Discussiones Mathematicae
Graph Theory 37 (2017) 301–313
doi:10.7151/dmgt.1953

RAINBOW CONNECTIVITY OF CACTI AND OF SOME INFINITE DIGRAPHS

Jesús Alva-Samos

and

Juan José Montellano-Ballesteros[1]

Instituto de Matemáticas, Universidad Nacional Autónoma de México
Área de la Investigación Científica, Ciudad Universitaria
Coyoacán 04510 México, CDMX, Mexico

e-mail: darkclaw35@gmail.com
juancho@im.unam.mx

Abstract

An arc-coloured digraph $D = (V, A)$ is said to be *rainbow connected* if for every pair $\{u, v\} \subseteq V$ there is a directed uv-path all whose arcs have different colours and a directed vu-path all whose arcs have different colours. The minimum number of colours required to make the digraph D rainbow connected is called the *rainbow connection number* of D, denoted $\overrightarrow{rc}(D)$. A cactus is a digraph where each arc belongs to exactly one directed cycle. In this paper we give sharp upper and lower bounds for the rainbow connection number of a cactus and characterize those cacti whose rainbow connection number is equal to any of those bounds. Also, we calculate the rainbow connection numbers of some infinite digraphs and graphs, and present, for each $n \geq 6$, a tournament of order n and rainbow connection number equal to 2.

Keywords: rainbow connectivity, cactus, arc-colouring.

2010 Mathematics Subject Classification: 05C20, 05C20, 05C40.

1. Introduction

Given a graph $G = (V, E)$, an edge-colouring of G is called *rainbow connected* (respectively *strongly rainbow connected*) if for every pair $\{u, v\} \subseteq V$ there is a uv-path (respectively a uv-geodesic) all whose edges receive different colours, and

[1]Research partially supported by PAPIIT-México project IN104915.

the *rainbow connection number* (respectively *strongly rainbow connection number*) of G is the minimum number k such that there is a rainbow connected (respectively strong rainbow connected) edge-colouring of G with k colours. These concepts were introduced by Chartrand *et al.* in [4]. A natural extension of this concept is that of the *rainbow connection* and *strong rainbow connection* in oriented graphs, which was introduced by Dorbec *et al.* in [5], and some work around these concepts have been done since then (see for instance [1, 6]).

Let $D = (V(D), A(D))$ be a strong connected digraph and $\rho : A(D) \to \{1, \dots, k\}$ be an arc-colouring of D. Given a pair $\{x, y\} \subseteq V(D)$, a directed xy-path T in D will be called *rainbow* if no two arcs of T receive the same colour. The arc colouring ρ will be called *rainbow connected* if for every pair of vertices $\{x, y\} \subseteq V(D)$ there is a rainbow xy-path and a rainbow yx-path. The *rainbow connection number* of D, denoted as $\overrightarrow{rc}(D)$, is the minimum number k such that there is a rainbow connected arc-colouring of D with k colours. Given a pair of vertices $\{x, y\} \subseteq V(D)$, an xy-path T will be called an *xy-geodesic* if the length of T is the distance from x to y in D. An arc-colouring of D will be called *strongly rainbow connected* if for every pair of distinct vertices $\{x, y\} \subseteq V(D)$ there is a rainbow xy-geodesic and a rainbow yx-geodesic. The *strong rainbow connection number of* D, denoted as $\overrightarrow{src}(D)$, is the minimum number k such that there is a strong rainbow connected arc-colouring of D with k colours.

In this paper we give sharp upper and lower bounds for the rainbow connection number of a cactus and characterize the cacti digraphs whose rainbow connection numbers are equal to any of those bounds. Also, we calculate the rainbow connection numbers of some infinite digraphs and graphs, and present, for each $n \geq 6$, a tournament of order n and rainbow connection number equal to 2. It will turn out that for each of the digraphs considered here, its rainbow connection number is equal to its strong rainbow connection number. For general concepts we refer the reader to [2, 3].

2. Notation and Some Previous Results

Let $D = (V(D), A(D)$ be a digraph. D will be called *simple* if it has no multi-arcs nor loops, *asymmetric* if D has no symmetric arcs, and *strong* if for every pair of vertices $\{x, y\}$ of D there is a directed xy-path and a directed yx-path in D. Given $x \in V(D)$, $N^+(x)$ and $N^-(x)$ will denote the out-neighborhood of x and the in-neighborhood of x in D, respectively. Similarily, $d^+(x)$ and $d^-(x)$ will denote the out-degree of x and the in-degree of x in D, respectively. Given $S \subseteq V(D)$, $D[S]$ will denote the subdigraph of D induced by S. Given $a = (x, y) \in A(D)$, the vertex x will be called the *tail* of a and y will be called the *head* of a. Also, a will be an *out-arc* of x and an *in-arc* of y. D/a will be the digraph obtained from D by the contraction of the arc a.

