INTERVAL INCIDENCE COLORING OF SUBCUBIC GRAPHS

ANNA MALAFIEJSKA

Department of Probability Theory and Biomathematics
Faculty of Physics and Applied Mathematics
Gdańsk University of Technology, Narutowicza 11/12, Gdańsk, Poland

e-mail: anna@animima.org

AND

Michał MALAFIEJSKI

Department of Algorithms and System Modeling
Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology, Narutowicza 11/12, Gdańsk, Poland

e-mail: michal@animima.org

Abstract

In this paper we study the problem of interval incidence coloring of subcubic graphs. In [14] the authors proved that the interval incidence 4-coloring problem is polynomially solvable and the interval incidence 5-coloring problem is NP-complete, and they asked if \( \chi_{ii}(G) \leq 2\Delta(G) \) holds for an arbitrary graph \( G \). In this paper, we prove that an interval incidence 6-coloring always exists for any subcubic graph \( G \) with \( \Delta(G) = 3 \).

Keywords: interval incidence coloring, incidence coloring, subcubic graph.

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1. Introduction

In the paper we consider simple nonempty graphs, and we use the standard notation of graph theory. Let \( G = (V, E) \) be a simple graph, and let \( X \subseteq V \) be a non-empty set. By \( N_G(X) = \{ v \in V : \exists u \in X \{v,u\} \in E \} \) we mean the open

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neighborhood of $X$, by $G[X]$ we mean the subgraph of $G$ induced by the set $X$, and by $G \setminus X$ we mean the graph $G[V \setminus X]$. We say that $X$ is a dominating set of $G$ if $V = N_G(X) \cup X$, and we say that $X$ is a total dominating set if $V = N_G(X)$. In what follows we use $N_G(v)$ instead of $N_G\{v\}$. Let $\deg_G(v) = |N_G(v)|$ be the degree of a vertex $v \in V(G)$. By $n(G), \Delta(G)$ and $\delta(G)$ we denote the number of vertices of $G$, the maximum and the minimum degree of a vertex of $G$, respectively. By a subcubic graph $G$ we mean a graph with $\Delta(G) \leq 3$. By an isolated vertex (in a graph $G$) we mean a vertex $v \in V(G)$ with $\deg_G(v) = 0$, and by an isolated edge (in a graph $G$) we mean an edge $e = \{u, v\}$ such that $\deg_G(u) = \deg_G(v) = 1$. We say that $X \subset V(G)$ is an independent set if each vertex of $G[X]$ is isolated in $G[X]$. By a pendant vertex we mean a vertex of degree 1.

For a given graph $G = (V, E)$, we define an incidence as a pair $(v, e)$, where vertex $v \in V$ is one of the endpoints of edge $e \in E$, i.e., $v \in e$. The set of all incidences of $G$ will be denoted by $I(G)$, thus $I(G) = \{(v, e): v \in V \land e \in E \land v \in e\}$. We say that two incidences $(v, e)$ and $(w, f)$ are adjacent if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}$, $f = \{w, u\}$ and $v \neq u$.

By an incidence coloring of $G$ we mean a function $c: I(G) \to \mathbb{N}$ such that $c((v, e)) \neq c((w, f))$ for any two adjacent incidences $(v, e)$ and $(w, f)$. The incidence coloring number of $G$, denoted by $\chi_i(G)$, is the smallest number of colors in an incidence coloring of $G$. In what follows we use the simplified notation $c(v, e)$ instead of $c((v, e))$.

A finite nonempty set $A \subset \mathbb{N}$ is an interval if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring $c$ of graph $G$ and $v \in V(G)$ let $A_c(v) = \{c(v, e): v \in e \land e \in E(G)\}$. By an interval incidence coloring of a graph $G$ we mean an incidence coloring $c$ of $G$ such that for each vertex $v \in V(G)$ the set $A_c(v)$ is an interval. By an interval incidence $k$-coloring we mean an interval incidence coloring using all colors from the set $\{1, \ldots, k\}$. The interval incidence coloring number of $G$, denoted by $\chi_{ii}(G)$, is the smallest number of colors in an interval incidence coloring of $G$.

1.1. Background and previous results

Alon et al. [1] defined the problem of partitioning a graph into the minimal number of star forests. Brualdi and Massey [3] formulated a model of incidence coloring of graphs with references to certain models of coloring of graphs, such as strong edge and vertex coloring of graphs. Guiduli [9] observed that the problem of incidence coloring of graphs is a special case of the problem of partitioning a symmetric digraph into directed star forests.

In [3] the authors conjectured that $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph $G$ (incidence coloring conjecture, shortly ICC). This conjecture was disproved by Guiduli in [9] who observed that Paley graphs have incidence coloring number at
least $\Delta + \Omega (\log \Delta)$. In fact, he used the crucial result from [1]. For many classes of graphs it is shown that the incidence coloring number is at most $\Delta + 2$, e.g., trees and cycles [3], complete graphs [3], complete bipartite graphs [3] (proof corrected in [19]), planar graphs with girth at least 11 or with girth at least 6 and maximum degree at least 5 [5], partial 2-trees (i.e., $K_4$-minor free graphs) [4], hypercubes [18], complete $k$-partite graphs [15].

In [17] the author proved that ICC holds for subcubic graphs. The incidence 4-colorability problem is $\text{NP}$-complete for semicubic graphs (i.e., subcubic graphs with vertex degrees equal to 1 or 3) [16] and for semicubic bipartite graphs [15].

In this paper we consider a restriction of the problem of incidence coloring of graphs in which the colors of incidences at a vertex form an interval. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring (see, e.g., [2, 6, 8]), which can be applied to the open-shop scheduling problem [6, 7]. In [11] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [12] the authors studied applications in one-multicast transmission in multifiber WDM networks.

In [13] the authors proved that the problem of interval incidence $k$-coloring of bipartite graphs is polynomial for each $k \leq 6$ and $\Delta \leq 3$, polynomial for $k = 5$ and $\Delta = 4$, and $\text{NP}$-complete for $k = 6$ and $\Delta = 4$. In [14] the authors proved certain lower and upper bounds on the interval incidence coloring number, e.g., $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$ for an arbitrary graph $G$, and they determined the exact values of $\chi_{ii}$ for some basic classes of graphs (e.g., complete $k$-partite graphs). In [14] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem of deciding whether $\chi_{ii} \leq 4$ is easy, and $\chi_{ii} \leq 5$ is $\text{NP}$-complete. The problem of interval incidence 6-coloring of subcubic graphs remained unsolved.

