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INTERVAL INCIDENCE COLORING OF SUBCUBIC GRAPHS[1]

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Abstract

In this paper we study the problem of interval incidence coloring of subcubic graphs. In [14] the authors proved that the interval incidence 4-coloring problem is polynomially solvable and the interval incidence 5-coloring problem is $\mathcal{NP}$-complete, and they asked if $\chi_{ii}(G) \leq 2\Delta(G)$ holds for an arbitrary graph G. In this paper, we prove that an interval incidence 6-coloring always exists for any subcubic graph G with $\Delta(G) = 3$.

Keywords: interval incidence coloring, incidence coloring, subcubic graph.

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1. Introduction

In the paper we consider simple nonempty graphs, and we use the standard notation of graph theory. Let $G = (V, E)$ be a simple graph, and let $X \subset V$ be a non-empty set. By $N_G(X) = \{v \in V : \exists_{u \in X} \{v, u\} \in E\}$ we mean the *open*

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neighborhood of X, by $G[X]$ we mean the subgraph of G induced by the set X, and by $G \setminus X$ we mean the graph $G[V \setminus X]$. We say that X is a *dominating set* of G if $V = N_G(X) \cup X$, and we say that X is a *total dominating set* if $V = N_G(X)$. In what follows we use $N_G(v)$ instead of $N_G(\{v\})$. Let $\deg_G(v) = |N_G(v)|$ be the degree of a vertex $v \in V(G)$. By $n(G), \Delta(G)$ and $\delta(G)$ we denote the number of vertices of G, the maximum and the minimum degree of a vertex of G, respectively. By a subcubic graph G we mean a graph with $\Delta(G) \leq 3$. By an *isolated vertex* (in a graph G) we mean a vertex $v \in V(G)$ with $\deg_G(v) = 0$, and by an *isolated edge* (in a graph G) we mean an edge $e = \{u, v\}$ such that $\deg_G(u) = \deg_G(v) = 1$. We say that $X \subset V(G)$ is an *independent set* if each vertex of $G[X]$ is isolated in $G[X]$. By a *pendant vertex* we mean a vertex of degree 1.

For a given graph $G = (V, E)$, we define an *incidence* as a pair (v, e), where vertex $v \in V$ is one of the endpoints of edge $e \in E$, i.e., $v \in e$. The set of all incidences of G will be denoted by $I(G)$, thus $I(G) = \{(v, e)\colon v \in V \wedge e \in E \wedge v \in e\}$. We say that two incidences (v, e) and (w, f) are *adjacent* if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}$, $f = \{w, u\}$ and $v \neq u$.

By an *incidence coloring* of G we mean a function $c\colon I(G) \to \mathbb{N}$ such that $c((v, e)) \neq c((w, f))$ for any two adjacent incidences (v, e) and (w, f). The *incidence coloring number* of G, denoted by $\chi_i(G)$, is the smallest number of colors in an incidence coloring of G. In what follows we use the simplified notation $c(v, e)$ instead of $c((v, e))$.

A finite nonempty set $A \subset \mathbb{N}$ is an *interval* if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring c of graph G and $v \in V(G)$ let $A_c(v) = \{c(v, e)\colon v \in e \wedge e \in E(G)\}$. By an *interval incidence coloring* of a graph G we mean an incidence coloring c of G such that for each vertex $v \in V(G)$ the set $A_c(v)$ is an interval. By an *interval incidence k-coloring* we mean an interval incidence coloring using all colors from the set $\{1, \ldots, k\}$. The *interval incidence coloring number* of G, denoted by $\chi_{ii}(G)$, is the smallest number of colors in an interval incidence coloring of G.

1.1. Background and previous results

Alon *et al.* [1] defined the problem of partitioning a graph into the minimal number of star forests. Brualdi and Massey [3] formulated a model of incidence coloring of graphs with references to certain models of coloring of graphs, such as strong edge and vertex coloring of graphs. Guiduli [9] observed that the problem of incidence coloring of graphs is a special case of the problem of partitioning a symmetric digraph into directed star forests.

In [3] the authors conjectured that $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph G (*incidence coloring conjecture*, shortly ICC). This conjecture was disproved by Guiduli in [9] who observed that Paley graphs have incidence coloring number at least $\Delta+\Omega(\log\Delta)$. In fact, he used the crucial result from [1]. For many classes of graphs it is shown that the incidence coloring number is at most $\Delta+2$, e.g., trees and cycles [3], complete graphs [3], complete bipartite graphs [3] (proof corrected in [19]), planar graphs with girth at least 11 or with girth at least 6 and maximum degree at least 5 [5], partial 2-trees (i.e., K_4-minor free graphs) [4], hypercubes [18], complete k-partite graphs [15].

In [17] the author proved that ICC holds for subcubic graphs. The incidence 4-colorability problem is $\mathcal{NP}$-complete for *semicubic* graphs (i.e., subcubic graphs with vertex degrees equal to 1 or 3) [16] and for semicubic bipartite graphs [15].

In this paper we consider a restriction of the problem of incidence coloring of graphs in which the colors of incidences at a vertex form an interval. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring (see, e.g., [2, 6, 8]), which can be applied to the open-shop scheduling problem [6, 7]. In [11] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [12] the authors studied applications in one-multicast transmission in multifiber WDM networks.

In [13] the authors proved that the problem of interval incidence k-coloring of bipartite graphs is polynomial for each $k\leq 6$ and $\Delta\leq 3$, polynomial for $k=5$ and $\Delta=4$, and $\mathcal{NP}$-complete for $k=6$ and $\Delta=4$. In [14] the authors proved certain lower and upper bounds on the interval incidence coloring number, e.g., $\Delta(G)+1\leq\chi_{ii}(G)\leq\chi(G)\cdot\Delta(G)$ for an arbitrary graph G, and they determined the exact values of χ_{ii} for some basic classes of graphs (e.g., complete k-partite graphs). In [14] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem of deciding whether $\chi_{ii}\leq 4$ is easy, and $\chi_{ii}\leq 5$ is $\mathcal{NP}$-complete. The problem of interval incidence 6-coloring of subcubic graphs remained unsolved.