Given an arc-colouring $\rho : A(D) \to \{1, \ldots, k\}$, a subdigraph H of D will be called *rainbow* if no two arcs of H receive the same colour. For each $i \in \{1, \ldots, k\}$, the set of arcs $\rho^{-1}(i)$ will be called the *chromatic class* of the colour i, and if $|\rho^{-1}(i)| = 1$, $\rho^{-1}(i)$ will be also called a *singular chromatic class.* If $|\rho[A(D)]| = k$ (that is, if ρ use k colours) ρ will be called a *k-colouring.*

Let Q be a strong asymmetric digraph. We say that Q is a *cactus* if each arc belongs to exactly one directed cycle.

Recall that a *block* is a maximal subdigraph without a cut-vertex. The *block graph* of D, denoted $B(D)$, is the graph with $V(B(D)) = \{B_i \mid B_i \text{ is block of } D\}$ and $B_iB_j \in E(B(D))$ if B_i and B_j shares a vertex in D.

From the definition of a cactus, it is not hard to obtain the following charaterization.

Lemma 1. *Let Q be a digraph with n vertices and m arcs. Then the following statements are equivalent.*

(I) *Q is a cactus.*

(II) *Q is a strong digraph in which every block is a directed cycle.*

(III) *Let q be the number of blocks in Q. Then Q has a decomposition into directed cycles $C_{n_1}, \ldots, C_{n_q}$ such that, for each $k = 2, \ldots, q$, we have*

$$\left| V(C_{n_k}) \cap \left(\bigcup_{i=1}^{k-1} V(C_{n_i}) \right) \right| = 1$$

and $q = m - n + 1$.

(IV) *There is exactly one directed path between each pair of vertices of Q.*

When a cactus on n vertices has decomposition into q cycles, we will say that such a digraph is an *(n, q)-cactus.* From now on, we always consider a cactus along with its cycle decomposition given in (III) of Lemma 1, and it is clear that such a decomposition is unique (up to the order of the cycles). Observe that since the directed path between any pair of vertices of Q is unique, in any rainbow connected colouring of Q such a path most be rainbow and also this implies that $\overrightarrow{rc}(Q) = \overrightarrow{src}(Q)$. Given $u, v \in V(Q)$, we will denote by uQv the unique directed uv-path in Q, and we will denote by K_Q the set formed by all the cut-vertices of Q. Also, for each $a \in A(Q)$ we denote by $C(a)$ the directed cycle containing a. By Lemma 1 we see that if Q is a cactus, then for every $u \in V(Q)$, $d^+(u) = d^-(u)$. Moreover, $u \in K_Q$ if and only if $d^+(u) = d^-(u) > 1$.

3. Cacti Digraphs

The aim of this section is to show the following.

Theorem 5. *Let Q be an (n, q)-cactus with $q \geq 2$. Then*

$$n - q + 1 \leq \overrightarrow{rc}(Q) \leq n - 1.$$

Theorem 6. *Let Q be an (n,q)-cactus. $\overrightarrow{rc}(Q) = n - q + 1$ if and only if K_Q is independent.*

Theorem 8. *Let Q be an (n,q)-cactus with $q \geq 2$. $\overrightarrow{rc}(Q) = n - 1$ if and only if $B(Q) \cong P_q$ and $Q[K_Q] \cong \overrightarrow{P_{q-1}}$.*

Theorem 9. *Let $q \geq 2$. For every $n \geq 2q+1$ there is an (n,q)-cactus with $\overrightarrow{rc}(Q) = n - q + k$ for $k \in \{1, 2, \ldots, q-1\}$.*

To show these theorems, we need some preparatory results.

Lemma 2. *Let ρ be a rainbow connected colouring on a cactus Q and let $(u,v) \in A(Q)$. If $\{u,v\} \cap K_Q = \emptyset$ then the arc (u,v) is a singular chromatic class. Moreover, if (u,v) is contained in a cycle of length at least four, then $\overrightarrow{rc}(Q) = \overrightarrow{rc}(Q/(u,v)) + 1$.*

Proof. Let $a = (u,v)$ with $\{u,v\} \cap K_Q = \emptyset$, let $b = (x,y)$ be any other arc of Q and let $C(a)$ and $C(b)$ be the cycles that contain a and b, respectively. If $C(a) = C(b)$, then $\rho(a) \neq \rho(b)$ by (IV) in Lemma 1. Otherwise, each $C(a)C(b)$-path contains a cut-vertex $w \in V(C(b))$. Without loss of generality $w \neq x$, then uQy contains both arcs, therefore $\rho(a) \neq \rho(b)$, and hence the arc a is a singular chromatic class.

