatrix} \frac{1}{x}\left(I_a + \frac{b}{x^2-\alpha\beta ab}J_a\right) & \frac{-\beta}{x^2-\alpha\beta ab}J_{a\times b} \\ \frac{-\alpha}{x^2-\alpha\beta ab}J_{b\times a} & \frac{1}{x}\left(I_b + \frac{a}{x^2-\alpha\beta ab}J_b\right) \end{pmatrix},$$

*where* $\alpha$, $\beta$ *and* $x$ *are real scalars.*

***Proof.*** Since $T$ is invertible, by Theorem 1, we see that $x \neq 0$ and $x^2 - \alpha\beta ab \neq 0$. The result is proved by a simple verification. ■

**Lemma 6.** *Suppose that* $\alpha, \beta$ *and* $r$ *are scalars. Then*

$$\begin{aligned}\det(M-\lambda I) = (-1)^{a+3b}\lambda^{a-1}(\lambda-r)(\lambda+r)^{b-2}(\lambda-rb+r)(\lambda^2-r\lambda-2)(\lambda^2+r\lambda-2)^{b-2}\\ \times(\lambda^3 - r\lambda^2(b-1) - \lambda(2+\alpha\beta ab) + \alpha\beta rab(b-1)),\end{aligned}$$

*where*

$$M = \begin{pmatrix} 0 & \beta J_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ \alpha J_{b\times a} & 0 & I_b & I_b \\ 0_{b\times a} & I_b & rB & 0_{b\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & rB \end{pmatrix}$$

*and* $B$ *is the matrix in* (2).

***Proof.*** First assume that $\lambda \neq 0$ and $\lambda^2 - \alpha\beta ab \neq 0$. By Lemma 5,

$$T = \begin{pmatrix} -\lambda I_a & \beta J_{a\times b} \\ \alpha J_{b\times a} & -\lambda I_b \end{pmatrix}$$

is invertible and we may use the Schur complement formula to compute determinant. We obtain

$$\begin{aligned}
&\det(M-\lambda I)\\
&= \det(T)\times\det\left(\begin{pmatrix} rB-\lambda I_b & 0_{b\times b}\\ 0_{b\times b} & rB-\lambda I_b\end{pmatrix}-\begin{pmatrix} 0_{b\times a} & I_b\\ 0_{b\times a} & I_b\end{pmatrix}T^{-1}\begin{pmatrix} 0_{b\times a} & 0\\ I_b & I_b\end{pmatrix}\right)\\
&= \det(T)\times\det\begin{pmatrix} rB+\left(\frac{1}{\lambda}-\lambda\right)I_b+\frac{a}{\lambda(\lambda^2-\alpha\beta ab)}J_b & \frac{1}{\lambda}\left(I_b+\frac{a}{\lambda^2-\alpha\beta ab}J_b\right)\\ \frac{1}{\lambda}\left(I_b+\frac{a}{\lambda^2-\alpha\beta ab}J_b\right) & rB+\frac{1-\lambda^2}{\lambda}I_b+\frac{a}{\lambda(\lambda^2-\alpha\beta ab)}J_b\end{pmatrix}\\
&= (-\lambda)^{a+b-2}(\lambda^2-\alpha\beta ab)\det(rB-\lambda I_b)\times\det\left(rB+\left(\frac{2}{\lambda}-\lambda\right)I_b+\frac{2a}{\lambda(\lambda^2-\alpha\beta ab)}J_b\right)\\
&= (-\lambda)^{a+b-2}(\lambda^2-\alpha\beta ab)(r-\lambda)(-\lambda-r)^{b-2}(-\lambda+rb-r)\\
&\times\left(\frac{2}{\lambda}-\lambda+r\right)\left(\frac{2}{\lambda}-\lambda-r\right)^{b-2}\left(\frac{2\alpha\beta ab}{\lambda(\lambda^2-\alpha\beta ab)}+\frac{2}{\lambda}-\lambda+rb-r\right)\\
&= (-1)^{a+3b}\lambda^{a-1}(\lambda-r)(\lambda+r)^{b-2}(\lambda-rb+r)(\lambda^2-r\lambda-2)\\
&\times(\lambda^2+r\lambda-2)^{b-2}[\lambda^3-r\lambda^2(b-1)-\lambda(2+\alpha\beta ab)+r\alpha\beta ab(b-1)].
\end{aligned}$$