1.2. Main results

Our main result in the paper is Theorem 21 which states $\chi_{ii}(G)\leq 6$ for every subcubic graph G. To prove it, we state and prove Theorem 8: in any subcubic graph G with $\delta(G)\geq 2$ there is a maximal induced bipartite subgraph of G without isolated vertices, or equivalently, G has a total dominating set S such that $G[S]$ is a bipartite graph.

2. Maximal Induced Bipartite Subgraphs Without Isolated Vertices

In this section we prove (in Theorem 8) that any subcubic graph G with $\delta(G)\geq 2$ contains a maximal induced bipartite subgraph without isolated vertices.

2.1. Introductory properties

By $H \subset G$ we mean that H is a subgraph of G. By $H \sqsubset G$ we mean that H is an induced subgraph of G, i.e., $H = G[V(H)]$.

Observation 1. *If $G_1 \sqsubset G_2$ and $G_2 \sqsubset G_3$, then $G_1 \sqsubset G_3$.*

Observation 2. *Let $G_1 \sqsubset G$ and $G_2 \sqsubset G$. If $G_1 \subset G_2$, then $G_1 \sqsubset G_2$.*

Let $\mathcal{B}(G) = \{H \sqsubset G \colon N_G(V(H)) = V(G) \wedge H$ is bipartite$\}$, i.e., the set of all induced bipartite subgraphs of a given graph G such that $V(H)$ is a total dominating set of G. If $H \in \mathcal{B}(G)$, then $V(H)$ is a total dominating set of G and, obviously, H has no isolated vertices.

In the following, let G be any graph. Let $\hat{\mathcal{B}}(G)$ be the subfamily of $\mathcal{B}(G)$ consisting of all the elements (graphs) in $\mathcal{B}(G)$ that are maximal with respect to the subgraph relation ($\subset$).

Observation 3. *If $H \in \mathcal{B}(G)$, then there is $H' \in \hat{\mathcal{B}}(G)$ such that $H \subset H'$.*

By Observations 2 and 3 we have

Observation 4. *Let $H \in \mathcal{B}(G)$. Then, $H \in \hat{\mathcal{B}}(G)$ if and only if for each $v \in V(G) \setminus V(H)$ the subgraph $G[V(H) \cup \{v\}]$ is not bipartite.*

Observation 5. *If $H \in \mathcal{B}(G) \setminus \hat{\mathcal{B}}(G)$, then there is a vertex $v \in V(G) \setminus V(H)$ such that $G[V(H) \cup \{v\}] \in \mathcal{B}(G)$.*

Since any dominating set $S \subset V(G)$ is a total dominating set if and only if $G[S]$ has no isolated vertices, we have

Observation 6. *Let G be an arbitrary graph and let $H \subset G$. Then, $H \in \hat{\mathcal{B}}(G)$ if and only if H is a maximal induced bipartite subgraph (of G) without isolated vertices.*

Let $\mathcal{G}_3^2$ be the family of subcubic graphs without isolated and pendant vertices, i.e., each vertex in a graph of this family has degree 2 or 3. Let $\mathcal{M}_3^2$ be the subfamily of $\mathcal{G}_3^2$ consisting of all the graphs for which there is no maximal induced bipartite subgraph without isolated vertices. Let us denote by $\mathcal{M}$ the set of elements in $\mathcal{M}_3^2$ that are minimal with respect to the subgraph relation ($\subset$). By Observation 6 we have

Observation 7. *Let $G \in \mathcal{G}_3^2$. Then, $G \in \mathcal{M}_3^2 \Leftrightarrow \mathcal{B}(G) = \emptyset \Leftrightarrow \hat{\mathcal{B}}(G) = \emptyset$.*

2.2. Main Theorem

Theorem 8. *Let G be a subcubic graph with $\delta(G) \geq 2$. Then, G has a maximal induced bipartite subgraph without isolated vertices.*

By Observation 7, Theorem 8 is equivalent to $\mathcal{M} = \emptyset$. First, we prove some structural properties of graphs from $\mathcal{M}$.

Lemma 9. *Let $G \in \mathcal{M}$. Then, G is a connected graph and $\Delta(G) = 3$.*

Proof. Let $G \in \mathcal{M}$. Let us assume to the contrary that $G = G_1 \cup G_2$, where G_1 and G_2 are disjoint graphs (without common vertices). Since $G_i \subsetneq G \in \mathcal{M}$ and $G_i \in \mathcal{G}_3^2$, we have $G_i \notin \mathcal{M}_3^2$, for $i \in \{1, 2\}$. Hence, there exist $H_1 \in \hat{\mathcal{B}}(G_1)$ and $H_2 \in \hat{\mathcal{B}}(G_2)$. Thus, $H_1 \cup H_2 \in \hat{\mathcal{B}}(G)$, a contradiction.

Since every cycle is either a bipartite graph or it becomes a bipartite graph after deleting an arbitrary vertex, G is not a cycle, which implies $\Delta(G) = 3$. ■

Lemma 10. *Let $G \in \mathcal{M}$ and let v be a vertex of degree 2 in G. Then, every neighbor of v in G has degree 3.*

Proof. Let $G \in \mathcal{M}$. Suppose to the contrary that there are two adjacent vertices of degree 2. Since G is not a cycle (by Lemma 9), there is a subgraph P of G with vertex set $\{v_0, \ldots, v_{k+1}\}$ and edges $\{v_i, v_{i+1}\}$, for $i \in \{0, \ldots, k\}$, such that $\deg_G(v_0) = \deg_G(v_{k+1}) = 3$, and $\deg_G(v_i) = 2$ for $i \in \{1, \ldots, k\}$, where $k \geq 2$.

Suppose $v_0 \neq v_{k+1}$. Since $G' = G \setminus \{v_1, \ldots, v_k\} \sqsubset G \in \mathcal{M}$ and $G' \in \mathcal{G}_3^2$, we have $G' \notin \mathcal{M}_3^2$. Hence, there exists $H' \in \hat{\mathcal{B}}(G')$, and $H' \sqsubset G$ by Observation 1. If $v_0 \in V(H')$, then let $H = G[V(H') \cup \{v_1, \ldots, v_{k-1}\}]$, otherwise, let $H = G[V(H') \cup \{v_1, \ldots, v_k\}]$. In both cases, $H \sqsubset G$, H is a bipartite graph, and $V(H)$ is a total dominating set, i.e., $H \in \mathcal{B}(G)$. By Observation 7 we get a contradiction.