Now, let h be the new vertex in Q/a. We define ρ_a to be the colouring induced by ρ on Q/a such that

$$\rho_a((x,y)) = \begin{cases} \rho((x,y)) & \text{if } x \neq h,\ y \neq h, \\ \rho((v,y)) & \text{if } x = h, \\ \rho((x,u)) & \text{if } y = h. \end{cases}$$

Clearly, ρ_a uses one colour less than ρ, and each rainbow path in Q corresponds to a rainbow path in Q/a. Thus $\overrightarrow{rc}(Q) \geq \overrightarrow{rc}(Q/(u,v)) + 1$. Analogously, we obtain that $\overrightarrow{rc}(Q) \leq \overrightarrow{rc}(Q/(u,v)) + 1$, and the result follows. ■

By the previous lemma we can restrict our study to cactus where each end-block is a 3-cycle, and for each pair of adjacent vertices contained in any other block, at least one is a cut-vertex.

Proposition 3. *Let ρ be an arc-colouring of Q. ρ is rainbow connected if and only if for any pair of distinct arcs $a = (u,v)$ and $b = (x,y)$ of Q such that $\rho(a) = \rho(b)$, it holds that*

(i) *$C(a) \neq C(b)$;*

(ii) *every $C(a)C(b)$-path contains the set of vertices $\{u,x\}$, or every $C(a)C(b)$-path contains the set of vertices $\{v,y\}$.*

Proof. Let ρ be a rainbow connected colouring on Q and let $a = (u, v)$ and $b = (x, y)$, with $\rho(a) = \rho(b)$. Notice that $C(a) \neq C(b)$ since each cycle has no two arcs sharing the same colour. Clearly, each $C(a)C(b)$-path contains two vertices (not necessarily different) $w_a \in K_Q \cap V(C(a))$ and $w_b \in K_Q \cap V(C(b))$. We claim that w_a and w_b are the tails (u and x) of a and b, or the heads (v and y) of a and b. Otherwise, without loss of generality, let $w_a \neq u$ and $w_b \neq y$, thus uQy contains both arcs a and b, contradicting that ρ is a rainbow connected colouring.

On the other hand, assume that if $\rho(a) = \rho(b)$, then $C(a) \neq C(b)$ and every $C(a)C(b)$-path contains the tails of a and b or every $C(a)C(b)$-path contains the heads of a and b. Suppose that ρ is not a rainbow connected colouring. Then there exists a directed path P containing two arcs (u, v) and (x, y) of the same colour. Thus $C((u, v)) \neq C((x, y))$ and vPx is a $C((u, v))C((x, y))$-path that contains no both tails nor both heads of (u, v) and (x, y), which is a contradiction. ∎

Let C_{n_i} be a block of an (n, q)-cactus Q. For each $u \in V(C_{n_i})$ we define the *i-branching* of u, denoted by $B_i(u)$, as the maximal cactus Q' contained in Q where $V(C_{n_i}) \subseteq V(Q')$ and u is not a cut-vertex of Q'. Notice that for each $u \in V(Q)$, the set $\{B_j(u) \,|\, u \in V(C_{n_j}) \text{ with } 1 \leq j \leq q\}$ is a decompositon of Q into the branchings of u.

Lemma 4. *Let C_{n_i} be an end-block of a cactus Q and let u be the only cut-vertex of Q contained in C_{n_i}. If $Q' = Q - (C_{n_i} - u)$, then*

$$\overrightarrow{rc}(Q') + n_i - 2 \leq \overrightarrow{rc}(Q) \leq \overrightarrow{rc}(Q') + n_i - 1.$$

Moreover, if $N(u) \cap K_Q = \emptyset$ or there is another end-block C_{n_j} contaning u, then $\overrightarrow{rc}(Q) = \overrightarrow{rc}(Q') + n_i - 2$.

Proof. The first inequality follows from Lemma 2, since each arc of $C_{n_i} - u$ has a unique colour. For the second inequality, since $u \in K_Q$, then there is C_{n_j} containing u where $j \neq i$. Denote by v and v' the in-neighbors of u in C_{n_i} and C_{n_j}, respectively, and let w and w' be the corresponding out-neighbors of u. By Lemma 2 we can assume that $C_{n_i} = uwvu$. Let ρ_0 be a rainbow connected $\overrightarrow{rc}(Q')$-colouring on Q' where $\rho_0(v', u) = c$ and define a colouring ρ on Q such that

$$\rho(a) = \begin{cases} r & \text{if } a = (w, v), \\ r + 1 & \text{if } a = (u, w) \text{ or } a \in \rho_0^{-1}(c) \setminus A(B_j(v')), \\ c & \text{if } a = (v, u), \\ \rho_0(a) & \text{otherwise,} \end{cases}$$

where r and $r + 1$ are two new colours. We will see that ρ is a rainbow connected colouring of Q. Let $\{a, b\} \subseteq A(Q)$ with $\rho(a) = \rho(b)$. If $\{a, b\} \subseteq A(Q')$, then the condition of Proposition 3 holds, so we can assume that $a \in A(C_{n_i})$ and $b \in A(Q')$. Let $\rho(a) = r + 1$. Then $a = (u, w)$ and $b \in A(B_k(v'))$ with $k \neq j$.