1.2. Main results

Our main result in the paper is Theorem 21 which states $\chi_{ii}(G) \leq 6$ for every subcubic graph $G$. To prove it, we state and prove Theorem 8: in any subcubic graph $G$ with $\delta(G) \geq 2$ there is a maximal induced bipartite subgraph of $G$ without isolated vertices, or equivalently, $G$ has a total dominating set $S$ such that $G[S]$ is a bipartite graph.

2. Maximal Induced Bipartite Subgraphs Without Isolated Vertices

In this section we prove (in Theorem 8) that any subcubic graph $G$ with $\delta(G) \geq 2$ contains a maximal induced bipartite subgraph without isolated vertices.
2.1. Introductory properties

By $H \subset G$ we mean that $H$ is a subgraph of $G$. By $H \subseteq G$ we mean that $H$ is an induced subgraph of $G$, i.e., $H = G[V(H)]$.

**Observation 1.** If $G_1 \subset G_2$ and $G_2 \subseteq G_3$, then $G_1 \subset G_3$.

**Observation 2.** Let $G_1 \subset G$ and $G_2 \subseteq G$. If $G_1 \subset G_2$, then $G_1 \subset G_2$.

Let $B(G) = \{H \subset G : N_G(V(H)) = V(G) \wedge H \text{ is bipartite}\}$, i.e., the set of all induced bipartite subgraphs of a given graph $G$ such that $V(H)$ is a total dominating set of $G$. If $H \in B(G)$, then $V(H)$ is a total dominating set of $G$ and, obviously, $H$ has no isolated vertices.

In the following, let $G$ be any graph. Let $\hat{B}(G)$ be the subfamily of $B(G)$ consisting of all the elements (graphs) in $B(G)$ that are maximal with respect to the subgraph relation ($\subset$).

**Observation 3.** If $H \in B(G)$, then there is $H' \in \hat{B}(G)$ such that $H \subset H'$.

By Observations 2 and 3 we have

**Observation 4.** Let $H \in B(G)$. Then, $H \in \hat{B}(G)$ if and only if for each $v \in V(G) \setminus V(H)$ the subgraph $G[V(H) \cup \{v\}]$ is not bipartite.

**Observation 5.** If $H \in B(G) \setminus \hat{B}(G)$, then there is a vertex $v \in V(G) \setminus V(H)$ such that $G[V(H) \cup \{v\}] \in B(G)$.

Since any dominating set $S \subset V(G)$ is a total dominating set if and only if $G[S]$ has no isolated vertices, we have

**Observation 6.** Let $G$ be an arbitrary graph and let $H \subset G$. Then, $H \in \hat{B}(G)$ if and only if $H$ is a maximal induced bipartite subgraph (of $G$) without isolated vertices.

Let $G_3^2$ be the family of subcubic graphs without isolated and pendant vertices, i.e., each vertex in a graph of this family has degree 2 or 3. Let $M_3^2$ be the subfamily of $G_3^2$ consisting of all the graphs for which there is no maximal induced bipartite subgraph without isolated vertices. Let us denote by $M$ the set of elements in $M_3^2$ that are minimal with respect to the subgraph relation ($\subset$). By Observation 6 we have

**Observation 7.** Let $G \in G_3^2$. Then, $G \in M_3^2 \iff B(G) = \emptyset \iff \hat{B}(G) = \emptyset$. 
2.2. Main Theorem

Theorem 8. Let $G$ be a subcubic graph with $\delta(G) \geq 2$. Then, $G$ has a maximal induced bipartite subgraph without isolated vertices.

By Observation 7, Theorem 8 is equivalent to $M = \emptyset$. First, we prove some structural properties of graphs from $M$.

Lemma 9. Let $G \in M$. Then, $G$ is a connected graph and $\Delta(G) = 3$.

Proof. Let $G \in M$. Let us assume to the contrary that $G = G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint graphs (without common vertices). Since $G_i \subseteq G \in M$ and $G_i \in G^3_i$, we have $G_i \notin M^3_i$, for $i \in \{1, 2\}$. Hence, there exist $H_1 \subseteq \mathcal{B}(G_1)$ and $H_2 \subseteq \mathcal{B}(G_2)$. Thus, $H_1 \cup H_2 \subseteq \mathcal{B}(G)$, a contradiction.

Since every cycle is either a bipartite graph or it becomes a bipartite graph after deleting an arbitrary vertex, $G$ is not a cycle, which implies $\Delta(G) = 3$. ■

Lemma 10. Let $G \in M$ and let $v$ be a vertex of degree 2 in $G$. Then, every neighbor of $v$ in $G$ has degree 3.

Proof. Let $G \in M$. Suppose to the contrary that there are two adjacent vertices of degree 2. Since $G$ is not a cycle (by Lemma 9), there is a subgraph $P$ of $G$ with vertex set $\{v_0, \ldots, v_{k+1}\}$ and edges $\{v_i, v_{i+1}\}$, for $i \in \{0, \ldots, k\}$, such that $\deg_G(v_0) = \deg_G(v_{k+1}) = 3$, and $\deg_G(v_i) = 2$ for $i \in \{1, \ldots, k\}$, where $k \geq 2$.

Suppose $v_0 \neq v_{k+1}$. Since $G' = G \setminus \{v_1, \ldots, v_k\} \subseteq G \in M$ and $G' \in G^2_3$, we have $G' \notin M^3_3$. Hence, there exists $H' \subseteq \mathcal{B}(G')$, and $H' \subseteq G$ by Observation 1. If $v_0 \in V(H')$, then let $H = G[V(H') \cup \{v_1, \ldots, v_{k-1}\}]$, otherwise, let $H = G[V(H') \cup \{v_1, \ldots, v_k\}]$. In both cases, $H \subseteq G$. $H$ is a bipartite graph, and $V(H)$ is a total dominating set, i.e., $H \subseteq \mathcal{B}(G)$. By Observation 7 we get a contradiction.