Suppose $v_0 = v_{k+1}$. Since $\deg_G(v_0) = 3$, there is $c \in N_G(v_0) \setminus \{v_1, v_k\}$. If $\deg_G(c) = 3$, then let $G' = G \setminus \{v_0, \ldots, v_k\}$. If $\deg_G(c) = 2$, then let $G' = G \setminus \{v_0, \ldots, v_k, c\}$. In both cases, $G' \sqsubset G$ and $G \neq G' \in \mathcal{G}_3^2$. Hence, there is $H' \in \hat{\mathcal{B}}(G')$. Let $H = G[V(H') \cup \{v_0, \ldots, v_{k-1}\}]$. Thus, $H \in \mathcal{B}(G)$, a contradiction. ■

Lemma 11. *If $G \in \mathcal{G}_3^2$ contains G_0 as a subgraph (see Figure 1), where vertices $v_2, v_3 \in V(G_0)$ are of degree 2 in G, then $G \notin \mathcal{M}$.*

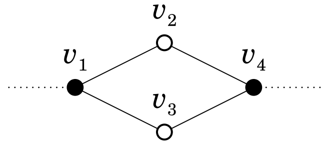

Figure 1. The subgraph G_0 of a graph G.

Proof. Suppose to the contrary that $G \in \mathcal{M}$. Suppose $G_0 \subset G$. The other possible edges in G are marked by the dotted lines (in Figure 1).

By $\deg_G(v_2) = deg_G(v_3) = 2$, from Lemma 10 we have $\deg_G(v_1) = \deg_G(v_4) = 3$. Since $G' = G \setminus \{v_3\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. Hence, $v_1 \in V(H')$ or $v_4 \in V(H')$. Thus, $H' \in \mathcal{B}(G)$, a contradiction. ■

Lemma 12. *Let $G \in \mathcal{M}$ and let v be a vertex of degree 3 in G. Then, at most one neighbor of v has degree 2.*

Proof. Let $G \in \mathcal{M}$ and let $N_G(v) = \{x, y, z\}$. Suppose to the contrary that at least two vertices from $N_G(v)$ have degree 2. Let $\deg_G(x) = \deg_G(y) = 2$. Let $\{v_x\} = N_G(x) \setminus \{v\}$ and $\{v_y\} = N_G(y) \setminus \{v\}$. By Lemma 10, $\deg_G(v_x) = \deg_G(v_y) = 3$.

Suppose $\deg_G(z) = 2$. Let $\{v_z\} = N_G(z) \setminus \{v\}$. By Lemma 10, $\deg_G(v_z) = 3$. If any two of the vertices v_x, v_y, v_z are equal, then by Lemma 11 (i.e., because $G_0 \sqsubset G$) we get a contradiction. Hence, vertices v_x, v_y, v_z are different. Since $G' = G \setminus \{x, y, z, v\} \in \mathcal{G}_3^2 \setminus M_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. Thus, $G[V(H') \cup \{v, x\}] \in \mathcal{B}(G)$, a contradiction.

Suppose $\deg_G(z) = 3$. If $v_x = v_y$, then by Lemma 11 we get a contradiction. Hence, $v_x \neq v_y$. Suppose $z = v_x$ (the case $z = v_y$ can be treated analogously). Since $G_x = G \setminus \{x\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H_x \in \hat{\mathcal{B}}(G_x)$. Since H_x is maximal in $\mathcal{B}(G)$, we have $v \in V(H_x)$ or $z \in V(H_x)$. Thus, $H_x \in \mathcal{B}(G)$, a contradiction. Then, vertices v_x, v_y, z are different. Since $G' = G \setminus \{x, y, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. If $z \in V(H')$, then let $A = V(H') \cup \{v\}$. If $z \notin V(H')$, then let $A = V(H') \cup \{v, x\}$. In both cases, $G[A] \in \mathcal{B}(G)$, a contradiction. ■

Let G be any subcubic graph. We say that $H \subset G$ is a *Q-cycle* (of G) if:

(q_1) for each $v \in V(H)$, $\deg_G(v) = 3$, and

(q_2) $H \sqsubset G$ and H is isomorphic to a cycle, i.e., H is an induced cycle, and

(q_3) for each vertex $v \in V(G) \setminus V(H)$, $|N_G(v) \cap V(H)| \leq 1$.

Lemma 13. *Let $G \in \mathcal{M}$. Let $v \in V(G)$ have all neighbors of degree 3. Then, for each $x \in N_G(v)$ there is a Q-cycle C_x such that $x \in V(C_x)$, $v \notin V(C_x)$ and $N_G(v) \cap V(C_x) = \{x\}$.*

Proof. Let $G \in \mathcal{M}$ and let $v \in V(G)$ be a vertex with all neighbors of degree 3. Since $G' = G \setminus \{v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. Hence, $N_G(v) \cap V(H') = \emptyset$.

Let $x \in N_G(v)$ and let $N_G(x) = \{a, b, v\}$. Since H' is bipartite and maximal in $\mathcal{B}(G)$, we have that a and b belong to the same connected component of H', and the length of each path in H' from a to b is odd. Let $P \subset H'$ be a path joining $x_1 = a$ and $x_{s-1} = b$ (s is odd), with vertex set $\{x_1, \ldots, x_{s-1}\}$ and edges $\{x_i, x_{i+1}\}$, for $i \in \{1, \ldots, s-2\}$. Let $x_0 = x$ and let C_x be the graph with $V(C_x) = V(P) \cup \{x_0\}$, and $E(C_x) = E(P) \cup \{\{x_{s-1}, x_0\}, \{x_0, x_1\}\}$. Since $P \subset H'$, we have $N_G(v) \cap V(C_x) = \{x\}$, and $v \notin V(C_x)$.