Since $\rho_0(b) = \rho_0((v', u))$ we have that each $C(b)C_{n_j}$-path contains the tails of b and (v', u), hence every $C(b)C_{n_i}$-path contains the tails of b and (u, w). Now, let $\rho(a) = \rho_0((v', u))$. Then $a = (v, u)$ and $b \in A(B_j(v'))$. If $b = (v', u)$, then clearly each $C_{n_i}C_{n_j}$-path contains u. If $b \neq (v', u)$, then each $C(b)C_{n_j}$-path contains the heads of b and (v', u) since $\rho_0(b) = \rho_0((v', u))$, therefore every $C(b)C_{n_i}$-path contains the heads of b and (v, u). Hence in any case the condition of Proposition 3 holds, thus ρ is a rainbow connected colouring on Q using $\overrightarrow{rc}(Q')+n_i-1$ colours, and the second inequality follows.

On the other hand, if C_{n_j} is another end-block containing u, let ρ' be a rainbow connected $\overrightarrow{rc}(Q')$-colouring on Q' and extend this colouring on Q such that the in-arcs and out-arcs of u in C_{n_i} receive the same colours as the in-arcs and out-arcs of u in C_{n_j}, respectively, and assign new colours to each arc in C_{n_i} not incident to u. Clearly such a colouring is a rainbow connected colouring on Q using $\overrightarrow{rc}(Q') + n_i - 2$ colours. Analogously if there is no other end-block containing u and $N(u) \cap K_Q = \emptyset$, then let ρ' be a rainbow connected $\overrightarrow{rc}(Q')$-colouring on Q', and let (u, v) and (w, u) be the arcs of Q' incident on u. Since u is not a cut-vertex in Q' and $N(u) \cap K_Q = \emptyset$, by Lemma 2, no other arc of Q' share the same colour with (u, v) or (w, u). Now, let (y, u) and (u, x) be the arcs of C_{n_i} incident on u, and define an extension ρ of ρ' on Q such that $\rho((y, u)) = \rho'((w, u))$, $\rho((u, x)) = \rho'((u, v))$, and assign $n_i - 2$ new colours to the arcs in $C_{n_i} - u$. Clearly ρ is a rainbow connected $(\overrightarrow{rc}(Q') + n_i - 2)$-colouring on Q. Since $\overrightarrow{rc}(Q) \geq \overrightarrow{rc}(Q') + n_i - 2$ in both cases, the equality holds. ∎

Now, we can prove a pair of main results of this section.

Theorem 5. *Let Q be an (n, q)-cactus with $q \geq 2$. Then*

$$n - q + 1 \leq \overrightarrow{rc}(Q) \leq n - 1.$$

Proof. Let Q be an (n, q)-cactus, $\{C_{n_1}, \ldots, C_{n_q}\}$ be a cycle decomposition of Q (see Lemma 1) and suppose that for each j, with $1 \leq j \leq q - 1$, C_{n_j} is an end-block of $\bigcup_{i=j}^{q} C_{n_i}$. By Lemma 4 it follows that

$$\overrightarrow{rc}(Q) \geq \overrightarrow{rc}(C_{n_q}) + \sum_{i=1}^{q-1}(n_i - 2) = n_q + \sum_{i=1}^{q-1}(n_i - 1) - (q - 1) = n - q + 1.$$

On the other hand, again by Lemma 4,

$$\overrightarrow{rc}(Q) \leq \overrightarrow{rc}(C_{n_q} \cup C_{n_{q-1}}) + \sum_{i=1}^{q-2}(n_i - 1) = n_q + n_{q-1} - 2 + \sum_{i=1}^{q-2}(n_i - 1) = n - 1$$

and the result follows. ∎

Theorem 6. *Let Q be an (n, q)-cactus. $\overrightarrow{rc}(Q) = n - q + 1$ if and only if K_Q is independent.*

Proof. Let K_Q be an independent set. We will proved the sufficiency by induction on q. If $q = 1$ the statement holds trivially. Then assume that the result is valid for any cactus with $q-1$ cycles. Now, let $\{C_{n_1}, \ldots, C_{n_q}\}$ be a cycle decomposition of Q (as in Lemma 1) where C_{n_q} is an end-block. Also, let $Q' = \bigcup_{i=1}^{q-1} C_{n_i}$. Then by induction hypothesis we have that $\overrightarrow{rc}(Q') = (n - n_q + 1) - (q - 1) + 1$. Now by Lemma 4 we obtain that $\overrightarrow{rc}(Q) = n - n_q - q + 3 + n_q - 2 = n - q + 1$.