Suppose $v_0 = v_{k+1}$. Since $\deg_G(v_0) = 3$, there is $c \in N_G(v_0) \setminus \{v_1, v_k\}$. If $\deg_G(c) = 3$, then let $G' = G \setminus \{v_0, \ldots, v_k\}$. If $\deg_G(c) = 2$, then let $G' = G \setminus \{v_0, \ldots, v_k\}$. In both cases, $G' \subseteq G$ and $G \neq G' \in G^2_3$. Hence, there is $H' \subseteq \mathcal{B}(G')$. Let $H = G[V(H') \cup \{v_0, \ldots, v_{k-1}\}]$. Thus, $H \subseteq \mathcal{B}(G)$, a contradiction. ■

Lemma 11. If $G \in G^3_3$ contains $G_0$ as a subgraph (see Figure 1), where vertices $v_2, v_3 \in V(G_0)$ are of degree 2 in $G$, then $G \notin M$.

![Figure 1. The subgraph $G_0$ of a graph $G$.](image-url)
Proof. Suppose to the contrary that \( G \in \mathcal{M} \). Suppose \( G_0 \subset G \). The other possible edges in \( G \) are marked by the dotted lines (in Figure 1).

By \( \deg_G(v_2) = \deg_G(v_3) = 2 \), from Lemma 10 we have \( \deg_G(v_1) = \deg_G(v_4) = 3 \). Since \( G' = G \setminus \{v_2\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2 \), there is \( H' \in \mathcal{B}(G') \). Hence, \( v_1 \in V(H') \) or \( v_4 \in V(H') \). Thus, \( H' \in \mathcal{B}(G) \), a contradiction. 

Lemma 12. Let \( G \in \mathcal{M} \) and let \( v \) be a vertex of degree 3 in \( G \). Then, at most one neighbor of \( v \) has degree 2.

Proof. Let \( G \in \mathcal{M} \) and let \( N_G(v) = \{x, y, z\} \). Suppose to the contrary that at least two vertices from \( N_G(v) \) have degree 2. Let \( \deg_G(x) = \deg_G(y) = 2 \). Let \( \{v_x\} = N_G(x) \setminus \{v\} \) and \( \{v_y\} = N_G(y) \setminus \{v\} \). By Lemma 10, \( \deg_G(v_x) = \deg_G(v_y) = 3 \).

Suppose \( \deg_G(z) = 2 \). Let \( \{v_z\} = N_G(z) \setminus \{v\} \). By Lemma 10, \( \deg_G(v_z) = 3 \).

If any two of the vertices \( v_x, v_y, v_z \) are equal, then by Lemma 11 (i.e., because \( G_0 \subset G \)) we get a contradiction. Hence, vertices \( v_x, v_y, v_z \) are different. Since \( G' = G \setminus \{x, y, z, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2 \), there is \( H' \in \mathcal{B}(G') \). Thus, \( G[V(H') \cup \{v, x\}] \in \mathcal{B}(G) \), a contradiction.

Suppose \( \deg_G(z) = 3 \). If \( v_x = v_y \), then by Lemma 11 we get a contradiction. Hence, \( v_x \neq v_y \). Suppose \( z = v_z \) (the case \( z = v_y \) can be treated analogously). Since \( G_x = G \setminus \{x\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2 \), there is \( H_x \in \mathcal{B}(G_x) \). Since \( H_x \) is maximal in \( \mathcal{B}(G) \), we have \( v \in V(H_x) \) or \( z \in V(H_x) \). Thus, \( H_x \in \mathcal{B}(G) \), a contradiction. Then, vertices \( v_x, v_y, z \) are different. Since \( G' = G \setminus \{x, y, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2 \), there is \( H' \in \mathcal{B}(G') \). If \( z \in V(H') \), then let \( A = V(H') \cup \{v\} \). If \( z \notin V(H') \), then let \( A = V(H') \cup \{v, x\} \). In both cases, \( G[A] \in \mathcal{B}(G) \), a contradiction. 

Let \( G \) be any subcubic graph. We say that \( H \subset G \) is a \textit{Q-cycle} (of \( G \)) if:

\[(q_1) \text{ for each } v \in V(H), \deg_G(v) = 3, \text{ and} \]
\[(q_2) H \subset G \text{ and } H \text{ is isomorphic to a cycle, i.e., } H \text{ is an induced cycle, and} \]
\[(q_3) \text{ for each vertex } v \in V(G) \setminus V(H), |N_G(v) \cap V(H)| \leq 1. \]

Lemma 13. Let \( G \in \mathcal{M} \). Let \( v \in V(G) \) have all neighbors of degree 3. Then, for each \( x \in N_G(v) \) there is a \textit{Q-cycle} \( C_x \) such that \( x \in V(C_x), v \notin V(C_x) \) and \( N_G(v) \cap V(C_x) = \{x\} \).

Proof. Let \( G \in \mathcal{M} \) and let \( v \in V(G) \) be a vertex with all neighbors of degree 3. Since \( G' = G \setminus \{v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2 \), there is \( H' \in \mathcal{B}(G') \). Hence, \( N_G(v) \cap V(H') = \emptyset \).

Let \( x \in N_G(v) \) and let \( N_G(x) = \{a, b, v\} \). Since \( H' \) is bipartite and maximal in \( \mathcal{B}(G) \), we have that \( a \) and \( b \) belong to the same connected component of \( H' \), and the length of each path in \( H' \) from \( a \) to \( b \) is odd. Let \( P \subset H' \) be a path joining \( x_1 = a \) and \( x_{s-1} = b \) (\( s \) is odd), with vertex set \( \{x_1, \ldots, x_{s-1}\} \) and edges \( \{x_i, x_{i+1}\} \), for \( i \in \{1, \ldots, s-2\} \). Let \( x_0 = x \) and let \( C_x \) be the graph with \( V(C_x) = V(P) \cup \{x_0\} \), and \( E(C_x) = E(P) \cup \{\{x_{s-1}, x_0\}, \{x_0, x_1\}\} \). Since \( P \subset H' \), we have \( N_G(v) \cap V(C_x) = \{x\} \), and \( v \notin V(C_x) \).
Claim 14. For each $i \in \{1, \ldots, s - 1\}$, the following properties are satisfied:

$(p_1)$ $\deg_G(x_i) = 3$,

$(p_2)$ $N_G(x_i) = \{a_i, x_{(i-1) \mod s}, x_{(i+1) \mod s}\}$, where $a_i \in V(H') \setminus V(C_2)$,

$(p_3)$ $N_G(a_i) \cap V(H') = \{x_i\}$.