Claim 14. *For each* $i \in \{1, \ldots, s-1\}$*, the following properties are satisfied:*

(p_1) $\deg_G(x_i) = 3$,
(p_2) $N_G(x_i) = \{a_i, x_{(i-1) \bmod s}, x_{(i+1) \bmod s}\}$, *where* $a_i \in V(H') \setminus V(C_x)$,
(p_3) $N_G(a_i) \cap V(H') = \{x_i\}$.

Proof. We proceed by induction on i. Suppose $i = 1$. Let $X = V(H') \setminus \{x_i\} \cup \{x, v\}$. Hence, $G[X]$ is bipartite. If $\deg_G(x_i) = 2$ or $N_G(a_i) \cap V(H') \neq \{x_i\}$, then $G[X] \in \mathcal{B}(G)$, a contradiction. If $a_i \notin V(H') \setminus V(C_x)$, then $a_i \notin V(H')$ or $a_i \in V(C_x)$. If $a_i \notin V(H')$, then $N_G(a_i) \cap V(H') \neq \{x_i\}$ (otherwise H' is not maximal in $\mathcal{B}(G)$), a contradiction. If $a_i \in V(C_x)$, then $G[X] \in \mathcal{B}(G)$, a contradiction.

Suppose the properties $(p_1), (p_2), (p_3)$ hold for $1, \ldots, i-1$ $(2 \leq i \leq s-1)$. Hence, each path joining x_1 and x_{s-1} in H' contains $x_1, \ldots, x_i$. Let $X = V(H') \setminus \{x_i\} \cup \{x, v\}$. Hence, $G[X]$ is bipartite. The rest of the proof of properties $(p_1), (p_2), (p_3)$ for i is literally the same as in the case $i = 1$. □

We show that C_x is a Q-cycle. Since $\deg_G(x) = 3$, by (p_1) we have (q_1). Since $v \notin V(C_x)$ and $a_i \notin V(C_x)$ (by (p_2)), for $i \in \{1, \ldots, s-1\}$, we have that C_x is an induced cycle of G. Since $a_i \in V(H')$ (by (p_2)), we have $a_i \neq v$. Thus, by (p_3) we get $|N_G(a_i) \cap V(C_x)| \leq 1$, for $i \in \{1, \ldots, s-1\}$. ■

We say that H is a *Q_2-cycle* (of G) if H is a Q-cycle of G, and it holds

(q_4) for each $v \in N_G(V(H)) \setminus V(H)$, $\deg_G(v) = 2$.

Lemma 15. *Let* $G \in \mathcal{M}$ *and let* C *be a* Q*-cycle of* G*. Then,* C *is a* Q_2*-cycle.*

Proof. Let $G \in \mathcal{M}$. Let C be a Q-cycle of G with the vertex set $\{x_0, \ldots, x_{s-1}\}$, and edges $\{x_0, x_1\}, \ldots, \{x_{s-2}, x_{s-1}\}, \{x_{s-1}, x_0\}$. Let $S = \{0, \ldots, s-1\}$. Let $\{a_i\} = N_G(x_i) \setminus V(C)$, for $i \in S$. If $\deg_G(a_i) = 2$, then let $\{b_i\} = N_G(a_i) \setminus \{x_i\}$. Hence, $b_i \notin V(C)$. By Lemma 10 we have $\deg_G(b_i) = 3$. Let $G' = G \setminus (V(C) \cup \{a_i \colon \deg_G(a_i) = 2 \wedge\ i \in S\})$. Since $G' \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$.

Suppose to the contrary that C is not a Q_2-cycle, i.e., there exists $r \in S$ such that $\deg_G(a_r) = 3$. Let $f \colon V(G') \to \{0, 1\}$ be the characteristic function of $V(H')$, i.e., $f(u) = 1$ if and only if $u \in V(H')$. Let us consider two cases.

(i) For each $i \in S$: $\deg_G(a_i) = 2 \Rightarrow f(b_i) = 0$ and $\deg_G(a_i) = 3 \Rightarrow f(a_i) = 0$.
(ii) For some $t \in S$: $\deg_G(a_t) = 2 \wedge f(b_t) = 1$ or $\deg_G(a_t) = 3 \wedge f(a_t) = 1$.

We construct a function $\tilde{f} \colon V(G) \to \{0, 1\}$ such that $\tilde{f}(u) = f(u)$ for each $u \in V(G')$. Let $u \in V(G) \setminus V(G')$. We define $\tilde{f}(u)$ depending on cases (i), (ii).

(i) Let $\tilde{f}(x_r) = 0$ and let $\tilde{f}(x_j) = 1$, for each $j \in S \setminus \{r\}$. For each $j \in S$, if $\deg_G(a_j) = 2$, then $\tilde{f}(a_j) = 1$,

(ii) Take any $t \in S$, if exists, such that $\deg_G(a_t) = 2 \wedge f(b_t) = 1$ and let $\tilde{f}(a_t) = 1$. Then, for each $j \in S, j \neq t$, if $\deg_G(a_j) = 2$, then $\tilde{f}(a_j) = 1 - f(b_j)$. Next, for each $j \in S$, if $\deg_G(a_j) = 2 \wedge f(b_j) = 0$, then $\tilde{f}(x_j) = 1$. Finally, for each $j \in S$, if $\deg_G(a_j) = 3$ or $\deg_G(a_j) = 2 \wedge f(b_j) = 1$, then $\tilde{f}(x_j) = 1 - \tilde{f}(a_{(j+1) \bmod s})$.

Let $H = G[\{u \in V(G) \colon \tilde{f}(u) = 1\}]$. In the case (i), $x_r \notin V(H) \cap V(C)$. Hence, H is a bipartite graph. For each $u \in V(G) \setminus V(G'), u \neq x_r$, we have that $u \in V(H)$. Thus, $V(H)$ is a total dominating set of G and $H \in \mathcal{B}(G)$, a contradiction.

In case (ii), if there is no $t \in S$ such that $\deg_G(a_t) = 2 \wedge f(b_t) = 1$, then, by assumption, there is $t \in S$ such that $\deg_G(a_t) = 3 \wedge f(a_t) = 1$, so finally $\tilde{f}(a_t) = 1$ for some $t \in S$. Hence, there is $p \in S$ such that $\tilde{f}(x_p) = 0$. Thus, $V(C) \setminus V(H) \neq \emptyset$.

