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ON GENERALIZED SIERPIŃSKI GRAPHS

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Abstract

In this paper we obtain closed formulae for several parameters of generalized Sierpiński graphs $S(G,t)$ in terms of parameters of the base graph G. In particular, we focus on the chromatic, vertex cover, clique and domination numbers.

Keywords: Sierpiński graphs, vertex cover number, independence number, chromatic number, domination number.

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1. Introduction

Let $G = (V, E)$ be a non-empty graph of order $n \geq 2$, and t a positive integer. We denote by V^t the set of words of length t on alphabet V. The letters of a word u of length t are denoted by $u_1 u_2 \cdots u_t$. The concatenation of two words u and v is denoted by uv. Klavžar and Milutinović introduced in [13] the graph $S(K_n, t)$, $t \geq 1$, whose vertex set is V^t, where $\{u, v\}$ is an edge if and only if there exists $i \in \{1, \ldots, t\}$ such that

(i) $u_j = v_j$, if $j < i$,
(ii) $u_i \neq v_i$,
(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

As noted in [10], in a compact form, the edge sets can be described as

$$\left\{\left\{wu_iu_j^{d-1}, wu_ju_i^{d-1}\right\} : u_i, u_j \in V,\ i \neq j;\ d \in [t];\ w \in V^{t-d}\right\}.$$

The graph $S(K_3, t)$ is isomorphic to the graph of the Tower of Hanoi with t disks [13]. Later, those graphs have been called Sierpiński graphs in [14] and they were studied by now from numerous points of view. For instance, the authors of [7] studied identifying codes, locating-dominating codes, and total-dominating codes in Sierpiński graphs. In [9] the authors propose an algorithm, which makes use of three automata and the fact that there are at most two internally vertex-disjoint shortest paths between any two vertices, to determine all shortest paths in Sierpiński graphs. The authors of [14] proved that for any $n \geq 1$ and $t \geq 1$, the Sierpiński graph $S(K_n, t)$ has a unique 1-perfect code (or efficient dominating set) if t is even, and $S(K_n, t)$ has exactly n 1-perfect codes if t is odd. The Hamming dimension of a graph G was introduced in [15] as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. That paper gives an upper bound for the Hamming dimension of the Sierpiński graphs $S(K_n, t)$ for $n \geq 3$. It also shows that the Hamming dimension of $S(K_3, t)$ grows as 3^{t-3}. The idea of almost-extreme vertex of $S(K_n, t)$ was introduced in [16] as a vertex that is either adjacent to an extreme vertex of $S(K_n, t)$ or is incident to an edge between two subgraphs of $S(K_n, t)$ isomorphic to $S(K_n, t-1)$. The authors of [16] deduced explicit formulas for the distance in $S(K_n, t)$ between an arbitrary vertex and an almost-extreme vertex. Also they gave a formula of the metric dimension of a Sierpiński graph, which was independently obtained by Parreau in her Ph.D. thesis. The eccentricity of an arbitrary vertex of Sierpiński graphs was studied in [12] where the main result gives an expression for the average eccentricity of $S(K_n, t)$. For a general background on Sierpiński graphs, the reader is invited to read the comprehensive survey [11] and references therein.

This construction was generalized in [8] for any graph $G = (V, E)$, by defining the t-th *generalized Sierpiński graph* of G, denoted by $S(G, t)$, as the graph with vertex set V^t and edge set defined as follows. $\{u, v\}$ is an edge if and only if there exists $i \in \{1, \ldots, t\}$ such that

(i) $u_j = v_j$, if $j < i$,
(ii) $u_i \neq v_i$ and $\{u_i, v_i\} \in E$,
(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

In a compact form, the edge sets can be described as

$$\left\{\left\{wu_iu_j^{d-1}, wu_ju_i^{d-1}\right\} : \{u_i, u_j\} \in E;\ d \in [t];\ w \in V^{t-d}\right\}.$$

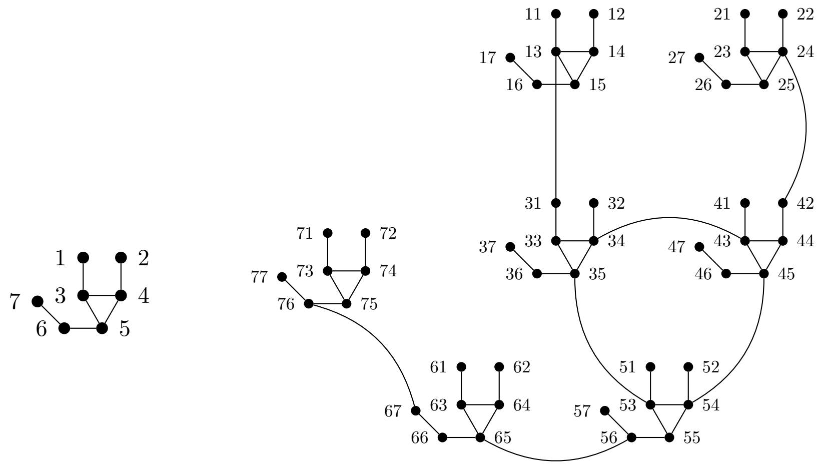

Figure 1. A graph G and the generalized Sierpiński graph $S(G,2)$.