On the other hand, if K_Q is not an independent set, then let $(u, v) \in A(Q)$ such that $u, v \in K_Q$. Notice that there are at least 3 cycles in Q: the first one containing (u, v), the second one containing u but not v, and the last one containing v but not u. Without loss of generality, let C_{n_1}, C_{n_2} and C_{n_3} be such cycles, respectively, and let $Q' = C_{n_1} \cup C_{n_2} \cup C_{n_3}$ where Q' has $n' = n_1 + n_2 + n_3 - 2$ vertices and $q' = 3$ cycles (see Figure 1). Denote by x_i^- and x_i^+ the predecessor and the successor of vertex $x \in \{u, v\}$ in C_{n_i}. Since $v_3^+ Q' u_2^-$ is a hamiltonian path and by Theorem 5 we have $rc(Q') = n' - 1$. Hence, by Lemma 4, we can see that $rc(Q) \geq rc(Q') + \sum_{i=4}^{q} (n_i - 2) = n' - 1 + \sum_{i=4}^{q} (n_i - 1) - (q - 3) = n - q + 2$. ■

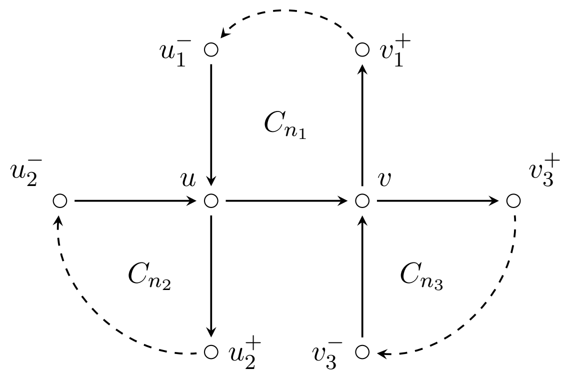

Figure 1. The cactus Q' in the proof of Theorem 6.

Lemma 7. *Let Q be an (n, q)-cactus and let K_Q not be an independent set. If K_Q lies on a cycle, then $\overrightarrow{rc}(Q) = n - q + 2$.*

Proof. By the previous theorem, it suffices to find a rainbow $(n-q+2)$-colouring on Q. Let $V(Q) = \{u_1, \ldots, u_n\}$ and assume, without loss of generality, that K_Q lies on C_{n_1}. Also, consider the set $U = \{(u_i, u_j) \in A(Q) \mid u_i \notin V(C_{n_1}), u_j \in V(C_{n_1})\}$. Now, define the $(n - q + 2)$-colouring ρ such that

$$\rho((u_i, u_j)) = \begin{cases} i & \text{if } (u_i, u_j) \notin U, \\ 0 & \text{if } (u_i, u_j) \in U. \end{cases}$$

We will see that ρ is a rainbow connected colouring. Let $\{a, b\} \subseteq A(Q)$ with the same colour. If $\rho(a) = \rho(b) \neq 0$, then a and b have the same tail. Otherwise,

$\{a, b\} \subseteq U$, then $C(a) \neq C(b)$ and the heads of both arcs lies on C_{n_1}. Clearly, each $C(a)C(b)$-path contains the heads of both arcs. Thus, by Proposition 3, ρ is a rainbow connected colouring and since $|U| = q - 1$, then ρ is a rainbow connected $(n - q + 2)$-colouring. ■

Now we characterize the family of cacti whose rainbow connection numbers are equal to the upper bound.

Theorem 8. *Let Q be an (n, q)-cactus with $q \geq 2$. $\overrightarrow{rc}(Q) = n - 1$ if and only if $B(Q) \cong P_q$ and $Q[K_Q] \cong \overrightarrow{P_{q-1}}$.*

Proof. First, suppose that $B(Q) \cong P_q$ and $Q[K_Q] \cong \overrightarrow{P_{q-1}}$ with $q \geq 2$. Since $B(Q)$ is a path, each non-end-block of Q contains exactly two cut-vertices of Q, and we can order the blocks of Q from one end-block to the other end-block. We denote by u_i the cut-vertex contained in $C_{n_i} \cap C_{n_{i+1}}$ for $i = 1, \ldots, q - 1$. Also, we denote by $v_{i,j}$ and $w_{i,j}$ the in-neighbor and the out-neighbor of u_i contained in C_{n_j}, respectively. If $Q[K_Q]$ is a directed path we can assume without loss of generality that $u_{i+1} = w_{i,i+1}$ for each $i = 1, \ldots, q - 2$ (see Figure 2). Notice that $w_{q-1,q}Qv_{1,1}$ is a path of length $n - 1$, so $\overrightarrow{rc}(Q) \geq n - 1$ which by Theorem 5 implies that $\overrightarrow{rc}(Q) = n - 1$.