Let us remind that for each $i \in S \setminus \{t\}$, if $\deg_G(a_i) = 2$ and $\tilde{f}(a_i) = 1$, then $f(b_i) = 0$. Let $X = \{i \in S : \deg_G(a_i) = 3 \wedge \tilde{f}(a_i) = 1\} \cup \{t\}$. Suppose that for some two $i, j \in X$, there is a path in H between a_i and a_j with successive vertices $x_i, x_{(i+1) \bmod s}, \ldots, x_j$. Hence, $\tilde{f}(x_i) = \tilde{f}(x_{(i+1) \bmod s}) = \cdots = \tilde{f}(x_j) = 1$, which implies that $\tilde{f}(a_{(i+1) \bmod s}) = 0, \tilde{f}(a_{(i+2) \bmod s}) = 0, \ldots, \tilde{f}(a_j) = 0$, a contradiction. Thus, H is a bipartite graph.

For every $j \in S$ we have $N_G(a_j) \cap V(H) \neq \emptyset$, and $\tilde{f}(a_j) = 1$ or $\tilde{f}(a_j) = 0 \wedge \tilde{f}(x_{(j-1) \bmod s}) = 1$. Hence, we get $N_G(x_j) \cap V(H) \neq \emptyset$. Thus, $V(H)$ is a total dominating set and $H \in \mathcal{B}(G)$, a contradiction. ■

By Lemmas 10, 12, 13 and Lemma 15, and by the definition of Q_2-cycle we have the following corollary.

Corollary 16. *Let $G \in \mathcal{M}$ and $v \in V(G)$. The following properties are satisfied:*

(i) $\deg_G(v) = 2$ *if and only if vertex v has all neighbors of degree* 3,

(ii) $\deg_G(v) = 3$ *if and only if exactly one neighbor of v has degree* 2,

(iii) *if* $\deg_G(v) = 3$, *then there is exactly one Q_2-cycle containing v,*

(iv) *if* $\deg_G(v) = 2$, *then vertex v has two neighbors from disjoint Q_2-cycles.*

By Corollary 16 we have the next corollary.

Corollary 17. *Let $G \in \mathcal{M}$. The graph G satisfies the following properties:*

(i) *there is an integer $q \geq 1$ such that $V(G) = D \cup \bigcup_{i=1}^{q} V(C_i)$, where for each $i \in \{1, \ldots, q\}$ the graph C_i is a Q_2-cycle and D is the set of all vertices of degree* 2,

(ii) $E(G) = \{\{u, v\} \colon \exists_{i \in \{1,\ldots,q\}} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\}$.

***Proof of Theorem* 8.** Suppose to the contrary that $G \in \mathcal{M}$.

By Corollary 17, there is $q \geq 1$ such that $V(G) = D \cup \bigcup_{i=1}^{q} V(C_i)$, where for each $i \in \{1, \ldots, q\}$ the graph C_i is a Q_2-cycle and D is the set of all vertices of degree 2, and

$$E(G) = \{\{u, v\} \colon \exists_{i \in \{1,\ldots,q\}} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\}.$$

Let $Q = (D \cup \bigcup_{i=1}^{q}\{c_i\}, E_Q)$, where for each $i \in \{1, \ldots, q\}$ vertex c_i corresponds to the cycle C_i and

$$E_Q = \{\{v, c_i\} \colon i \in \{1, \ldots, q\} \wedge v \in D \wedge \exists_{x \in V(C_i)} \{v, x\} \in E(G)\}.$$

By Corollary 16 and Corollary 17 we have that Q is a simple bipartite graph with partitions D and $C = \bigcup_{i=1}^{q}\{c_i\}$. Obviously, for all vertices $v \in D$ and $c \in C$ we have that $\deg_Q(v) = 2 < \deg_Q(c)$. Thus, by Hall's Marriage Theorem [10] there is a matching S in Q covering all vertices from partition C.

Let

$$S' = \{\{v, x\} \in E(G) \colon v \in D \wedge \exists_{i \in \{1,\ldots,q\}} \{v, c_i\} \in S \wedge x \in V(C_i)\}$$

and let

$$V' = \left\{ x \in \bigcup_{i=1}^{q} V(C_i) \colon \exists_{e \in S'} x \in e \right\}.$$

Let $H = G[V(G) \setminus (D \cup V')]$. For each $i \in \{1, \ldots, q\}$ there is x such that $\{x\} = V(C_i) \cap V'$ and $N_G(x) \cap V(H) \neq \emptyset$. If $y \in V(C_i)$ and $x \neq y$, then $N_G(y) \cap V(H) \neq \emptyset$. Hence, H is an induced bipartite graph without isolated vertices. Since for each $v \in D$ at most one neighbor of v belongs to V', we have $N_G(v) \cap V(H) \neq \emptyset$. Thus, $N_G(V(H)) = V(G)$ and $H \in \mathcal{B}(G)$, a contradiction. ■

3. Interval Incidence 6-Coloring of Subcubic Graphs

In this section we prove our main result, i.e., Theorem 21, which states $\chi_{ii}(G) \leq 2\Delta(G)$ for each subcubic graph G. By Theorem 8 we have the following lemma.

Lemma 18. *Let G be a connected graph and $G \in \mathcal{G}_3^2$. Let $H \in \hat{\mathcal{B}}(G)$ and let $A, B \subset V(H)$ be any partition of $V(H)$, such that A and B are disjoint independent sets and $A \cup B = V(H)$. Then, A and B are disjoint independent dominating sets, and the graph $G[V(G) \setminus V(H)]$ has only isolated vertices and isolated edges.*

Proof. Let $v \in V(G) \setminus V(H)$. If $N_G(v) \cap V(H) \subset A$ or $N_G(v) \cap V(H) \subset B$, then $G[V(H) \cup \{v\}]$ is a bipartite graph, a contradiction. Thus, $N_G(v) \cap A \neq \emptyset$ and $N_G(v) \cap B \neq \emptyset$. Let $v \in A$ ($v \in B$). Since H is an induced graph without isolated vertices, we have $v \in N_G(B)$ ($v \in N_G(A)$). Hence, A and B are disjoint independent dominating sets.