Figure 1 shows a graph G and the generalized Sierpiński graph $S(G,2)$, while Figure 2 shows the Sierpiński graph $S(G,3)$.

Notice that if $\{u,v\}$ is an edge of $S(G,t)$, then there is an edge $\{x,y\}$ of G and a word w such that $u = wxyy\cdots y$ and $v = wyxx\cdots x$. In general, $S(G,t)$ can be constructed recursively from G with the following process: $S(G,1) = G$ and, for $t \geq 2$, we copy n times $S(G,t-1)$ and add the letter x at the beginning of each label of the vertices belonging to the copy of $S(G,t-1)$ corresponding to x. Then for every edge $\{x,y\}$ of G, add an edge between vertex $xyy\cdots y$ and vertex $yxx\cdots x$. See, for instance, Figure 2. Vertices of the form $xx\cdots x$ are called *extreme vertices* of $S(G,t)$. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G,t)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx\cdots x$ of $S(G,t)$ also has degree $d(x)$. Moreover, the degrees of two vertices $yxx\cdots x$ and $xyy\cdots y$, which connect two copies of $S(G,t-1)$, are equal to $d(x)+1$ and $d(y)+1$, respectively.

For any $w \in V^{t-1}$ and $t \geq 2$, the subgraph $\langle V_w \rangle$ of $S(G,t)$, induced by $V_w = \{wx : x \in V\}$, is isomorphic to G. Notice that there exists only one vertex $u \in V_w$ of the form $w'xx\cdots x$, where $w' \in V^r$ for some $r \leq t-2$. We will say that $w'xx\cdots x$ is *the extreme vertex* of $\langle V_w \rangle$, which is an extreme vertex in $S(G,t)$ whenever $r = 0$. By definition of $S(G,t)$ we deduce the following remark.

Remark 1. Let $G = (V,E)$ be a graph, let $t \geq 2$ be an integer and $w \in V^{t-1}$. If $u \in V_w$ and $v \in V^t - V_w$ are adjacent in $S(G,t)$, then either u is the extreme vertex of $\langle V_w \rangle$ or u is adjacent to the extreme vertex of $\langle V_w \rangle$.

The authors of [8] announced some results about generalized Sierpiński graphs concerning their automorphism groups and perfect codes. These results definitely deserve to be published. To the best of our knowledge, [19] was the first published paper studying the generalized Sierpiński graphs. In that article, the authors obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs, while in [2] this work was extended to the so-called generalized Randić index. Also, the total chromatic number of generalized Sierpiński graphs was studied in [6] and the strong metric dimension has recently been studied in [18]. In this paper we obtain closed formulae for several parameters of generalized Sierpiński graphs $S(G,t)$ in terms of parameters of the base graph G. In particular, we focus on the chromatic, vertex cover, clique and domination numbers. Throughout the paper G will denote a simple graph of order $n \geq 2$.