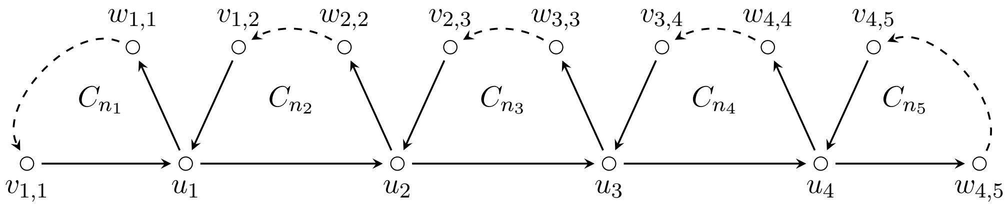

Figure 2. A cactus Q with $B(Q) \cong P_5$ and $Q[K_Q] \cong \overrightarrow{P_4}$.

For the sufficiency we proceed by induction on q. The cases $q \in \{2, 3\}$ follow from Theorem 6 and Lemma 7, respectively. Suppose that the statement holds for any cactus with q cycles and let Q be an $(n, q+1)$-cactus with $\overrightarrow{rc}(Q) = n-1$. By Lemma 2 we can assume that the end-blocks of Q have lenght 3. Let C_{n_i} be an end-block where x_i is the only cut-vertex of Q contained in such a cycle, and let $Q_i = Q - (C_{n_i} - x_i)$. Observe that $\overrightarrow{rc}(Q_i) = n - 3$ (otherwise, by Lemma 4, $\overrightarrow{rc}(Q) < n - 1$ which is a contradiction). Thus, by induction hypothesis, $B(Q_i) \cong P_q$ and $Q_i[K_{Q_i}] \cong \overrightarrow{P_{q-1}}$. Therefore, for each end-block C_{n_i} of Q, $B(Q_i)$ is a path. It follows that either Q has three end-blocks (and $B(Q) \cong K_{1,3}$), or two end-blocks, but if $B(Q) \cong K_{1,3}$ then either by Theorem 6 or by Lemma 7, $\overrightarrow{rc}(Q) < n - 1$ which is not possible. Thus Q has exactly two end-blocks, and therefore $B(Q) \cong P_{q+1}$. Moreover, if C_{n_1} and $C_{n_{q+1}}$ are the end-blocks of Q (and

x_1 and x_q are the only cut-vertices of Q contained in C_{n_1} and $C_{n_{q+1}}$, respectively), by induction hypothesis $Q_1[K_{Q_1}] \cong Q_{q+1}[K_{Q_{q+1}}] \cong \overrightarrow{P_{q-1}}$ and then it follows that $Q[K_Q] \cong \overrightarrow{P_q}$. ■

We now construct cacti digraphs on n vertices with rainbow connection number equal to any value between the lower and upper bounds.

Theorem 9. *Let $q \geq 2$. For every $n \geq 2q+1$ there is an (n,q)-cactus with $\overrightarrow{rc}(Q) = n - q + k$ for $k \in \{1, 2, \ldots, q-1\}$.*

Proof. Let $q \geq 2$ and $k \in \{1, 2, \ldots, q-1\}$. Consider a $(2q+1, q)$-cactus Q where $C_{n_i} = u_i v_i w_i u_i$ for $1 \leq i \leq q$, such that $\{u_j\} = C_{n_j} \cap (\bigcup_{r=1}^{j-1} C_{n_r})$ for $2 \leq j \leq q$; $u_t = v_{t-1}$ for $2 \leq t \leq k+1$ and $u_t = v_1$ for $k+2 \leq t \leq q$. Now let $Q' = \bigcup_{r=1}^{k+1} C_{n_r}$. Notice that $Q'[K_{Q'}] \cong \overrightarrow{P_k}$ and $B(Q') \cong P_{k+1}$, then, by Theorem 8, $\overrightarrow{rc}(Q') = 2k+2$. Also, since C_{n_t} is an end-block containing u_1 for $k+2 \leq t \leq q-1$, then, by Lemma 4, $\overrightarrow{rc}(Q) = \overrightarrow{rc}(Q') + q - (k+1)$. Therefore $\overrightarrow{rc}(Q) = 2k+2+q-k-1 = q+k+1 = (2q+1) - q + k$. Now, for any $n > 2q+1$ we subdivide $n - 2q - 1$ times the arc (v_1, w_1), so, by Lemma 2, the resulting (n,q)-cactus has rainbow connection number $n - q + k$. ■

4. TOURNAMENTS

In [5] the following two theorems were proven.