Since G is subcubic and $|N_G(v) \cap V(H)| \geq 2$ for any $v \in V(G) \setminus V(H)$, graph $G[V(G) \setminus V(H)]$ has only isolated vertices and isolated edges. ■

Lemma 19. *Let G be a subcubic non-bipartite graph with $\Delta(G) = 3$. Then, there is a vertex coloring $c\colon V(G) \to \{1,2,3,4\}$ such that for each $v \in V(G)$ the following properties hold:*

(i) *if* $\deg_G(v) = 1$, *then* $c(v) \in \{1,4\}$,

(ii) *if* $\deg_G(v) \geq 2$ *and* $c(v) \neq p$, *then* $a_p(v) \geq 1$, *for* $p \in \{1,4\}$,

(iii) $a_i(v) \leq |c(v) - i|$, *for* $i \in \{1,2,3,4\}$,

where $a_i(v) = |\{w \in N_G(v)\colon c(w) = i\}|$, *for* $i \in \{1,2,3,4\}$.

Proof. If $\delta(G) = 1$, then we successively remove pendant vertices from graph G, until there is no pendant vertex. Let us denote the resulting graph by G'. Obviously, $\delta(G') \geq 2$. Let us observe that we cut off all trees attached to G.

By Theorem 8 we have $\hat{\mathcal{B}}(G') \neq \emptyset$. Let H be any element of $\hat{\mathcal{B}}(G')$ with the largest possible number of vertices.

Let $A, B \subset V(H)$ be any two partite sets of $V(H)$, i.e., A and B are disjoint independent sets and $A \cup B = V(H)$. By Lemma 18, A and B are disjoint independent dominating sets of G', and the graph $G[V(G') \setminus V(H)]$ has only isolated vertices and isolated edges. Let $I_i \subset V(G') \setminus V(H)$ be the set of all vertices of degree i in G', for $i \in \{2,3\}$. Let us define the partition $I_3 = I_3^A \cup I_3^B \cup I_3^2$:

- $I_3^A = \{v \in I_3\colon |N_{G'}(v) \cap A| = 2 \wedge |N_{G'}(v) \cap B| = 1\}$,
- $I_3^B = \{v \in I_3\colon |N_{G'}(v) \cap A| = 1 \wedge |N_{G'}(v) \cap B| = 2\}$,
- $I_3^2 = \{v \in I_3\colon |N_{G'}(v) \cap A| = 1 \wedge |N_{G'}(v) \cap B| = 1\}$.

Note that I_2, I_3^A, I_3^B are independent sets in G', each vertex $v \in I_3^2$ belongs to an isolated edge in $G'[I_3^2]$, and each vertex from I_2 has neighbors from A and B.

Let us define a coloring $c\colon V(G) \to \{1,2,3,4\}$ in the following steps.

(C_1) If $v \in A$, then $c(v) = 1$, and if $v \in B$, then $c(v) = 4$.

(C_2) If $v \in I_3^B$, then $c(v) = 2$, and if $v \in I_3^A$, then $c(v) = 3$.

(C_3) For each successive $v \in I_2$ we assign a color following the algorithm: if $c(v)$ is not determined, then let $\{u\} = N_{G'}(v) \cap A$. If there is $x \in N_{G'}(u)$ such that $c(x) = 2$, then let $c(v) = 3$. Otherwise, for each vertex $x \in N_{G'}(u)$ either $c(x) \in \{3,4\}$ or $c(x)$ is not determined, and then let $c(v) = 2$.

(C_4) For each successive $\{v,w\} \in E(G'[I_3^2])$ we assign colors to both v and w following the algorithm: if $c(v)$ and $c(w)$ are not determined, then let $\{u\} = N_{G'}(v) \cap A$. If there is $x \in N_{G'}(u)$ such that $c(x) = 2$, then let $c(v) = 3$ and $c(w) = 2$. Otherwise, for each vertex $x \in N_{G'}(u)$ either $c(x) \in \{3,4\}$ or $c(x)$ is not determined, and then let $c(v) = 2$ and $c(w) = 3$.

(C_5) For each $v \in V(G')$ such that $\deg_{G'}(v) < \deg_G(v)$, there is a tree T_v such that $V(T_v) \subset V(G) \setminus V(G')$ and let $\{w\} = V(T_v) \cap N_G(v)$. Let $d\colon V(T_v) \to$

$\{a, b\}$ be a 2-coloring of T_v such that $d(w) = a$. Suppose $c(v) \leq 2$. For each $u \in V(T_v)$, if $d(u) = a$, then let $c(u) = 4$, and if $d(u) = b$, then let $c(u) = 1$. Suppose $c(v) \geq 3$. For each $u \in V(T_v)$, if $d(u) = a$, then let $c(u) = 1$, and if $d(u) = b$, then let $c(u) = 4$.

In step (C_1) we colored $V(H) = A \cup B$ with colors 1 and 4, in steps (C_2)–(C_4) we colored vertices from $I_2 \cup I_3$ with colors 2 or 3, and in step (C_5) we colored vertices from $V(G) \setminus V(G')$ with colors 1 or 4. Since vertices colored with an arbitrary color form an independent set, c is a vertex 4-coloring of G.

Let $v \in V(G)$ and let $\deg_G(v) = 1$. Then, $v \in V(G) \setminus V(G')$ and, by (C_5), $c(v) \in \{1, 4\}$. Thus, we get the property (i). Let $\deg_G(v) \geq 2$. If $v \in V(G) \setminus V(G')$, then, by (C_5), the property (ii) holds. Let $v \in V(G')$. Since A and B are disjoint independent dominating sets of G', the property (ii) holds.

Since c is a proper coloring of G, there is $a_{c(v)}(v) = 0$ for each $v \in V(G)$.

Let $v \in V(G) \setminus V(G')$. By step (C_5), $c(v) \in \{1, 4\}$. If $c(v) = 1$, then $a_2(v) = 0$, $a_3(v) \leq 1$ and $a_4(v) \leq 3$. If $c(v) = 4$, then $a_3(v) = 0$, $a_2(v) \leq 1$ and $a_1(v) \leq 3$.