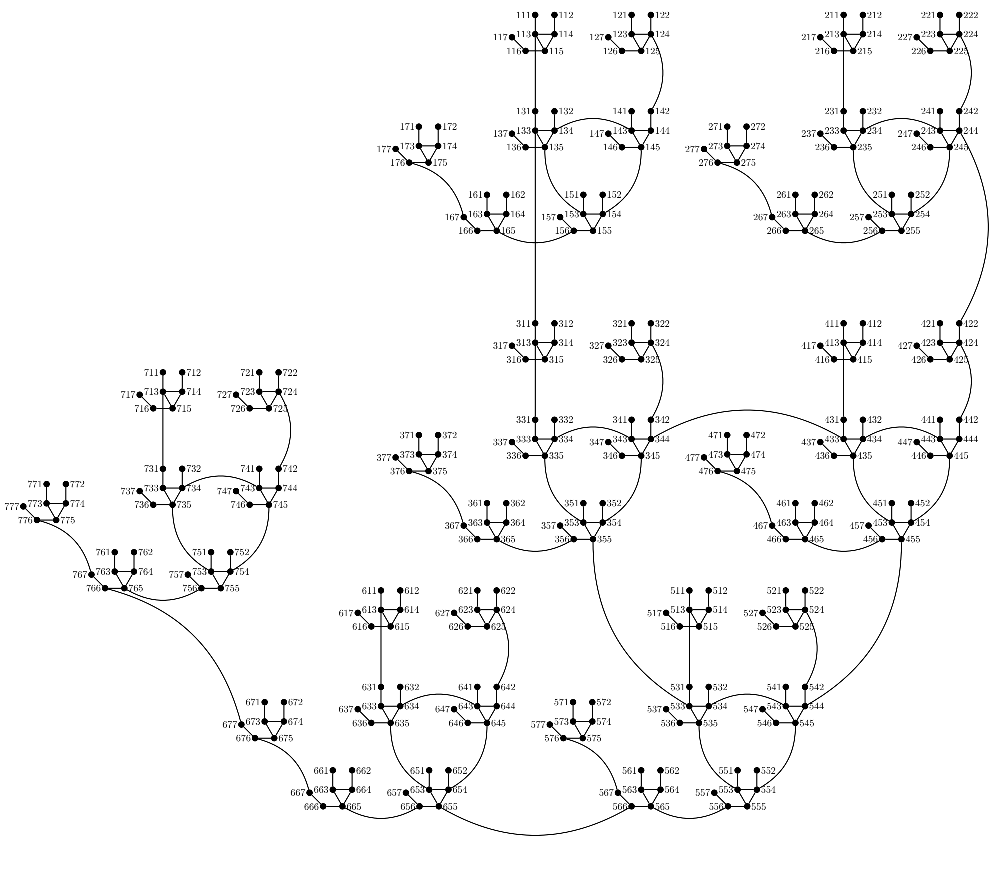

Figure 2. The generalized Sierpiński graph $S(G,3)$.
The base graph G is shown in Figure 1.

2. Some Remarks on Trees

Given a graph G, the order and size of $S(G,t)$ is obtained in the following remark.

Remark 2. Let graph G be a graph of order $n \geq 2$ and size m, and let t be a positive integer. Then the order of $S(G,t)$ is n^t and the size is $m\frac{n^t-1}{n-1}$.

Proof. By definition of $S(G,t)$, for any $t \geq 2$ we have that the order of $S(G,t)$ is $|S(G,t)| = n|S(G,t-1)|$ and $|S(G,1)| = n$. Hence, $|S(G,t)| = n^t$. Analogously, the size of $S(G,t)$ is $\|S(G,t)\| = n\|S(G,t-1)\| + m$ and $\|S(G,1)\| = m$. Then $\|S(G,t)\| = m\left(n^{t-1} + n^{t-2} + \cdots + 1\right) = m\frac{n^t-1}{n-1}$. ■

Corollary 3. *For any tree T and any positive integer t, $S(T,t)$ is a tree.*

Proof. Let n be the order of T. By the connectivity of T we have that $S(T,t)$ is connected. On the other hand, by Remark 2, $S(T,t)$ has order n^t and size $n^t - 1$. Therefore, the result follows. ■

The next result gives a formula for the number of leaves in a generalized Sierpiński tree. A vertex with degree one in a tree T is called a *leaf* and a vertex adjacent to a leaf is called a *support*. The number of leaves of a tree T will be denoted by $\varepsilon(T)$ and the set of support vertices of T by $\mathrm{Sup}(T)$. Also, if $x \in \mathrm{Sup}(T)$, then $\varepsilon_T(x)$ will denote the number of leaves of T which are adjacent to x.

Theorem 4. *Let T be a tree of order $n \geq 2$ having $\varepsilon(T)$ leaves. For any positive integer t, the number of leaves of $S(T,t)$ is*

$$\varepsilon(S(T,t)) = \frac{\varepsilon(T)\left(n^t - 2n^{t-1} + 1\right)}{n-1}.$$

Proof. Let $t \geq 2$. For any $x \in V$, we denote by $S_x(T,t-1)$ the copy of $S(T,t-1)$ corresponding to x in $S(T,t)$, i.e., $S_x(T,t-1)$ is the subgraph of $S(T,t)$ induced by the set $\{xw : w \in V^{t-1}\}$, which is isomorphic to $S(T,t-1)$. To obtain the result, we only need to determine the contribution of $S_x(T,t-1)$ to the number of leaves of $S(T,t)$, for all $x \in V$. By definition of $S(T,t)$, there exists an edge of $S(T,t)$ connecting the vertex $xy\cdots y$ of $S_x(T,t-1)$ with the vertex $yx\cdots x$ of $S_y(T,t-1)$ if and only if x and y are adjacent in T. Hence, a leaf $xy\cdots y$ of $S_x(S(T,t-1))$ is adjacent in $S(T,t)$ to a vertex $yx\cdots x$ of $S_y(T,t-1)$ if and only if y is a leaf of T and x is its support vertex. Thus, if $x \in \mathrm{Sup}(T)$, then the contribution of $S_x(T,t-1)$ to the number of leaves of $S(T,t)$ is $\varepsilon(S(T,t-1)) - \varepsilon_T(x)$ and, if $x \notin \mathrm{Sup}(T)$, then the contribution of $S_x(T,t-1)$ to the number of leaves of $S(T,t)$ is $\varepsilon(S(T,t-1))$. Then, we obtain