To complete the proof, we must show that the assumptions $\lambda\neq 0$ and $\lambda^2-ab\neq 0$ can be relaxed. Towards this, first note that $3b+1$ nonzero eigenvalues of $M$ are the roots of

$$\begin{aligned}
&(\lambda-r)(\lambda+r)^{b-2}(\lambda-rb+r)(\lambda^2-r\lambda-2)(\lambda^2+r\lambda-2)^{b-2}\\
&\times\left(\lambda^3-r\lambda^2(b-1)-\lambda(2+\alpha\beta ab)+r\alpha\beta ab(b-1)\right).
\end{aligned}$$

It remains to find the other $a-1$ eigenvalues. It is clear that $\lambda=0$ is an eigenvalue of $M$ of multiplicity at least $a-1$. This means that $\lambda=\pm\sqrt{\alpha\beta ab}$ is not an eigenvalue, and so $\lambda^2-\alpha\beta ab\neq 0$ is acceptable. Since $\lambda^{a-1}$ appears in the computations, we also obtain $a-1$ zero eigenvalues. ■

## 3. THE CONSTRUCTION AND MAIN RESULTS

We now present the construction. Suppose that $G$ is a $q$-regular graph of order $n$ and $H$ is a $p$-regular graph of order $m$. Graphs $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are constructed as follows. For a given integer $b\geqslant 3$ and $j=1,2$ and 3 assume that $U_j=\{H_i^j|i=1,2,\ldots,b\}$ and $V_j=\{K_i^j|i=1,2,\ldots,b\}$ are classes of $b$ copies of $H$, and let $G_1$ and $G_2$ be two copies of $G$. By the following instructions we join these graphs together to obtain $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ which are graphs of order $2n+6bm$.

**Step 1.** First do the following graph operations.

(i) $G_1\vee(H_1^1+H_2^1+\cdots+H_b^1)$.

(ii) $G_2 \vee (K_1^1 + K_2^1 + \cdots + K_b^1)$.
(iii) $H_1^2 \vee K_i^2$ for $i = 1, 2, \ldots, b-1$ and $H_b^2 \vee K_i^2$ for $i = 2, \ldots, b$.
(iv) $H_i^2 \vee K_j^2$ for $i \neq j$ and $2 \leq i, j \leq b-1$.
(v) $H_1^3 \vee K_i^3$ for $i = 1, 2, \ldots, b-1$ and $H_b^3 \vee K_i^3$ for $i = 2, \ldots, b$.
(vi) $H_i^3 \vee K_j^3$ for $i \neq j$ and $2 \leq i, j \leq b-1$.

**Step 2.** To construct $\mathcal{G}_4(G, H)$ in addition to Step 1, do the following graph operations (Figure 1).
(i) $H_i^1 \vee (H_i^2 + K_i^1)$ for $i = 1, 2, \ldots, b$.
(ii) $K_i^1 \vee (H_i^1 + H_i^3)$ for $i = 1, 2, \ldots, b$.
(iii) $K_i^2 \vee K_i^3$ $i = 2, \ldots, b$.