Proof. We proceed by induction on $i$. Suppose $i = 1$. Let $X = V(H') \setminus \{x_1\} \cup \{x, v\}$. Hence, $G[X]$ is bipartite. If $\deg_G(x_1) = 2$ or $N_G(a_1) \cap V(H') \neq \{x_1\}$, then $G[X] \in \mathcal{B}(G)$, a contradiction. If $a_i \notin V(H') \setminus V(C_2)$, then $g_i \notin V(H')$ or $a_i \notin V(G_2)$. If $a_i \notin V(H')$, then $N_G(a_i) \cap V(H') \neq \{x_i\}$ (otherwise $H'$ is not maximal in $\mathcal{B}(G)$), a contradiction. If $a_i \in V(C_2)$, then $G[X] \in \mathcal{B}(G)$, a contradiction.

Suppose the properties $(p_1), (p_2), (p_3)$ hold for $1, \ldots, i-1$ ($2 \leq i \leq s-1$). Hence, each path joining $x_1$ and $x_{i-1}$ in $H'$ contains $x_1, \ldots, x_i$. Let $X = V(H') \setminus \{x_1\} \cup \{x, v\}$. Hence, $G[X]$ is bipartite. The rest of the proof of properties $(p_1), (p_2), (p_3)$ for $i$ is literally the same as in the case $i = 1$.

We show that $C_x$ is a $Q_2$-cycle. Since $\deg_G(x) = 3$, by $(p_1)$ we have $(q_1)$. Since $v \notin V(C_2)$ and $a_i \notin V(C_2)$ (by $(p_2)$), for $i \in \{1, \ldots, s-1\}$, we have that $C_x$ is an induced cycle of $G$. Since $a_i \in V(H')$ (by $(p_2)$), we have $a_i \neq v$. Thus, by $(p_3)$ we get $|N_G(a_i) \cap V(C_2)| \leq 1$, for $i \in \{1, \ldots, s-1\}$.

We say that $H$ is a $Q_2$-cycle (of $G$) if $H$ is a $Q$-cycle of $G$, and it holds $(q_4)$ for each $v \in N_G(V(H)) \setminus V(H)$, $\deg_G(v) = 2$.

Lemma 15. Let $G \in \mathcal{M}$ and let $C$ be a $Q$-cycle of $G$. Then, $C$ is a $Q_2$-cycle.

Proof. Let $G \in \mathcal{M}$. Let $C$ be a $Q$-cycle of $G$ with the vertex set $\{x_0, \ldots, x_{s-1}\}$, and edges $\{x_0, x_1\}, \ldots, \{x_{s-2}, x_{s-1}\}, \{x_{s-1}, x_0\}$. Let $S = \{0, \ldots, s-1\}$. Let $\{a_i\} = N_G(x_i) \setminus V(C)$, for $i \in S$. If $\deg_G(a_i) = 2$, then let $\{b_i\} = N_G(a_i) \setminus \{x_i\}$. Hence, $b_i \notin V(C)$. By Lemma 10 we have $\deg_G(b_i) = 3$. Let $G' = G \setminus (V(C) \cup \{a_i\colon \deg_G(a_i) = 2 \land i \in S\})$. Since $G' \in G_2^2 \setminus M_3^2$, there is $H' \in \mathcal{B}(G')$.

Suppose to the contrary that $C$ is not a $Q_2$-cycle, i.e., there exists $r \in S$ such that $\deg_G(a_r) = 3$. Let $f: V(G') \rightarrow \{0, 1\}$ be the characteristic function of $V(H')$, i.e., $f(u) = 1$ if and only if $u \in V(H')$. Let us consider two cases.

(i) For each $i \in S$: $\deg_G(a_i) = 2 \Rightarrow f(b_i) = 0$ and $\deg_G(a_i) = 3 \Rightarrow f(a_i) = 0$.

(ii) For some $t \in S$: $\deg_G(a_i) = 2 \land f(b_i) = 1$ or $\deg_G(a_i) = 3 \land f(a_i) = 1$.

We construct a function $\tilde{f}: V(G) \rightarrow \{0, 1\}$ such that $\tilde{f}(u) = f(u)$ for each $u \in V(G')$. Let $u \in V(G) \setminus V(G')$. We define $\tilde{f}(u)$ depending on cases (i), (ii).

(i) Let $\tilde{f}(x_r) = 0$ and let $\tilde{f}(x_j) = 1$, for each $j \in S \setminus \{r\}$. For each $j \in S$, if $\deg_G(a_j) = 2$, then $\tilde{f}(a_j) = 1$. 


(ii) Take any \( t \in S \), if exists, such that \( \deg_G(a_t) = 2 \) and let \( \tilde{f}(a_t) = 1 \). Then, for each \( j \in S, j \neq t \), if \( \deg_G(a_j) = 2 \), then \( \tilde{f}(a_j) = 1 - f(b_j) \). Next, for each \( j \in S \), if \( \deg_G(a_j) = 3 \) or \( \deg_G(a_j) = 2 \), then \( \tilde{f}(a_j) = 1 - f(a_{(j+1) \mod s}) \).

Let \( H = G[\{u \in V(G) : \tilde{f}(u) = 1\}] \). In the case (i), \( x_r \notin V(H) \cap V(C) \). Hence, \( H \) is a bipartite graph. For each \( u \in V(G) \setminus V(G'), u \neq x_r \), we have that \( u \in V(H) \). Thus, \( V(H) \) is a total dominating set of \( G \) and \( H \in \mathcal{B}(G) \), a contradiction.