$$\begin{aligned}\varepsilon(S(T,t)) &= (n - |\operatorname{Sup}(T)|)\varepsilon(S(T,t-1)) + \sum_{x\in \operatorname{Sup}(T)} (\varepsilon(S(T,t-1)) - \varepsilon_T(x)) \\ &= n\varepsilon(S(T,t-1)) - \varepsilon(T).\end{aligned}$$

Now, since $\varepsilon(S(T,1)) = \varepsilon(T)$, we have that

$$\varepsilon(S(T,t)) = \varepsilon(T)\left(n^{t-1} - n^{t-2} - \cdots - n - 1\right) = \varepsilon(T)\left(n^{t-1} - \frac{(n^{t-1}-1)}{n-1}\right).$$

Therefore, the result follows. ■

3. Chromatic Number and Clique Number

The *chromatic number* of a graph $G = (V,E)$, denoted by $\chi(G)$, is the smallest number of colours needed to colour the vertices of G so that no two adjacent vertices share the same colour. A *proper vertex-colouring* of G is a map $f : V \longrightarrow \{1,2,\ldots,k\}$ such that for any edge $\{u,v\}$ of G, $f(u) \neq f(v)$. The elements of $\{1,2,\ldots,k\}$ are called *colours*, the vertices of one colour form a *colour class* and we say that f is a *k-colouring*. So the chromatic number of G is the minimum k such that there exists a k-colouring. For instance, for any bipartite graph G, $\chi(G) = 2$. Since every tree is a bipartite graph, by Corollary 3 we conclude that for any tree T and any positive integer t, $\chi(S(T,t)) = 2$.

The problem of finding the chromatic number of a graph is NP-hard, [5]. This suggests finding the chromatic number for special classes of graphs or obtaining good bounds on this invariant. As shown in [17], $\chi(S(K_n,t)) = n$. We shall show that the chromatic number of a generalized Sierpiński graph is determined by the chromatic number of its base graph.

Theorem 5. *For any graph G and any positive integer t,*

$$\chi(S(G,t)) = \chi(G).$$

Proof. Let w be a word of length $t-1$ on the alphabet V. By definition of $S(G,t)$, the subgraph $\langle V_w\rangle$ of $S(G,t)$ induced by the set $V_w = \{wx : x \in V(G)\}$ is isomorphic to G. Hence, $\chi(S(G,t)) \geq \chi(\langle V_w\rangle) = \chi(G)$.

Now, let $f : V \longrightarrow \{1,2,\ldots,k\}$ be a proper vertex-colouring of G and let $f_1 : V^t \longrightarrow \{1,2,\ldots,k\}$ be a map defined by $f_1(wx) = f(x)$, for all $wx \in V^t$. If two vertices $wx, w'y \in V^t$ are adjacent in $S(G,t)$, then x and y are adjacent in G. Hence, if $wx, w'y \in V^t$ are adjacent in $S(G,t)$, then $f_1(wx) = f(x) \neq f(y) = f_1(w'y)$ and, as a consequence, f_1 is a proper vertex-colouring of $S(G,t)$. Therefore, $\chi(S(G,t)) \leq \chi(G)$. ■

As a direct consequence of Theorem 5 we deduce the following result.

Corollary 6. *For any bipartite graph G and any positive integer t, $S(G,t)$ is bipartite.*

A *clique* of a graph $G = (V,E)$ is a subset $C \subseteq V$ such that for any pair of different vertices $v, w \in C$, there exists an edge $\{v,w\} \in E$, i.e., the subgraph induced by C is complete. The *clique number* of a graph G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. The chromatic number of a graph is equal to or greater than its clique number, i.e., $\chi(G) \geq \omega(G)$.

It is well-known that the problem of finding a maximum clique is NP-complete [5]. We shall show that the clique number of a generalized Sierpiński graph is equal to the clique number of its base graph.

Theorem 7. *For any graph G and any positive integer t,*

$$\omega(S(G,t)) = \omega(G).$$

Proof. We shall show that for any $t \geq 2$, $\omega(S(G,t)) = \omega(S(G,t-1))$. Let $x \in V$ and let $\langle V_x \rangle$ be the subgraph of $S(G,t)$ induced by the set $V_x = \{xw : \ w \in V^{t-1}\}$. Since $\langle V_x \rangle \cong S(G,t-1)$, $\omega(S(G,t)) \geq \omega(\langle V_x \rangle) = \omega(S(G,t-1))$.