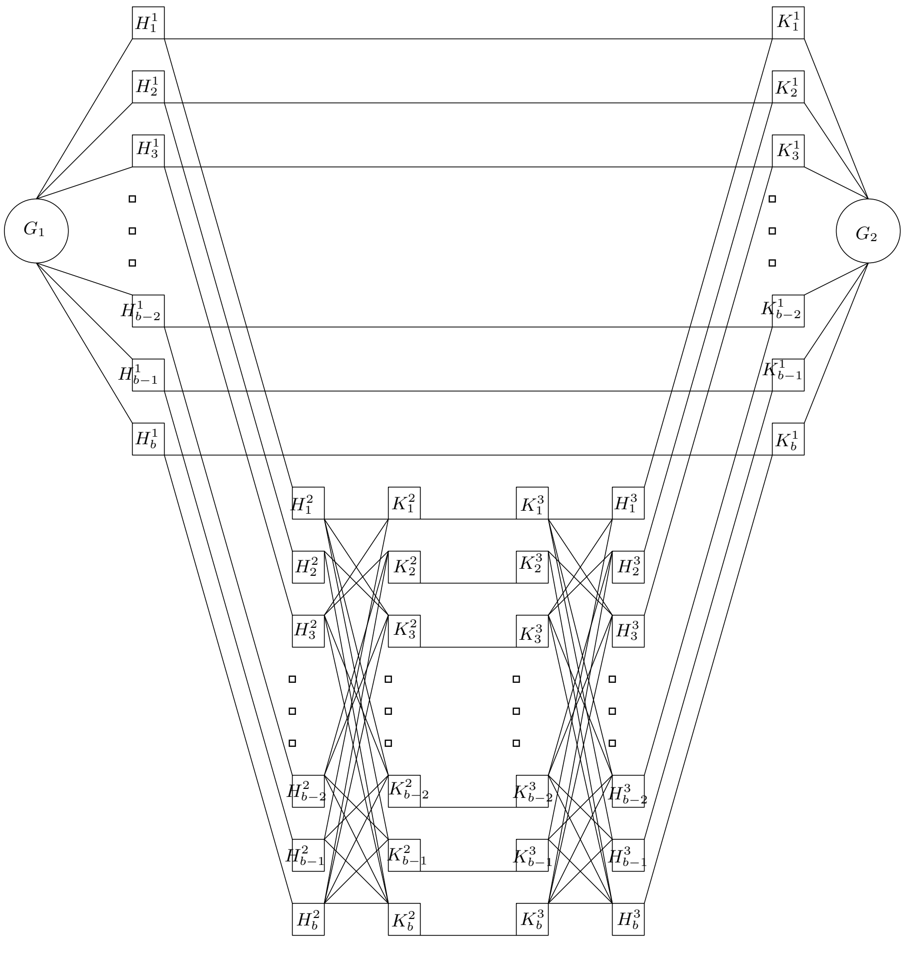


Figure 1. $\mathcal{G}_4(G, H)$.

**Step 3.** To construct $\mathcal{G}_5(G,H)$ in addition to Step 1, do the following graphs operations (Figure 2).

(i) $H_i^1 \vee (H_i^2 + H_i^3)$ for $i = 1, 2, \ldots, b$.

(ii) $K_i^1 \vee (K_i^2 + K_i^3)$ for $i = 1, 2, \ldots, b$.