Theorem 10 (Dorbec *et al.* [5])**.** *If T is a strong tournament with $n \geq 5$ vertices, then $2 \leq \overrightarrow{rc}(T) \leq n-1$.*

Theorem 11 (Dorbec *et al.* [5])**.** *For every n and k such that $3 \leq k \leq n-1$, there exists a tournament T on n vertices such that $\overrightarrow{rc}(T) = k$.*

When $n \in \{4, 5\}$ it is easy to verified that $\overrightarrow{rc}(T) \geq 3$ for each tournament T on n vertices. Here we will show that this is not true for tournaments of order $n \geq 6$.

Theorem 12. *For every $n \geq 6$, there is a tournament T of order n with $\overrightarrow{src}(T) = \overrightarrow{rc}(T) = 2$.*

Proof. For $n = 2k+1$, with $k \geq 3$, let $S = \{1, 2, 4, \ldots, 2(k-1)\}$ and let $T = (V(T), A(T))$ be the tournament such that $V(T) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $A(T) = \{(u_i, u_j) \mid j - i \overset{n}{\equiv} s,\ s \in S\}$ (observe that T is the circulant tournament $C_{2k+1}(S)$). Now consider the partition of $A(T)$ into the sets

$$\begin{aligned} A_0 = {} & \{(u_0, u_1), (u_0, u_2), (u_1, u_{2k-1})\} \\ & \cup \big(\{(u_r, u_s) \mid r \equiv 0 \bmod 2, r \geq 2\} \setminus \{(u_2, u_{2k})\}\big) \end{aligned}$$

and $A_1 = A(T) \setminus A_0$; and let ρ be the colouring where $\rho(a) = i$ if $a \in A_i$ for $i = 0, 1$.

Observe that it suffices to check that there is a rainbow u_iu_j-path for each pair (i, j) with $0 \leq i \leq n-1$ and $j = i+3, i+5, \dots, i+2k-1, i+2k$. We will denote such a path by $P_{i,j}$. First, let $P_{i,i+3} = u_iu_{i+1}u_{i+3}$ for $i \in \{0, \dots, 2k-1\}$ and $P_{2k,2} = u_{2k}u_1u_2$. Now, let $3 \leq i \leq n-1$. Then $P_{i,i+r} = u_iu_{i+1}u_{i+r}$ for each $r = 5, \dots, 2k-1$ and $P_{i,i+2k} = u_iu_{i+2k-2}u_{i+2k}$ (observe that $i + 2k - 2 \cong i - 3$). For $i = 1$ we have $P_{1,j} = u_1u_2u_j$ with $j \in \{6, \dots, 2k-2\}$ and $P_{1,j} = u_1u_{2k-1}u_j$ if $j \in \{2k, 2k+1\}$. For $i = 2$, $P_{2,j} = u_2u_3u_j$ with $j \in \{7, \dots, 2k+1\}$ and $P_{2,2k+2} = P_{2,1} = u_2u_{2k}u_1$. Finally, for $i = 0$, $P_{0,j} = u_0u_{j-1}u_j$ with $j \geq 5$ and j odd, and $P_{0,2k} = u_0u_2u_{2k}$. In any case, $P_{i,j}$ is a rainbow path by the definition of ρ. Hence $\overrightarrow{rc}(T) = 2$.