Now, let C be a maximum clique of $S(G,t)$ and $xw_1, yw_2 \in C$. If $x \neq y$, then $yw_2 = yxx\cdots x$ is the only vertex not belonging to V_x which is adjacent to $xw_1 = xyy\cdots y$ and, analogously, $xw_1 = xyy\cdots y$ is the only vertex not belonging to V_y which is adjacent to $yw_2 = yxx\cdots x$. Hence, $x \neq y$ leads to $|C| = 2 = \omega(S(G,t-1))$ and so $|C| > 2$ leads to $x = y$ which implies that $C \subseteq V_x$. Thus, $\omega(S(G,t)) = |C| \leq \omega(\langle V_x \rangle) = \omega(S(G,t-1))$.

Therefore, $\omega(S(G,t)) = \omega(S(G,t-1)) = \cdots = \omega(S(G,1)) = \omega(G)$. ■

4. Vertex Cover Number and Independence Number

A *vertex cover* of a graph G is a set of vertices such that each edge of G is incident with at least one vertex of the set. The *vertex cover number* of G, denoted by $\beta(G)$, is the cardinality of a smallest vertex cover of G. For example, in the graph G of Figure 1, $\{3,4,6\}$ is a vertex cover of minimum cardinality and so $\beta(G) = 3$.

It is well-known that the problem of finding a minimum vertex cover is a classical optimization problem in computer science and is a typical example of an NP-hard optimization problem [5]. As the next result shows, the vertex cover number of a generalized Sierpiński graph can be computed from the vertex cover number and the order of the base graph.

Theorem 8. *For any graph G of order n and any positive integer t,*

$$\beta(S(G,t)) = n^{t-1}\beta(G).$$

Proof. Let $w \in V^{t-1}$ be a word of length $t-1$ on the alphabet V. By definition of $S(G,t)$, the subgraph $\langle V_w \rangle$ of $S(G,t)$ induced by the set $V_w = \{wx : x \in V\}$ is isomorphic to G. Hence,

$$\beta(S(G,t)) \geq \sum_{w \in V^{t-1}(G)} \beta(\langle V_w \rangle) = n^{t-1}\beta(G).$$

Now, let $C \subseteq V$ be a vertex cover of G of cardinality $|C| = \beta(G)$ and let

$$C' = \left\{ wv : v \in C \text{ and } w \in V^{t-1} \right\}.$$

Since $\langle V_w \rangle \cong G$, for any $w \in V^{t-1}$ we have that $C'_w = \{wv : v \in C\} \subseteq C'$ is a vertex cover of $\langle V_w \rangle$. In addition, if two vertices $w_1 y, w_2 x \in V^t$, $w_1 \neq w_2$, are adjacent in $S(G,t)$, then x and y are adjacent in G and so $x \in C$ or $y \in C$. Hence, $w_1 y \in C'$ or $w_2 x \in C'$. Therefore, C' is a vertex cover of $S(G,t)$ and, as a consequence, $\beta(S(G,t)) \leq |C'| = n^{t-1}|C| = n^{t-1}\beta(G)$. ■

A set X of vertices of a graph G is called *independent* if no two distinct vertices of X are adjacent. The cardinality of a largest independent set is called the *independence number* of G and denoted by $\alpha(G)$. For example, in the graph G of Figure 1, $\{1,2,5,7\}$ is an independent set of maximum cardinality and so $\alpha(G) = 4$.

The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph. Such a result will provide us another very useful result on generalized Sierpiński graphs.

Theorem 9 [4]**.** *For any graph G, $\beta(G) + \alpha(G) = |G|$.*

By using this result and Theorem 8 we obtain a formula for the independence number of $S(G,t)$.

Theorem 10. *For any graph G of order n and any positive integer t,*

$$\alpha(S(G,t)) = n^{t-1}\alpha(G).$$

5. Domination Number

For a vertex v of $G = (V,E)$, $N_G(v)$ denotes the set of neighbours that v has in G and $N_G[v] = N_G(v) \cup \{v\}$ denotes the closed neighbourhood of v. A set $D \subseteq V$ is *dominating* in G if every vertex of $V - D$ has at least one neighbour in D, i.e., $D \cap N_G(u) \neq \emptyset$, for all $u \in V - D$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. A dominating set of cardinality $\gamma(G)$ is called a *$\gamma(G)$-set.* Let $\mathcal{D}(G)$ be the set of all $\gamma(G)$-sets. The dominating set problem, which concerns testing whether $\gamma(G) \leq k$ for a given graph G and input k, is a classical NP-complete decision problem in computational complexity theory [5]. In this section we obtain an upper bound on the domination number of $S(G,t)$ and we show that the bound is tight.