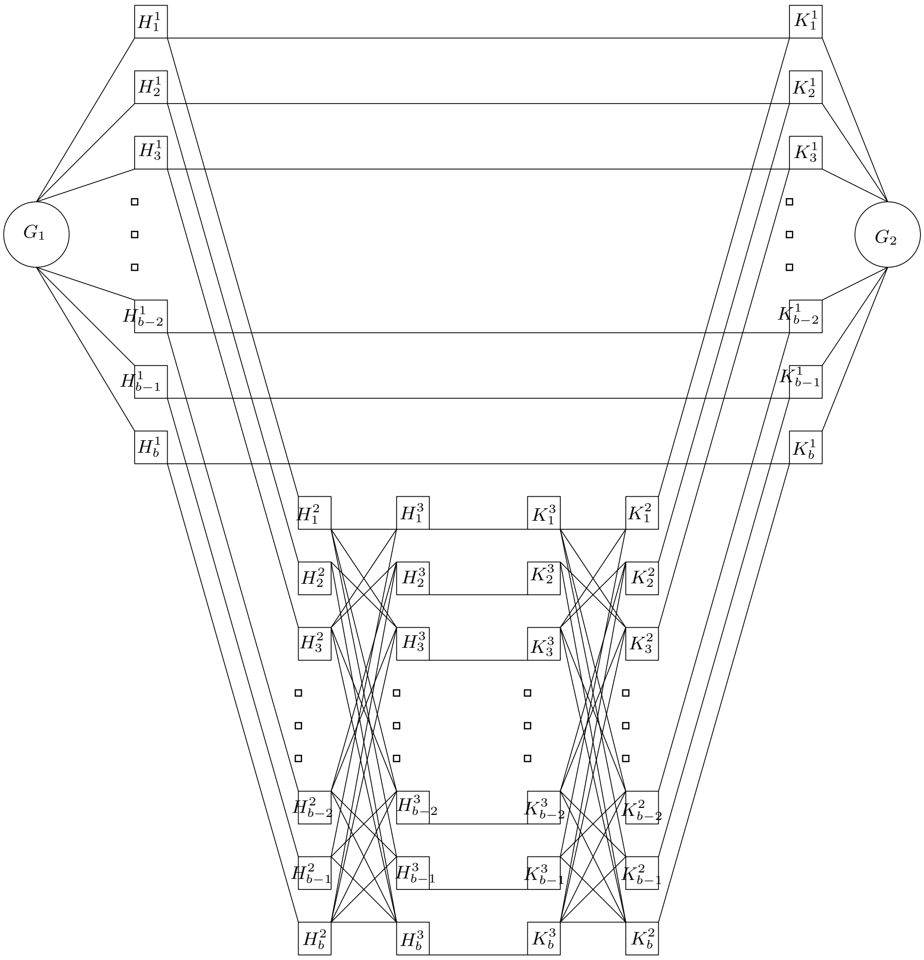


Figure 2. $\mathcal{G}_5(G,H)$.

Accordingly, the adjacency matrices of $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are as follows. $A(\mathcal{G}_4(G,H)) = \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_1 \\ \mathcal{A}_1 & \mathcal{A}_0 \end{pmatrix}$ where

$$(5)\qquad \mathcal{A}_0 = \begin{pmatrix} A(G) & J & 0 & 0 \\ J & I_b \otimes A(H) & I_b \otimes J & 0 \\ 0 & I_b \otimes J & I_b \otimes A(H) & B \otimes J \\ 0 & 0 & B \otimes J & I_b \otimes A(H) \end{pmatrix},$$

$$(6)\qquad \mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_b \otimes J & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_b \otimes J \end{pmatrix},$$

and $A(\mathcal{G}_5(G,H)) = \begin{pmatrix} \mathcal{M}_0 & \mathcal{M}_1 \\ \mathcal{M}_1 & \mathcal{M}_0 \end{pmatrix}$, where

$$(7)\qquad \mathcal{M}_0 = \begin{pmatrix} A(G) & J & 0 & 0 \\ J & I_b \otimes A(H) & I_b \otimes J & I_b \otimes J \\ 0 & I_b \otimes J & I_b \otimes A(H) & 0 \\ 0 & I_b \otimes J & 0 & I_b \otimes A(H) \end{pmatrix},$$

$$(8)\qquad \mathcal{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B \otimes J_m & 0 \\ 0 & 0 & 0 & B \otimes J_m \end{pmatrix},$$

and $B$ is the same as in (2).

**Lemma 7.** *Graphs $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are regular if and only if $p = q$ and $n = (b-2)m$. In this case they are $(q+mb)$-regular graphs.*

***Proof.*** We show the result for $\mathcal{G}_4(G,H)$, and analogously it can be proved for $\mathcal{G}_5(G,H)$. Multiplying matrix $\mathcal{A}_0 + \mathcal{A}_1$ by vector $\mathbf{1}$ yields

$$\mathbf{1}'(\mathcal{A}_0+\mathcal{A}_1) = (q+mb, \dots, q+mb, 2m+p+n, \dots, 2m+p+n, p+bm, \dots, p+bm).$$

This means that the degrees of vertices of $\mathcal{G}_4(G,H)$ are $q+mb$, $n+p+2m$ or $p+mb$. Thus $\mathcal{G}_4(G,H)$ is regular if and only if $q+mb = n+p+2m = p+mb$. Hence $\mathcal{G}_4(G,H)$ if and only if $p = q$ and $n = (b-2)m$. ∎

Henceforth we assume that $p = q$ and $n = (b-2)m$ so that $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are regular graphs.