In case (ii), if there is no \( t \in S \) such that \( \deg_G(a_t) = 2 \), then, by assumption, there is \( t \in S \) such that \( \deg_G(a_t) = 3 \), and finally \( \tilde{f}(a_t) = 1 \) for some \( t \in S \). Hence, there is \( p \in S \) such that \( \tilde{f}(x_p) = 0 \). Thus, \( V(C) \setminus V(H) \neq \emptyset \).

Let us remind that for each \( i \in S \setminus \{t\} \), if \( \deg_G(a_i) = 2 \) and \( \tilde{f}(a_i) = 1 \), then \( f(b_i) = 0 \). Let \( X = \{i \in S : \deg_G(a_i) = 3 \} \cup \{t\} \). Suppose that for some two \( i, j \in X \), there is a path in \( H \) between \( a_i \) and \( a_j \) with successive vertices \( x_i, x_{(i+1) \mod s}, \ldots, x_j \). Hence, \( \tilde{f}(x_i) = \tilde{f}(x_{(i+1) \mod s}) = \cdots = \tilde{f}(x_j) = 1 \), which implies that \( \tilde{f}(a_{(i+1) \mod s}) = 0, \tilde{f}(a_{(i+2) \mod s}) = 0, \ldots, \tilde{f}(a_j) = 0 \), a contradiction. Thus, \( H \) is a bipartite graph.

For every \( j \in S \) we have \( N_G(a_j) \cap V(H) \neq \emptyset \), and \( \tilde{f}(a_j) = 1 \) or \( \tilde{f}(a_j) = 0 \). Hence, we get \( N_G(a_j) \cap V(H) \neq \emptyset \). Thus, \( V(H) \) is a total dominating set and \( H \in \mathcal{B}(G) \), a contradiction.

By Lemmas 10, 12, 13 and Lemma 15, and by the definition of \( Q_2 \)-cycle we have the following corollary.

**Corollary 16.** Let \( G \in \mathcal{M} \) and \( v \in V(G) \). The following properties are satisfied:

(i) \( \deg_G(v) = 2 \) if and only if vertex \( v \) has all neighbors of degree 3,
(ii) \( \deg_G(v) = 3 \) if and only if exactly one neighbor of \( v \) has degree 2,
(iii) if \( \deg_G(v) = 3 \), then there is exactly one \( Q_2 \)-cycle containing \( v \),
(iv) if \( \deg_G(v) = 2 \), then vertex \( v \) has two neighbors from disjoint \( Q_2 \)-cycles.

By Corollary 16 we have the next corollary.

**Corollary 17.** Let \( G \in \mathcal{M} \). The graph \( G \) satisfies the following properties:

(i) there is an integer \( q \geq 1 \) such that \( V(G) = D \cup \bigcup_{i=1}^q V(C_i) \), where for each \( i \in \{1, \ldots, q\} \) the graph \( C_i \) is a \( Q_2 \)-cycle and \( D \) is the set of all vertices of degree 2,
(ii) \( E(G) = \{\{u, v\} : \exists i \in \{1, \ldots, q\} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\} \).

**Proof of Theorem 8.** Suppose to the contrary that \( G \in \mathcal{M} \).
By Corollary 17, there is \( q \geq 1 \) such that \( V(G) = D \cup \bigcup_{i=1}^{q} V(C_i) \), where for each \( i \in \{1, \ldots, q\} \) the graph \( C_i \) is a \( Q_2 \)-cycle and \( D \) is the set of all vertices of degree 2, and

\[
E(G) = \{\{u, v\} : \exists i \in \{1, \ldots, q\} \{\{u, v\} \in E(C_i) \lor (u \in V(C_i) \land v \in D)\}\}.
\]

Let \( Q = (D \cup \bigcup_{i=1}^{q} \{c_i\}, E_Q) \), where for each \( i \in \{1, \ldots, q\} \) vertex \( c_i \) corresponds to the cycle \( C_i \) and

\[
E_Q = \{\{v, c_i\} : i \in \{1, \ldots, q\} \land v \in D \land \exists x \in V(C_i) \{v, x\} \in E(G)\}.
\]

By Corollary 16 and Corollary 17 we have that \( Q \) is a simple bipartite graph with partitions \( D \) and \( C = \bigcup_{i=1}^{q} \{c_i\} \). Obviously, for all vertices \( v \in D \) and \( e \in C \) we have that \( \deg_Q(v) = 2 < \deg_Q(e) \). Thus, by Hall’s Marriage Theorem [10] there is a matching \( S \) in \( Q \) covering all vertices from partition \( C \).

Let

\[
S' = \{\{v, x\} \in E(G) : v \in D \land \exists i \in \{1, \ldots, q\} \{v, c_i\} \in S \land x \in V(C_i)\}
\]

and let

\[
V' = \left\{ x \in \bigcup_{i=1}^{q} V(C_i) : \exists e \in S, x \in e \right\}.
\]

Let \( H = G[V(G) \setminus (D \cup V')] \). For each \( i \in \{1, \ldots, q\} \) there is \( x \) such that \( \{x\} = V(C_i) \cap V' \) and \( N_G(x) \cap V(H) \neq \emptyset \). If \( y \in V(C_i) \) and \( x \neq y \), then \( N_G(y) \cap V(H) \neq \emptyset \). Hence, \( H \) is an induced bipartite graph without isolated vertices. Since for each \( v \in D \) at most one neighbor of \( v \) belongs to \( V' \), we have \( N_G(v) \cap V(H) \neq \emptyset \). Thus, \( N_G(V(H)) = V(G) \) and \( H \in B(G) \), a contradiction.

3. Interval Incidence 6-Coloring of Subcubic Graphs

In this section we prove our main result, i.e., Theorem 21, which states \( \chi_{ii}(G) \leq 2\Delta(G) \) for each subcubic graph \( G \). By Theorem 8 we have the following lemma.

**Lemma 18.** Let \( G \) be a connected graph and \( G \in \mathcal{G}^3 \). Let \( H \in \hat{B}(G) \) and let \( A, B \subset V(H) \) be any partition of \( V(H) \), such that \( A \) and \( B \) are disjoint independent sets and \( A \cup B = V(H) \). Then, \( A \) and \( B \) are disjoint independent dominating sets, and the graph \( G[V(G) \setminus V(H)] \) has only isolated vertices and isolated edges.