Let $v \in V(G') \setminus V(H)$. If $v \in I_3^A$, then $c(v) = 3, a_1(v) = 2, a_2(v) = 0, a_4(v) = 1$. If $v \in I_3^B$, then $c(v) = 2, a_1(v) = 1, a_3(v) = 0, a_4(v) = 2$. If $v \in I_2$, then $c(v) \in \{2, 3\}$. If $\deg_{G'}(v) = \deg_G(v)$, then $a_1(v) = a_4(v) = 1$, and $a_2(v) = a_3(v) = 0$. If $\deg_{G'}(v) < \deg_G(v)$, then if $c(v) = 2$, then $a_1(v) = 1, a_2(v) = a_3(v) = 0, a_4(v) = 2$, and if $c(v) = 3$, then $a_1(v) = 2, a_2(v) = a_3(v) = 0, a_4(v) = 1$. If $v \in I_3^2$, then $c(v) \in \{2, 3\}$. If $c(v) = 2$, then $a_1(v) = a_3(v) = a_4(v) = 1$. If $c(v) = 3$, then $a_1(v) = a_2(v) = a_4(v) = 1$.

Let $v \in A \cup B$. Since A and B are disjoint dominating sets of G' and $H \in \hat{\mathcal{B}}(G')$, it suffices to prove that if $c(v) = 1$, then $a_2(v) \leq 1$, and if $c(v) = 4$, then $a_3(v) \leq 1$.

Suppose to the contrary that $c(v) = 1$ and $a_2(v) = 2$ for some $v \in A$. The case $c(v) = 4$ and $a_3(v) = 2$, for some $v \in B$, is analogous. Let $x, y \in N_{G'}(v)$ such that $c(x) = c(y) = 2$. Since B is a dominating set of G', there is $w \in N_{G'}(v) \cap B$ with $c(w) = 4$. By the definition of coloring c, we have $v, x, y, w \in V(G')$ and $v, w \in V(H)$.

Since $c(x) = c(y) = 2$, we have $a_1(x) = a_1(y) = 1$, $a_3(x) \leq 1, a_3(y) \leq 1$, $1 \leq a_4(x) \leq 2$ and $1 \leq a_4(y) \leq 2$. Let us consider the following cases:

- $x \notin N_{G'}(w)$ and $y \notin N_{G'}(w)$. If edge $\{v, w\}$ is isolated in H, then let $W = V(H) \cup \{x, y\}$. Otherwise, let $W = V(H) \cup \{x, y\} \setminus \{v\}$.
- $x \in N_{G'}(w)$ or $y \in N_{G'}(w)$. Let $W = V(H) \cup \{x, y\} \setminus \{v\}$.

In both cases, the graph $G'[W] \in \mathcal{B}(G')$ and $|V(G'[W])| > |V(H)|$, a contradiction. Thus, the coloring c satisfies the property (iii). ■

Proposition 20. [14] *For any graph G, $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$.*

We prove that an interval incidence 6-coloring always exists for any subcubic graph G with $\Delta(G) = 3$.

Theorem 21. *Let G be a subcubic graph. Then, $\chi_{ii}(G) \leq 2\Delta(G)$.*

Proof. If G is a subcubic bipartite graph, then by Proposition 20 we have $\chi_{ii}(G) \leq 2\Delta(G)$. If $\Delta(G) = 2$, then one can easily construct an interval incidence 4-coloring. Thus, $\chi_{ii}(G) \leq 2\Delta(G)$. Let G be a subcubic non-bipartite graph with $\Delta(G) = 3$. By Lemma 19, there is a vertex coloring $c\colon V(G) \to \{1,2,3,4\}$ satisfying the properties (i), (ii), (iii) from Lemma 19.

We construct an incidence coloring $f\colon I(G) \to \{1,2,3,4,5,6\}$ in three steps.

In the first step, using the coloring c, we define the interval $A_f(v)$ for each vertex $v \in V(G)$, as follows. If $\deg_G(v) = 2$ and $c(v) \in \{2,3\}$, then let $A_f(v) = \{3,4\}$. If $c(v) = 4$ and $\deg_G(v) = 1$, then $A_f(v) = \{6\}$. If $c(v) = 4$ and $\deg_G(v) = 2$, then $A_f(v) = \{5,6\}$. In the other cases, let $A_f(v) = \{c(v), \ldots, c(v) + \deg_G(v) - 1\}$. Thus, by Lemma 19 (i)–(iii) we get

(a_1) if $\deg_G(v) = 1$, then $c(v) \in \{1,4\}$ and $A_f(v) = \{c(v)\}$,

(a_2) if $\deg_G(v) = 2$, then if $c(v) \in \{1,3\}$, then $A_f(v) = \{c(v), c(v)+1\}$ and if $c(v) \in \{2,4\}$, then $A_f(v) = \{c(v)+1, c(v)+2\}$,

(a_3) if $\deg_G(v) = 3$, then $A_f(v) = \{c(v), c(v)+1, c(v)+2\}$.

In the second step, for each $v \in V(G)$, we construct a sequence $L_f(v)$ (i.e., a linear ordered set) from elements of $N_G(v)$, as follows (see Figure 2).

(l_1) Suppose $\deg_G(v) = 1$. If $N_G(v) = \{x\}$, then let $L_f(v) = (x)$.

(l_2) Suppose $\deg_G(v) = 2$. Let $N_G(v) = \{x, y\}$, where $c(x) \leq c(y)$. Then,
- if $c(v) \in \{1,4\}$, then let $L_f(v) = (x, y)$,
- if $c(v) \in \{2,3\}$, then let $L_f(v) = (y, x)$.

(l_3) Suppose $\deg_G(v) = 3$. Let $N_G(v) = \{x, y, z\}$, where $c(x) \leq c(y) \leq c(z)$. Then,
- if $c(v) \in \{1,4\}$, then let $L_f(v) = (x, y, z)$,
- if $c(v) = 2$, then let $L_f(v) = (y, z, x)$,
- if $c(v) = 3$, then let $L_f(v) = (z, x, y)$.

By v_i we mean the i-th element of the sequence $L_f(v)$, i.e., $L_f(v) = (v_1, \ldots)$.