We recall the notion of equitable partition. Suppose $A$ is a symmetric matrix whose rows and columns are indexed by $\{1,\dots,n\}$. Let $X = \{X_1,\dots,X_s\}$ be a partition of $\{1,\dots,n\}$. By definition, $X$ is an equitable partition if $A[X_i|X_j]\mathbf{1} = b_{i,j}\mathbf{1}$ for $i,j = 1,\dots,s$, where $A[X_i|X_j]$ is the submatrix of $A$ determined by the row corresponding to $X_i$ and the columns corresponding to $X_j$. Let $H$ denotes the $n \times s$ matrix whose $j^{\text{th}}$ column is the eigenvector of $A[X_j|X_j]$ for $j = 1, \ldots, s$, and $B$ denotes the $s \times s$ matrix with components $b_{i,j}$. Then we have $AH = HB$, and so eigenvalues of $A$ consist of eigenvalues of $B$ together with the eigenvalues belonging to eigenvectors orthogonal to the columns of $H$ (i.e., summing to zero on each part of the partition). See [4, page 24].

**Theorem 8.** *Suppose that $q = \lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $G$ and $q = \mu_1, \mu_2, \ldots, \mu_m$ are eigenvalues of $H$. Then*

$$\begin{aligned}\varphi_{\mathcal{G}_4(G,H)}(\lambda) = \varphi_{\mathcal{G}_5(G,H)}(\lambda) = \frac{1}{(\lambda - q)^{6b+2}} \varphi_G(\lambda)^2 \varphi_H(\lambda)^{6b} \\ \times (\lambda - b + 2 - q)^{2b-2}(\lambda + b - 2 - q)^{2b-2}(\lambda - 2(b-2) - q)^{b-1} \\ \times (\lambda + (1-b)(b-2) - q)^2(\lambda + (b-1)(b-2) - q)^2 \\ \times (\lambda + (b-2)^2 - q)(\lambda - (b-2)^2 - q)(\lambda + 2(b-2) - q)^{b-1} \\ \times (\lambda + b(b-2) - q)(\lambda - b(b-2) - q).\end{aligned}$$

***Proof.*** In view of Lemma 2, $\varphi_{\mathcal{G}_4(G,H)}(\lambda) = \varphi_{\mathcal{A}_0+\mathcal{A}_1}(\lambda)\varphi_{\mathcal{A}_0-\mathcal{A}_1}(\lambda)$. We consider two cases.

*Case* 1. $\varphi_{\mathcal{A}_0+\mathcal{A}_1}(\lambda)$. By (6) and (5) we have

$$(9) \quad \mathcal{A}_0 + \mathcal{A}_1 = \begin{pmatrix} A(G) & J & 0 & 0 \\ J & I_b \otimes (A(H) + J) & I_b \otimes J & 0 \\ 0 & I_b \otimes J & I_b \otimes A(H) & B \otimes J \\ 0 & 0 & B \otimes J & I_b \otimes (A(H) + J) \end{pmatrix}.$$

Suppose that $u$ is an eigenvector of $A(G)$ corresponding to an eigenvalue $\lambda \neq q$ of $A(G)$. Then $\mathbf{1}'u = 0$ implies that $[u', \mathbf{0}', \mathbf{0}', \mathbf{0}'](\mathcal{A}_0 + \mathcal{A}_1) = \lambda[u', \mathbf{0}', \mathbf{0}', \mathbf{0}']$ in which $\mathbf{0}$ is the $bm$-tuple zero vector. This means that every eigenvalue $\lambda \neq q$ of $A(G)$ is an eigenvalue of $\mathcal{A}_0 + \mathcal{A}_1$.