The domination number of $S(K_n,t)$ was previously studied by Klavžar, Milutinović and Petr in [14] where they obtained the following result.

Theorem 11 [14]**.** *For any integers $n \geq 2$ and $t \geq 1$,*

$$\gamma(S(K_n,t)) = \begin{cases} \dfrac{n^t+n}{n+1}, & t \text{ even;} \\ \dfrac{n^t+1}{n+1}, & t \text{ odd.} \end{cases}$$

In order to state our results we need to introduce some additional notation and terminology. We define the parameter $\xi(G)$ as follows:

$$\xi(G) = \max_{D \in \mathcal{D}(G)} \left\{ |D'| : \ D' \subseteq D \text{ and } \langle D' \rangle \text{ has no isolated vertices} \right\}.$$

Notice that $0 \leq \xi(G) \leq \gamma(G)$. In particular, $\xi(G) = \gamma(G)$ if and only if there exists a $\gamma(G)$-set whose induced subgraph has no isolated vertices, while $\xi(G) = 0$ if and only if any $\gamma(G)$-set is independent. A subset S of vertices of G for which $|N_G[v] \cap S| = 1$ for every $v \in V(G)$ is called a 1-*perfect code* or an *efficient dominating set*. As noted in [1], if a graph has an efficient dominating set, then every such set has cardinality $\gamma(G)$. Therefore, if G has an efficient dominating set, then $\xi(G) = 0$. The converse is not true, as the graphs shown in Figure 3 have no efficient dominating sets and in both cases $\xi(G) = 0$.

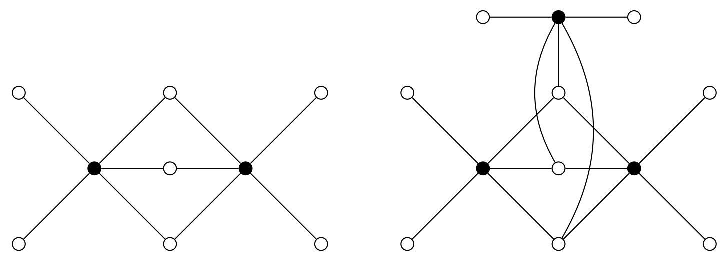

Figure 3. Two graphs where the set of black-coloured vertices is the only $\gamma(G)$-set. In both cases $\xi(G) = 0$ and $\gamma(G) = \beta(G)$.

Theorem 12. *For any graph G of order n and any integer $t \geq 2$,*

$$\gamma(S(G,t)) \leq n^{t-2}(n\gamma(G) - \xi(G)).$$

Proof. Let D be a $\gamma(G)$-set and let

$$D_{t-1} = \left\{wx: \ w \in V^{t-1} \text{ and } x \in D\right\}.$$

If $u \in V$ is adjacent to $v \in D$, then for any $w \in V^{t-1}$, we have that $wu \in V^t$ is adjacent to $wv \in D_{t-1}$ and, as a consequence, D_{t-1} is a dominating set in $S(G,t)$.

Now, assume that there exists $D' \subseteq D$ such that the subgraph induced by D' has no isolated vertices and $|D'| = \xi(G)$. Let D_{t-2} be the set

$$D_{t-2} = \left\{w'uu: \ w' \in V^{t-2} \text{ and } u \in D'\right\}.$$

We shall show that $D^* = D_{t-1} - D_{t-2}$ is a dominating set in $S(G,t)$. To this end, taking $w' \in V^{t-2}$ and $u \in D'$, we only need to show that each $x \in N_{S(G,t)}(w'uu) \cup \{w'uu\}$ is dominated by some vertex belonging to D^*. Since $w'uu$ is dominated by $w'uv \in D^*$, for some $v \in D' \cap N_G(u)$, from now on we assume that $x \in N_{S(G,t)}(w'uu)$. Now, if $x = w'uz$, for some $z \in N_G(u)$, then x is dominated by $w'zu \in D^*$, so we assume that $x = w''zz$, where $w'' \in V^{t-2}$ and $z \in N_G(u)$. Thus, for $z \notin D$ or $z \in D \cap D'$ we have that $x = w''zz$ is dominated by $w''zu \in D^*$. Finally, if $z \in D - D'$, then $x = w''zz \in D^*$.