In the final step, for each vertex v, we define the incidence coloring f as follows: $f(v, \{v, v_i\}) = \min A_f(v) + i - 1$, for $i \in \{1, \ldots, \deg_G(v)\}$.

In Figure 2 the *white* vertex is the vertex v, and the list above is $L_f(v)$. By Lemma 19 (i)–(iii), the set of all possible values of c of a vertex is as given in the curly brackets below the vertex. The colors of incidences at the white vertex (i.e., v) are given at the edges adjacent to v.

Obviously, all the incidences at vertex v are colored with different colors from $A_f(v)$. Observe that the set of colors $A_f(v)$ is an interval of integers.

We prove that the coloring f is an incidence coloring. It is enough to prove that for each vertex $v \in V(G)$ and each vertex $w \in N_G(v)$ we have $f(v, \{v, w\}) \notin A_f(w)$, or, equivalently, $f(v, \{v, w\}) < \min A_f(w)$ or $f(v, \{v, w\}) > \max A_f(w)$.

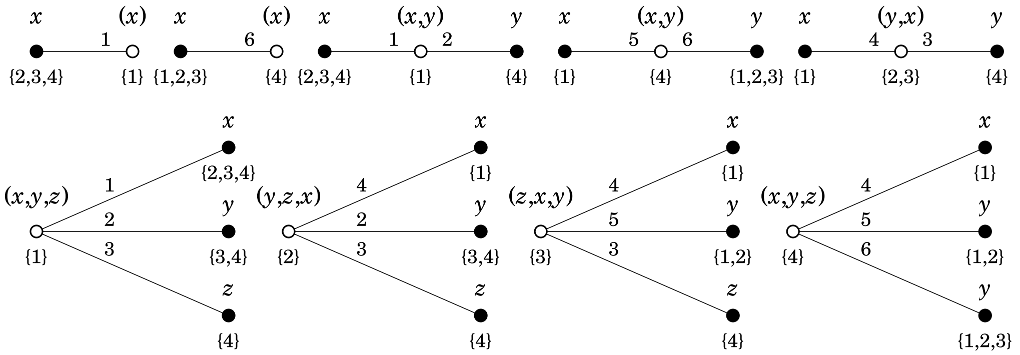

Figure 2. Interval coloring of incidences at the white vertex v, according to its degree and the values of c at the neighbors x, y, z of v. The set of possible values of c of a vertex is given in the curly brackets below the vertex. The list $L_f(v)$ is given above the white vertex v.

Suppose that $c(v) = 1$. Then, $A_f(v) \subset \{1,2,3\}$ and $\min A_f(v) = 1$. By the construction of $L_f(v)$ we have: if $\deg_G(v) \geq 1$, then $c(v_1) \in \{2,3,4\}$, and if $\deg_G(v) = 2$, then $c(v_2) = 4$, and if $\deg_G(v) = 3$, then $c(v_2) \in \{3,4\}$ and $c(v_3) = 4$ (see Figure 2). Hence, for each $i \in \{1, \ldots, \deg_G(v)\}$ we have $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 < i + 1 \leq \min A_f(v_i)$.

Suppose that $c(v) = 2$. Then, $A_f(v) \subset \{2,3,4\}$. Let $\deg_G(v) = 3$. Hence, $\min A_f(v) = 2$, and $c(v_1) \in \{3,4\}$ and $c(v_2) = 4 \wedge c(v_3) = 1$. Thus, $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 = i + 1 < i + 2 \leq \min A_f(v_i)$, for $i \in \{1,2\}$, and $f(v, \{v, v_3\}) = \min A_f(v) + 2 = 4 > 3 \geq \max A_f(v_3)$. Let $\deg_G(v) = 2$. Hence, $\min A_f(v) = 3$, and $c(v_1) = 4$ and $c(v_2) = 1$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$ and $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$.

Suppose that $c(v) = 3$. Then, $A_f(v) \subset \{3,4,5\}$ and $\min A_f(v) = 3$. Let $\deg_G(v) = 3$. Hence, $c(v_1) = 4$ and $c(v_2) = 1$ and $c(v_3) \in \{1,2\}$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$, and $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 > i + 1 \geq \max A_f(v_i)$, for $i \in \{2,3\}$. Let $\deg_G(v) = 2$. Hence, $c(v_1) = 4$ and $c(v_2) = 1$. Thus, $f(v, \{v, v_1\}) = 3 < 4 \leq \min A_f(v_1)$ and $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$.

Suppose that $c(v) = 4$. Then, $A_f(v) \subset \{4,5,6\}$. Let $\deg_G(v) = 3$. Hence, $c(v_1) = 1$ and $c(v_2) \in \{1,2\}$ and $c(v_3) \in \{1,2,3\}$ and $c(v_2) \leq c(v_3)$. Thus, $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 \geq i + 3 > i + 2 \geq \max A_f(v_i)$, for each $i \in \{1,2,3\}$. Let $\deg_G(v) = 2$. Hence, $c(v_1) = 1$ and $c(v_2) \in \{1,2,3\}$, and $A_f(v) = \{5,6\}$. Thus, $f(v, \{v, v_1\}) = 5 > \max A_f(v_1)$ and $f(v, \{v, v_2\}) = 6 > \max A_f(v_2)$. Let $\deg_G(v) = 1$. Hence, $c(v_1) \in \{1,2,3\}$. Thus, $f(v, \{v, v_1\}) = 6 > 5 \geq \max A_f(v_1)$.

In all the cases we proved that $f(v, \{v, v_i\}) \notin A_f(v_i)$ for each $v_i \in N_G(v)$. Thus, f is an interval incidence 6-coloring of G. ■

4. Summary

In this paper we proved that for any subcubic graph G, $\chi_{ii}(G) \leq 2\Delta(G)$. In [14] we proved that the upper bound of $2\Delta(G)$ on $\chi_{ii}(G)$ holds for each complete k-partite graph G and this bound is valid for other classes of graphs. Thus, we state the following

Conjecture 22 [Interval Incidence Coloring Conjecture (IICC)]**.** *For any graph G, $\chi_{ii}(G) \leq 2\Delta(G)$.*

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