Now, let $v$ be an eigenvector of $A(H)$ corresponding to an eigenvalue $\mu \neq q$ of $A(H)$. Considering the vector $[\underbrace{0, \ldots, 0}_{n\text{-tuple}}, \mathbf{0}', \ldots, \mathbf{0}', v', \mathbf{0}', \ldots, \mathbf{0}']$, in which $\mathbf{0}$ is an $m$-tuple zero vector and $v'$ stands in $i^{th}$ place for $i = 1, 2, \ldots, 3b$, we have $v'\mathbf{1} = 0$, and so

$$[\underbrace{0, \ldots, 0}_{n\text{-tuple}}, \mathbf{0}', \ldots, \mathbf{0}', v', \mathbf{0}', \ldots, \mathbf{0}'](\mathcal{A}_0 + \mathcal{A}_1) = \mu[\underbrace{0, \ldots, 0}_{n\text{-tuple}}, \mathbf{0}', \ldots, \mathbf{0}', v', \mathbf{0}', \ldots, \mathbf{0}'].$$

Thus $\mu$ is an eigenvalue of $\mathcal{A}_0 + \mathcal{A}_1$ of multiplicity $3b$. Accordingly, $\mathcal{A}_0 + \mathcal{A}_1$ has $3b(m-1)$ eigenvalues derived from spectrum of $H$. These, together with the $n-1$ eigenvalues of $A(G)$, account for $(3bm - 1) + (n - 1)$ eigenvalues of

$\mathcal{A}_0 + \mathcal{A}_1$. Since $\mathcal{A}_0 + \mathcal{A}_1$ has $n + 3bm$ eigenvalues, it remains to find $1 + 3b$ eigenvalues. To determine these eigenvalues, we consider an equitable partition of $\mathcal{A}_0 + \mathcal{A}_1$. In view of the form of $\mathcal{A}_0 + \mathcal{A}_1$ there is a partition of vertices, say $\{X_0, X_1, X_2, \ldots, X_{bm}\}$, such that $|X_0| = n$ and $|X_i| = m$ for $i = 1, 2, \ldots, bm$. Moreover, we have $(\mathcal{A}_0 + \mathcal{A}_1)[X_i|X_j]\mathbf{1} = s_{i,j}\mathbf{1}$ where scalars $s_{i,j}$ are entries of the matrix $S$

$$S = \begin{pmatrix} q & m\mathbf{1}' & \mathbf{0}' & \mathbf{0}' \\ n\mathbf{1} & (q+m)I_b & mI_b & 0 \\ \mathbf{0} & mI_b & qI_b & mB \\ \mathbf{0} & 0 & mB & (q+m)I_b \end{pmatrix}. \tag{10}$$

Therefore $(\mathcal{A}_0 + \mathcal{A}_1)L = LS$ where the columns of the matrix $L$ consist of characteristic vectors of $(\mathcal{A}_0 + \mathcal{A}_1)[X_i|X_i]$ for $i = 0, 1, 2, \ldots, 3b$. It follows that the eigenvalues of $S$ are eigenvalues of $\mathcal{A}_0 + \mathcal{A}_1$ as well. In Lemma 3, if we put $a = r = 1$, $x = \frac{n}{m}$ and $y = 1$, then $S = \frac{n}{m}A + qI_{3b+1}$, and use $n = (b-2)m$, then

$$S = (b-2)A + qI_{3b+1}. \tag{11}$$

Further, applying these values in Lemma 3 we get

$$\varphi_A(\lambda) = (-1)^{1-b}(\lambda-1)^{b-1}(\lambda+1)^{b-1}(\lambda-2)^{b-1}(\lambda+1-b)(\lambda-1+b)(\lambda-2+b)(\lambda-b).$$

Thus in view of relation (11) the characteristic polynomial of $S$ can be computed as follows.

$$\begin{aligned}\varphi_S(\lambda) &= (-1)^{1-b}(\lambda - b + 2 - q)^{b-1}(\lambda + b - 2 - q)^{b-1}(\lambda - 2(b-2) - q)^{b-1} \\ &\quad \times (\lambda + (1-b)(b-2) - q)(\lambda + (b-1)(b-2) - q) \\ &\quad \times (\lambda + (b-2)^2 - q)(\lambda - b(b-2) - q).\end{aligned}$$