**Proof.** Let \( v \in V(G) \setminus V(H) \). If \( N_G(v) \cap V(H) \subset A \) or \( N_G(v) \cap V(H) \subset B \), then \( G[V(H) \cup \{v\}] \) is a bipartite graph, a contradiction. Thus, \( N_G(v) \cap A \neq \emptyset \) and \( N_G(v) \cap B \neq \emptyset \). Let \( v \in A \) (\( v \in B \)). Since \( H \) is an induced graph without isolated vertices, we have \( v \in N_G(B) \) (\( v \in N_G(A) \)). Hence, \( A \) and \( B \) are disjoint independent dominating sets.
Since \( G \) is subcubic and \( |N_G(v) \cap V(H)| \geq 2 \) for any \( v \in V(G) \setminus V(H) \), graph \( G[V(G) \setminus V(H)] \) has only isolated vertices and isolated edges.

**Lemma 19.** Let \( G \) be a subcubic non-bipartite graph with \( \Delta(G) = 3 \). Then, there is a vertex coloring \( c: V(G) \to \{1, 2, 3, 4\} \) such that for each \( v \in V(G) \) the following properties hold:

(i) if \( \deg_G(v) = 1 \), then \( c(v) \in \{1, 4\} \),

(ii) if \( \deg_G(v) \geq 2 \) and \( c(v) \neq p \), then \( a_p(v) \geq 1 \), for \( p \in \{1, 4\} \),

(iii) \( a_i(v) \leq |c(v) - i| \), for \( i \in \{1, 2, 3, 4\} \),

where \( a_i(v) = |\{w \in N_G(v) : c(w) = i\}| \), for \( i \in \{1, 2, 3, 4\} \).

**Proof.** If \( \delta(G) = 1 \), then we successively remove pendant vertices from graph \( G \), until there is no pendant vertex. Let us denote the resulting graph by \( G' \). Obviously, \( \delta(G') \geq 2 \). Let us observe that we cut off all trees attached to \( G \).

By Theorem 8 we have \( \mathcal{B}(G') \neq \emptyset \). Let \( H \) be any element of \( \mathcal{B}(G') \) with the largest possible number of vertices.

Let \( A, B \subset V(H) \) be any two partite sets of \( V(H) \), i.e., \( A \) and \( B \) are disjoint independent sets and \( A \cup B = V(H) \). By Lemma 18, \( A \) and \( B \) are disjoint independent dominating sets of \( G' \), and the graph \( G[V(G') \setminus V(H)] \) has only isolated vertices and isolated edges. Let \( I \subset V(G') \setminus V(H) \) be the set of all vertices of degree \( i \) in \( G' \), for \( i \in \{2, 3\} \). Let us define the partition \( I = I_A \cup I_B \cup I_3 \):

- \( I_A = \{v \in I_3 : |N_{G'}(v) \cap A| = 2 \& |N_{G'}(v) \cap B| = 1\} \),
- \( I_B = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \& |N_{G'}(v) \cap B| = 2\} \),
- \( I_3 = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \& |N_{G'}(v) \cap B| = 1\} \).

Note that \( I_A, I_B, I_3 \) are independent sets in \( G' \), each vertex \( v \in I_3 \) belongs to an isolated edge in \( G'[I_3] \), and each vertex from \( I_2 \) has neighbors from \( A \) and \( B \).

Let us define a coloring \( c: V(G) \to \{1, 2, 3, 4\} \) in the following steps.

\((C_1)\) If \( v \in A \), then \( c(v) = 1 \), and if \( v \in B \), then \( c(v) = 4 \).

\((C_2)\) If \( v \in I_2 \), then \( c(v) = 2 \), and if \( v \in I_3 \), then \( c(v) = 3 \).

\((C_3)\) For each successive \( v \in I_2 \) we assign a color following the algorithm: if \( c(v) \) is not determined, then let \( \{u\} = N_{G'}(v) \cap A \). If there is \( x \in N_{G'}(u) \) such that \( c(x) = 2 \), then let \( c(v) = 3 \). Otherwise, for each vertex \( x \in N_{G'}(u) \) either \( c(x) \in \{3, 4\} \) or \( c(x) \) is not determined, and then let \( c(v) = 2 \).

\((C_4)\) For each successive \( \{v, w\} \in E[G'[I_3^2]) \) we assign colors to both \( v \) and \( w \) following the algorithm: if \( c(v) \) and \( c(w) \) are not determined, then let \( \{u\} = N_{G'}(v) \cap A \). If there is \( x \in N_{G'}(u) \) such that \( c(x) = 2 \), then let \( c(v) = 3 \) and \( c(w) = 2 \). Otherwise, for each vertex \( x \in N_{G'}(u) \) either \( c(x) \in \{3, 4\} \) or \( c(x) \) is not determined, and then let \( c(v) = 2 \) and \( c(w) = 3 \).

\((C_5)\) For each \( v \in V(G') \) such that \( \deg_G(v) < \deg_G(v) \), there is a tree \( T_v \) such that \( V(T_v) \subset V(G) \setminus V(G') \) and let \( \{w\} = V(T_v) \cap N_G(v) \). Let \( d: V(T_v) \to \)
\{a, b\} be a 2-coloring of \(T_v\) such that \(d(w) = a\). Suppose \(c(v) \leq 2\). For each \(u \in V(T_v)\), if \(d(u) = a\), then let \(c(u) = 4\), and if \(d(u) = b\), then let \(c(u) = 1\).

Suppose \(c(v) \geq 3\). For each \(u \in V(T_v)\), if \(d(u) = a\), then let \(c(u) = 1\), and if \(d(u) = b\), then let \(c(u) = 4\).

In step (\(C_1\)) we colored \(V(H) = A \cup B\) with colors 1 and 4, in steps (\(C_2\))–(\(C_4\)) we colored vertices from \(I_2 \cup I_3\) with colors 2 or 3, and in step (\(C_5\)) we colored vertices from \(V(G) \setminus V(G')\) with colors 1 or 4. Since vertices colored with an arbitrary color form an independent set, \(c\) is a vertex 4-coloring of \(G\).