Hence, $D_{t-1} - D_{t-2}$ is a dominating set in $S(G,t)$ and, as a consequence,

$$\gamma(S(G,t)) \leq |D_{t-1} - D_{t-2}| = n^{t-1}|D| - n^{t-2}|D'|.$$

Therefore, the result follows. ■

From now on $\Omega(G)$ denotes the set of vertices of degree one in G.

Lemma 13. *Let G be a non-trivial graph such that $\gamma(G) = \beta(G)$. The following assertions are equivalent.*

(a) *There exists a unique $\gamma(G)$-set.*
(b) *There exists a $\gamma(G)$-set D such that $|\Omega(G) \cap N_G(x)| \geq 2$, for every $x \in D$.*

Proof. Notice that G does not have isolated vertices, as $\gamma(G) = \beta(G)$. Assume that there exists a unique $\gamma(G)$-set. Let D be a $\beta(G)$-set. Since $\gamma(G) = \beta(G)$, D is the only $\gamma(G)$-set and $V - D$ is an $\alpha(G)$-set. Suppose that $|\Omega(G) \cap N_G(v)| \leq 1$, for some $v \in D$. Let $x \in N_G(v) - D$. If $x \notin \Omega(G)$, then there exists $v' \in D$ such that $x \in N_G(v')$. Hence, if $\Omega(G) \cap N_G(v) = \emptyset$, then v does not have private neighbours (with respect to D) and, as a result, $(D - \{v\}) \cup \{x\}$ is a dominating set, which is a contradiction. Also, if $\Omega(G) \cap N_G(v) = \{y\}$, then y is the only private neighbour of v (with respect to D) and, as a consequence, $(D - \{v\}) \cup \{y\}$ is a dominating set, which is a contradiction again. Therefore, (b) follows.

Now, assume that S is a $\gamma(G)$-set such that $|\Omega(G) \cap N_G(x)| \geq 2$, for every $x \in S$. Let S' be a $\gamma(G)$-set. As S' is a minimum dominating set, if a vertex

$v \in V(G)$ is adjacent to at least two vertices of degree one, then v must belong to S', which implies that $S \subseteq S'$. Therefore $S' = S$. ■

We shall show a class of graphs for which $\gamma = \beta$ and there exists a unique γ-set. Let G be a graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of graphs. The *corona product* $G \odot \mathcal{H}$ is the graph obtained from G and $\mathcal{H}$ by joining by an edge each vertex of H_i with the i-th vertex of G. This class of graphs was introduced by Frucht and Harary in [3] for the case of families $\mathcal{H}$ composed by isomorphic graphs. If $\mathcal{H}$ is composed by empty graphs of order greater than one, then $\gamma(G \odot \mathcal{H}) = \beta(G \odot \mathcal{H}) = n$ and $V(G)$ is the only $\gamma(G \odot \mathcal{H})$-set. Notice that, in this case, $\xi(G \odot \mathcal{H})$ is equal to n minus the number of isolated vertices of G.

Theorem 14. *Let G be a graph of order $n \geq 2$ such that there exists a unique $\gamma(G)$-set and $\gamma(G) = \beta(G)$. Then for any integer $t \geq 2$*

$$\gamma(S(G,t)) = n^{t-2}(n\gamma(G) - \xi(G)).$$

Proof. Let $D \subseteq V$ be the only $\gamma(G)$-set. As we have shown in the proof of Theorem 12, $D^* = D_{t-1} - D_{t-2}$ is a dominating set of $S(G,t)$ and $|D^*| = n^{t-2}(n\gamma(G) - \xi(G))$. Let D^π be a dominating set of $S(G,t)$ of minimum cardinality. If $D^* - D^\pi = \emptyset$, then $|D^*| \leq |D^\pi|$ and, as a consequence, $\gamma(S(G,t)) = |D^*|$. Let $w \in V^{t-1}$ and assume that $wx \in D^* - D^\pi$. Since $x \in D$, by Lemma 13 we have $|\Omega(G) \cap N_G(x)| \geq 2$. Thus, if wx is not the extreme vertex of $\langle V_w \rangle$, then there are at least two vertices of degree one adjacent to wx in $S(G,t)$, which implies that $wx \in D^\pi$ and it is a contradiction. Thus, wx is the extreme vertex of $\langle V_w \rangle$, i.e., $wx = w'xx$, for some $w' \in V^{t-2}$. Notice the following.

- $w'xx$ is dominated by a vertex $w''y \in D^\pi$. Now, if $w''y \in D^*$, then x and y are adjacent in G, $x \in D'$ and $w'xx \notin D^*$, which is a contradiction. Hence, $w''y \in D^\pi - D^*$.
- The only vertex in $D^* - D^\pi$ dominated by $w''y$ is $w'xx$, as a vertex in $S(G,t)$ can only be adjacent to one extreme vertex.