We conclude that

$$\begin{aligned}\varphi_{\mathcal{A}_0+\mathcal{A}_1}(\lambda) &= \tfrac{(-1)^{1-b}}{(\lambda-q)^{3b+1}}\varphi_G(\lambda)\varphi_H(\lambda)^{3b}(\lambda - b + 2 - q)^{b-1}(\lambda + b - 2 - q)^{b-1} \\ &\quad \times (\lambda - 2(b-2) - q)^{b-1}(\lambda + (1-b)(b-2) - q)(\lambda + (b-1)(b-2) - q) \\ &\quad \times (\lambda + (b-2)^2 - q)(\lambda - b(b-2) - q).\end{aligned}$$

*Case* 2. $\varphi_{\mathcal{A}_0-\mathcal{A}_1}(\lambda)$. By a similar method that we used for $\mathcal{A}_0 + \mathcal{A}_1$, it can be shown that if in Lemma 3 we put $r = -1$, $a = 1$, $y = 1$ and $x = b-2$, then

$$\begin{aligned}\varphi_{\mathcal{A}_0-\mathcal{A}_1}(\lambda) &= \frac{(-1)^{1-b}}{(\lambda-q)^{3b+1}}\varphi_G(\lambda)\varphi_H(\lambda)^{3b} \times (\lambda + b - 2 - q)^{b-1}(\lambda + 2(b-2) - q)^{b-1} \\ &\quad \times (\lambda - b + 2 - q)^{b-1}(\lambda + (1-b)(b-2) - q)(\lambda - (b-2)^2 - q) \\ &\quad \times (\lambda + b(b-2) - q)(\lambda + (b-1)(b-2) - q).\end{aligned}$$

We remark that if $G$ and $H$ are empty graphs, then $\mathcal{G}_4(G,H)$ reduces to the known graph $G_4(a,b)$. We now turn to $\varphi_{\mathcal{G}_5(G,H)}(\lambda)$.

From Lemma 2 $\varphi_{\mathcal{G}_5(G,H)}(\lambda) = \varphi_{\mathcal{M}_0+\mathcal{M}_1}(\lambda)\varphi_{\mathcal{M}_0-\mathcal{M}_1}(\lambda)$. Therefore we again have two cases.

*Case* 3. $\varphi_{\mathcal{M}_0+\mathcal{M}_1}(\lambda)$. By (7) and (8) we have

$$\mathcal{M}_0+\mathcal{M}_1 = \begin{pmatrix} A(G) & J & 0 & 0 \\ J & I_b \otimes A(H) & I_b \otimes J & I_b \otimes J \\ 0 & I_b \otimes J & I_b \otimes A(H) + B \otimes J_m & 0 \\ 0 & I_b \otimes J & 0 & I_b \otimes A(H) + B \otimes J_m \end{pmatrix}.$$

Similarly to Case 1, we can deduce that $\lambda_2, \ldots, \lambda_n$ are eigenvalues of $\mathcal{M}_0 + \mathcal{M}_1$ derived from spectrum of $G$ and $\mu_2, \ldots, \mu_n$ are also eigenvalues of multiplicity $3b$ derived from spectrum of $H$. Therefore we have found $3b(m-1)$ eigenvalues from $3mb + n$. As before, it remains to find $3b + 1$ eigenvalues.

Considering an equitable partition for matrix $\mathcal{M}_0 + \mathcal{M}_1$ we obtain these eigenvalues. By the structure of $\mathcal{M}_0 + \mathcal{M}_1$ let $\{X_0, X_1, X_2, \ldots, X_{bm}\}$ be a partition of vertices such that $|X_0| = n$ and $|X_i| = m$ for $i = 1, 2, \ldots, bm$. Moreover, we have $(\mathcal{M}_0 + \mathcal{M}_1)[X_i|X_j]\mathbf{1} = s_{i,j}\mathbf{1}$ where scalars $s_{i,j}$ are the entries of the matrix $S$

$$S = \begin{pmatrix} q & m\mathbf{1}' & \mathbf{0}' & \mathbf{0}' \\ n\mathbf{1} & qI_b & mI_b & mI_b \\ \mathbf{0} & mI_b & qI_b + mB & 0 \\ \mathbf{0} & mI_b & 0 & qI_b + mB \end{pmatrix}. \tag{12}$$