Let \(v \in V(G)\) and let \(\deg_G(v) = 1\). Then, \(v \in V(G) \setminus V(G')\) and, by (\(C_5\)), \(c(v) \in \{1, 4\}\). Thus, we get the property (i). Let \(\deg_G(v) \geq 2\). If \(v \in V(G) \setminus V(G')\), then, by (\(C_5\)), the property (ii) holds. Let \(v \in V(G')\). Since \(A\) and \(B\) are disjoint independent dominating sets of \(G'\), the property (ii) holds.

Since \(c\) is a proper coloring of \(G\), there is \(a_{c(v)}(v) = 0\) for each \(v \in V(G)\).

Let \(v \in V(G') \setminus V(G')\). By step (\(C_5\)), \(c(v) \in \{1, 4\}\). If \(c(v) = 1\), then \(a_2(v) = 0\), \(a_3(v) \leq 1\) and \(a_4(v) \leq 3\). If \(c(v) = 4\), then \(a_3(v) = 0\), \(a_2(v) \leq 1\) and \(a_1(v) \leq 3\).

Let \(v \in V(G') \setminus V(H)\). If \(v \in I_4^3\), then \(c(v) = 3\), \(a_1(v) = 2\), \(a_2(v) = 0\), \(a_4(v) = 1\). If \(v \in I_4^3\), then \(c(v) = 2\), \(a_1(v) = 1\), \(a_3(v) = 0\), \(a_4(v) = 2\). If \(v \in I_2\), then \(c(v) \in \{2, 3\}\). If \(\deg_G(v) = \deg_G(v)\), then \(a_1(v) = a_4(v) = 1\), and \(a_2(v) = a_3(v) = 0\). If \(\deg_G(v) < \deg_G(v)\), then \(c(v) = 2\), then \(a_1(v) = 1\), \(a_2(v) = a_3(v) = 0\), \(a_4(v) = 2\), and if \(c(v) = 3\), then \(a_1(v) = 2\), \(a_2(v) = a_3(v) = 0\), \(a_4(v) = 1\). If \(v \in I_2^4\), then \(c(v) \in \{2, 3\}\). If \(c(v) = 2\), then \(a_1(v) = a_3(v) = a_4(v) = 1\). If \(c(v) = 3\), then \(a_1(v) = a_2(v) = a_4(v) = 1\).

Let \(v \in A \cup B\). Since \(A\) and \(B\) are disjoint dominating sets of \(G'\) and \(H \in \mathcal{B}(G')\), it suffices to prove that if \(c(v) = 1\), then \(a_2(v) \leq 1\), and if \(c(v) = 4\), then \(a_3(v) \leq 1\).

Suppose to the contrary that \(c(v) = 1\) and \(a_2(v) = 2\) for some \(v \in A\). The case \(c(v) = 4\) and \(a_3(v) = 2\), for some \(v \in B\), is analogous. Let \(x, y \in N_{G'}(v)\) such that \(c(x) = c(y) = 2\). Since \(B\) is a dominating set of \(G'\), there is \(w \in N_{G'}(v) \cap B\) with \(c(w) = 4\). By the definition of coloring \(c\), we have \(v, x, y, w \in V(G')\) and \(v, w \in V(H)\).

Since \(c(x) = c(y) = 2\), we have \(a_1(x) = a_1(y) = 1\), \(a_3(x) \leq 1\), \(a_3(y) \leq 1\), \(1 \leq a_4(x) \leq 2\) and \(1 \leq a_4(y) \leq 2\). Let us consider the following cases:

- \(x \notin N_{G'}(w)\) and \(y \notin N_{G'}(w)\). If edge \(\{v, w\}\) is isolated in \(H\), then let \(W = V(H) \cup \{x, y\}\). Otherwise, let \(W = V(H) \cup \{x, y\} \setminus \{v\}\).
- \(x \in N_{G'}(w)\) or \(y \in N_{G'}(w)\). Let \(W = V(H) \cup \{x, y\} \setminus \{v\}\).

In both cases, the graph \(G'[W] \in \mathcal{B}(G')\) and \(|V(G'[W])| > |V(H)|\), a contradiction. Thus, the coloring \(c\) satisfies the property (iii).

\[\square\]

**Proposition 20.** [14] For any graph \(G\), \(\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)\).

We prove that an interval incidence 6-coloring always exists for any subcubic graph \(G\) with \(\Delta(G) = 3\).
Theorem 21. Let $G$ be a subcubic graph. Then, $\chi_{ii}(G) \leq 2\Delta(G)$.

Proof. If $G$ is a subcubic bipartite graph, then by Proposition 20 we have $\chi_{ii}(G) \leq 2\Delta(G)$. If $\Delta(G) = 2$, then one can easily construct an interval incidence 4-coloring. Thus, $\chi_{ii}(G) \leq 2\Delta(G)$. Let $G$ be a subcubic non-bipartite graph with $\Delta(G) = 3$. By Lemma 19, there is a vertex coloring $c : V(G) \to \{1, 2, 3, 4\}$ satisfying the properties (i), (ii), (iii) from Lemma 19.

We construct an incidence coloring $f : I(G) \to \{1, 2, 3, 4, 5, 6\}$ in three steps.

In the first step, using the coloring $c$, we define the interval $A_f(v)$ for each vertex $v \in V(G)$, as follows. If $\deg_G(v) = 2$ and $c(v) \in \{2, 3\}$, then let $A_f(v) = \{3, 4\}$. If $c(v) = 4$ and $\deg_G(v) = 1$, then $A_f(v) = \{6\}$. If $c(v) = 4$ and $\deg_G(v) = 2$, then $A_f(v) = \{5, 6\}$. In the other cases, let $A_f(v) = \{c(v), \ldots, c(v) + \deg_G(v) - 1\}$. Thus, by Lemma 19 (i)–(iii) we get

$(a_1)$ if $\deg_G(v) = 1$, then $c(v) \in \{1, 4\}$ and $A_f(v) = \{c(v)\}$,

$(a_2)$ if $\deg_G(v) = 2$, then if $c(v) \in \{1, 3\}$, then $A_f(v) = \{c(v), c(v) + 1\}$ and if $c(v) \in \{2, 4\}$, then $A_f(v) = \{c(v) + 1, c(v) + 2\}$,

$(a_3)$ if $\deg_G(v) = 3$, then $A_f(v) = \{c(v), c(v) + 1, c(v) + 2\}$.