Let $S \subseteq D^\pi - D^*$ such that each vertex of $D^* - D^\pi$ is dominated by a vertex in S. Then we can define a mapping $f : S \longrightarrow D^* - D^\pi$, where $f(u) = v$ means that v is dominated by u. As f is a surjective mapping, $|D^* - D^\pi| \leq |S| \leq |D^\pi - D^*|$. Therefore, $|D^*| \leq |D^\pi|$, and we can conclude that $\gamma(S(G,t)) = |D^\pi| = |D^*| = n^{t-2}(n\gamma(G) - \xi(G))$. ■

We can take $G = P_4$ and $t = 2$ to show that in Theorem 14 we do not have equivalence. In this case $\gamma(G) = \beta(G) = 2$, $\xi(G) = 2$ and $\gamma(S(G,t)) = n^{t-2}(n\gamma(G) - \xi(G)) = 6$, while G has four different $\gamma(G)$-sets.

Lemma 15. *Let G be a graph of order n and let $t \geq 3$ be an integer. If $\gamma(S(G,t)) = n^{t-1}\gamma(G)$, then there exists a unique $\gamma(G)$-set.*

Proof. Assume that $\gamma(S(G,t)) = n^{t-1}\gamma(G)$. Notice that, by Theorem 12, we have that $\xi(G) = 0$. Suppose, for the purpose of contradiction, that A and B are two different $\gamma(G)$-sets. In such a case, there exist $a \in A - B$ and $b \in B$ such that b dominates a. Now, if $b \in A$, then $\xi(G) > 0$, which is a contradiction. So, $b \notin A$. Following a procedure analogous to that used in the proof of Theorem 12 we see that $A_{t-1} = \{wv : \ w \in V^{t-1} \text{ and } v \in A\}$ is a dominating set of $S(G,t)$. Hence,

$$A' = (A_{t-1} - (\{abb\cdots bx : \ x \in A\} \cup \{baa\cdots a\})) \cup \{abb\cdots bx : \ x \in B\}$$

is a dominating set of $S(G,t)$ as any vertex in $\{abb\cdots bx : \ x \in V\}$ is dominated by some vertex in $\{abb\cdots bx : \ x \in B\} \subseteq A'$, $baa\cdots a$ is dominated by $abb\cdots b \in A'$ and, for any $z \in N_G(a)$, $baa\cdots az$ is dominated by $baa\cdots aza \in A'$. Thus, $\gamma(S(G,t)) \leq |A'| = n^{t-1}\gamma(G) - 1$, which is a contradiction. Therefore, the result follows. ■

Theorem 16. *Let G be a graph of order n such that $\gamma(G) = \beta(G)$ and let $t \geq 3$ be an integer. The following assertions are equivalent.*

(a) $\gamma(S(G,t)) = n^{t-1}\gamma(G)$.

(b) $\xi(G) = 0$ *and there exists a unique $\gamma(G)$-set.*

Proof. Assume that $\gamma(S(G,t)) = n^{t-1}\gamma(G)$. By Theorem 12, we have that $\xi(G) = 0$ and by Lemma 15 we have that there exists a unique $\gamma(G)$-set.

Now, if $\xi(G) = 0$ and there exists a unique $\gamma(G)$-set, then by Theorem 14 we conclude that $\gamma(S(G,t)) = n^{t-1}\gamma(G)$. ■

It is easy to see that $\gamma(S(K_{1,r},2)) = r+1$. Hence, from Theorem 16 we deduce that $\gamma(S(K_{1,r},t)) = (r+1)^{t-1}$ for all positive integers r and t.

We now proceed to construct a family $\mathcal{G}$ of graphs fulfilling the conditions of Lemma 15 and Theorems 14 and 16. We say that a graph G belongs to $\mathcal{G}$ if there exist a multigraph G' (without loops) of order n and a family $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ of non-trivial empty graphs such that G is constructed from the corona product $G' \odot \mathcal{H}$ by subdividing the edges of $G' \odot \mathcal{H}$ corresponding to the edges of G'. Figure 3 (left hand side) shows a graph $G \in \mathcal{G}$ obtained from the multigraph G' of order two and size three and a family $\mathcal{H}$, which is composed by two empty graphs of order two. The set of black-coloured vertices is the only $\gamma(G)$-set, which is the vertex set of G, and the vertices of degree two correspond to the subdivisions of the edges of G'. Lemma 15 and Theorems 14 and 16 hold for all $G \in \mathcal{G}$, as $\gamma(G) = \beta(G) = n$, $\xi(G) = 0$ and the vertex set of G' is the only $\gamma(G)$-set. Notice that the right hand side graph of Figure 3 satisfies the conditions above and it does not belong to $\mathcal{G}$.

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