Now we have $(\mathcal{M}_0 + \mathcal{M}_1)L = LS$ where columns of the matrix $L$ consist of eigenvectors of $(\mathcal{M}_0+\mathcal{M}_1)[X_i|X_i]$ for $i = 0, 1, 2, \ldots, 3b$. It follows that eigenvalues of $S$ are eigenvalues of $\mathcal{M}_0 + \mathcal{M}_1$ as well. Now considering Lemma 6 with $r = 1$, $\beta = 1$, $a = 1$ and $\alpha = \frac{n}{m}$ it follows that $S = \frac{n}{m}M + qI_{3b+1}$. Since we have assumed that $\mathcal{G}_5(G,H)$ is regular, by Lemma 7 we have $S = (b-2)M + qI_{3b+1}$. Now setting $r = 1$, $\beta = 1$, $a = 1$ and $\alpha = b - 2$ in Lemma 6 it follows that $\varphi_M(\lambda) = (-1)^{1+3b}(\lambda^2-1)^{b-1}(\lambda+2)^{b-2}(\lambda-2)(\lambda-b)(\lambda-(b-1))(\lambda+b-1)(\lambda-b+2)$. Therefore we have

$$\begin{aligned} \varphi_S(\lambda) = \; & (-1)^{1+3b}(\lambda - b + 2 - q)(\lambda + b - q - 2))^{b-1}(\lambda + 2(b-2) - q)^{b-2} \\ & \times (\lambda - 2(b-2) - q)(\lambda - b(b-2) - q)(\lambda - (b-1)(b-2) - q) \\ & \times (\lambda + (b-1)(b-2) - q)(\lambda - (b-2)^2 - q). \end{aligned}$$

This is enough to conclude that

$$\varphi_{\mathcal{M}_0+\mathcal{M}_1}(\lambda) = \frac{(-1)^{1+3b}}{(\lambda-q)^{3b+1}}\varphi_G(\lambda)\varphi_H(\lambda)^{3b}$$
$$\times ((\lambda-b+2-q)(\lambda+b-q-2))^{b-1}(\lambda+2(b-2)-q)^{b-2}$$
$$\times (\lambda-2(b-2)-q)(\lambda-b(b-2)-q)(\lambda-(b-1)(b-2)-q)$$
$$\times (\lambda+(b-1)(b-2)-q)(\lambda-(b-2)^2-q).$$

*Case* 4. $\varphi_{\mathcal{M}_0-\mathcal{M}_1}(\lambda)$. To evaluate this polynomial, we merely follow the approach that was used in Case 3 and we set $r=-1$, $a=1$, $\beta=1$ and $\alpha=b-2$ in Lemma 6. Then we obtain

$$\varphi_{\mathcal{M}_0-\mathcal{M}_1}(\lambda) = \frac{(-1)^{1+3b}}{(\lambda-q)^{3b+1}}\varphi_G(\lambda)\varphi_H(\lambda)^{3b}((\lambda-b+2-q)(\lambda+b-q-2))^{b-1}$$
$$\times (\lambda-2(b-2)-q)^{b-2}(\lambda+2(b-2)-q)(\lambda+b(b-2)-q)$$
$$\times (\lambda-(b-1)(b-2)-q)(\lambda+(b-1)(b-2)-q)(\lambda+(b-2)^2-q).$$

■

In view of Theorem 8 we have the following.

**Corollary 9.** *$\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are cospectral regular graphs. Furthermore, if $G$ and $H$ are integral, then $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are integral.*

**Example 10.** Let $G$ and $H$ be empty graphs of orders $m$ and $n$, respectively. Then by Lemma 7, $\mathcal{G}_4(G,H)$ and $\mathcal{G}_5(G,H)$ are cospectral regular integral graphs if and only if $n=(b-2)m$. Assuming $m=1$, yields $\mathcal{G}_4(n,n+2)$ and $\mathcal{G}_5(n,n+2)$ to which has been mentioned in [10].

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