In the second step, for each $v \in V(G)$, we construct a sequence $L_f(v)$ (i.e., a linear ordered set) from elements of $N_G(v)$, as follows (see Figure 2).

$(l_1)$ Suppose $\deg_G(v) = 1$. If $N_G(v) = \{x\}$, then let $L_f(v) = \{x\}$.

$(l_2)$ Suppose $\deg_G(v) = 2$. Let $N_G(v) = \{x, y\}$, where $c(x) \leq c(y)$. Then,

- if $c(v) \in \{1, 4\}$, then let $L_f(v) = \{x, y\}$,
- if $c(v) \in \{2, 3\}$, then let $L_f(v) = \{y, x\}$.

$(l_3)$ Suppose $\deg_G(v) = 3$. Let $N_G(v) = \{x, y, z\}$, where $c(x) \leq c(y) \leq c(z)$. Then,

- if $c(v) \in \{1, 4\}$, then let $L_f(v) = \{x, y, z\}$,
- if $c(v) = 2$, then let $L_f(v) = \{y, z, x\}$,
- if $c(v) = 3$, then let $L_f(v) = \{z, x, y\}$.

By $v_i$ we mean the $i$-th element of the sequence $L_f(v)$, i.e., $L_f(v) = (v_1, \ldots)$. In the final step, for each vertex $v$, we define the incidence coloring $f$ as follows: $f(v, \{v, v_i\}) = \min A_f(v) + i - 1$, for $i \in \{1, \ldots, \deg_G(v)\}$.

In Figure 2 the white vertex is the vertex $v$, and the list above is $L_f(v)$. By Lemma 19 (i)–(iii), the set of all possible values of $c$ of a vertex is as given in the curly brackets below the vertex. The colors of incidences at the white vertex (i.e., $v$) are given at the edges adjacent to $v$.

Obviously, all the incidences at vertex $v$ are colored with different colors from $A_f(v)$. Observe that the set of colors $A_f(v)$ is an interval of integers.

We prove that the coloring $f$ is an incidence coloring. It is enough to prove that for each vertex $v \in V(G)$ and each vertex $w \in N_G(v)$ we have $f(v, \{v, w\}) \notin A_f(w)$, or, equivalently, $f(v, \{v, w\}) < \min A_f(w)$ or $f(v, \{v, w\}) > \max A_f(w)$.
Figure 2. Interval coloring of incidences at the white vertex v, according to its degree and the values of c at the neighbors x, y, z of v. The set of possible values of c of a vertex is given in the curly brackets below the vertex. The list $L_f(v)$ is given above the white vertex v.

Suppose that $c(v) = 1$. Then, $A_f(v) \subseteq \{1, 2, 3\}$ and $\min A_f(v) = 1$. By the construction of $L_f(v)$ we have: if $\deg_G(v) \geq 1$, then $c(v_1) \in \{2, 3, 4\}$, and if $\deg_G(v) = 2$, then $c(v_2) = 4$, and if $\deg_G(v) = 3$, then $c(v_2) \in \{3, 4\}$ and $c(v_3) = 4$ (see Figure 2). Hence, for each $i \in \{1, \ldots, \deg_G(v)\}$ we have $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 < i + 1 \leq \min A_f(v_i)$.

Suppose that $c(v) = 2$. Then, $A_f(v) \subseteq \{2, 3, 4\}$. Let $\deg_G(v) = 3$. Hence, $\min A_f(v) = 2$, and $c(v_1) \in \{3, 4\}$. By $\deg_G(v) = 2$, then $c(v_2) = 4$ and $\deg_G(v) = 3$. Hence, for each $i \in \{1, 2\}$, and $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 < i + 1 \leq \min A_f(v_i)$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$ and $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$.

Suppose that $c(v) = 3$. Then, $A_f(v) \subseteq \{3, 4, 5\}$ and $\min A_f(v) = 3$. Let $\deg_G(v) = 3$. Hence, $c(v_1) = 4$ and $c(v_2) = 1$ and $c(v_3) \in \{1, 2\}$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$, and $f(v, \{v, v_1\}) = \min A_f(v) + i - 1 < i + 1 \geq \max A_f(v_i)$, for $i \in \{2, 3\}$. Let $\deg_G(v) = 2$. Hence, $c(v_1) = 4$ and $c(v_2) = 1$. Thus, $f(v, \{v, v_i\}) = 3 < 4 \leq \min A_f(v_1)$ and $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$.

Suppose that $c(v) = 4$. Then, $A_f(v) \subseteq \{4, 5, 6\}$. Let $\deg_G(v) = 3$. Hence, $c(v_1) = 4$ and $c(v_2) \in \{1, 2\}$ and $c(v_3) \in \{1, 2, 3\}$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) + i - 1 < i + 1 \geq \max A_f(v_i)$, for each $i \in \{1, 2, 3\}$. Let $\deg_G(v) = 2$. Hence, $c(v_1) = 4$ and $c(v_2) \in \{1, 2, 3\}$, and $A_f(v) = \{5, 6\}$. Thus, $f(v, \{v, v_1\}) = 5 > \max A_f(v_1)$ and $f(v, \{v, v_2\}) = 6 > \max A_f(v_2)$. Let $\deg_G(v) = 1$. Hence, $c(v_1) \in \{1, 2, 3\}$. Thus, $f(v, \{v, v_i\}) = 6 > 5 \geq \max A_f(v_1)$.

In all the cases we proved that $f(v, \{v, v_i\}) \notin A_f(v_i)$ for each $v_i \in N_G(v)$. Thus, $f$ is an interval incidence 6-coloring of $G$. ■
4. Summary

In this paper we proved that for any subcubic graph $G$, $\chi_{ii}(G) \leq 2\Delta(G)$. In [14] we proved that the upper bound of $2\Delta(G)$ on $\chi_{ii}(G)$ holds for each complete $k$-partite graph $G$ and this bound is valid for other classes of graphs. Thus, we state the following

**Conjecture 22** [Interval Incidence Coloring Conjecture (IICC)]. For any graph $G$, $\chi_{ii}(G) \leq 2\Delta(G)$.

**References**


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