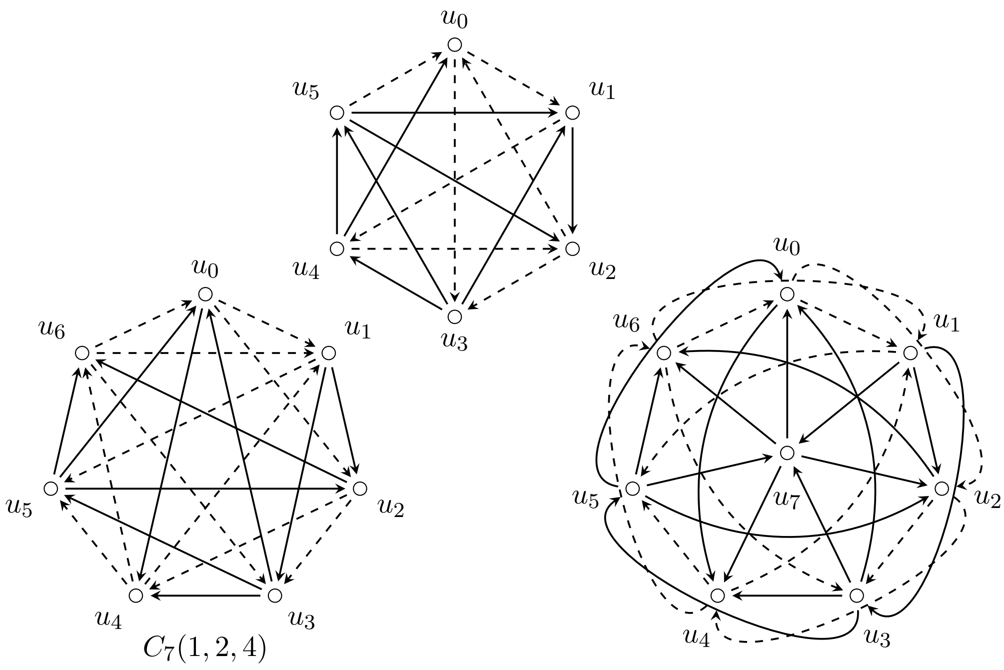

Figure 3. Tournaments on n vertices with $\overrightarrow{rc}(T_n) = 2$ with $n \in \{6, 7, 8\}$.

For $n = 6$, let T be the first tournament in Figure 3. If $n = 2k \geq 8$, we consider the tournament T obtained by adding a new vertex u_{n-1} to $C_{2k-1}(1, 2, 4, \dots, 2(k-2))$ in such a way that $(u_{n-1}, u_i) \in A(T)$ if i is even and $(u_i, u_{n-1}) \in A(T)$ if i is odd (see the third tournament in Figure 3). Also, let ρ be the rainbow colouring on $C_{2k-1}(1, 2, 4, \dots, 2(k-2))$ described above and extend it to T in such a way that $\rho(a) = 1$ for each arc a incident to u_{n-1}. Clearly, there is a rainbow u_iu_j-path if $i, j \neq n-1$, and for the vertex u_{n-1} we have to consider

$$P_{n-1,i} = \begin{cases} u_{n-i}u_i & \text{if } i \text{ is even,} \\ u_{n-1}u_{i-1}u_i & \text{if } i \text{ is odd,} \end{cases}$$

$$\text{and} \quad P_{i,n-1} = \begin{cases} u_i u_{n-i} & \text{if } i \text{ is odd,} \\ u_{n-2}u_1u_{n-1} & \text{if } i = n-2 = 2k-2, \\ u_i u_{i+1} u_{n-1} & \text{otherwise.} \end{cases}$$

By construction such paths are rainbow paths, therefore $\overrightarrow{rc}(T) = 2$.

Finally, just observe that if $\overrightarrow{rc}(T) = 2$ then $\overrightarrow{src}(T) = 2$ and the result follows. ∎

From the above the next result follows.

Corollary 13. *For every $n \geq 6$ and every k such that $2 \leq k \leq n-1$, there exists a tournament T on n vertices such that $\overrightarrow{rc}(T) = k$.*

5. Infinite Digraphs

Now, we focus our attention on infinite digraphs. For this, first we define the following. Given a pair of positive integers k and b, let $k_b(0), k_b(1), \ldots, k_b(j), \ldots$ be the b-base expansion of k, that is to say, $k = \sum_{i=0} k_b(i)b^i$ where, for every $i \geq 0$, $k_b(i)$ is an integer such that $0 \leq k_b(i) \leq b-1$.

Now we will define some infinite digraphs, all of them inspired in the definition of the Rado Graph, and each of them with $\mathbb{N} \cup \{0\}$ as its set of vertices. Let D_1 be the digraph such that $(i,j) \in A(D_1)$ if and only if $j_4(i) \in \{1,3\}$ or $i_4(j) \in \{2,3\}$; let D_2 be the symmetric digraph such that $\{(i,j),(j,i)\} \subseteq A(D_2)$ if and only if $j_2(i) = 1$ or $i_2(j) = 1$; let D_3 be the digraph such that $(i,j) \in A(D_3)$ if and only if $j_3(i) = 1$ or $i_3(j) = 2$; and let D_4 be the digraph such that $A(D_4) = A(D_3) \cup \{(i,j),(j,i) \,|\, j_3(i) = i_3(j) = 0\}$. Finally, let D_5 be the tournament such that for every pair i,j, with $i > j$, if $i_2(j) = 1$ then $(i,j) \in A(D_5)$, and if $i_2(j) = 0$ then $(j,i) \in A(D_5)$.

Theorem 14. *Let $D \in \{D_1, D_2, D_3, D_4, D_5\}$. Thus $\overrightarrow{rc}(D) = \overrightarrow{src}(D) = 2$.*

Proof. Let ρ be a colouring defined for each $(i,j) \in A(D)$ as

$$\rho(i,j) = \begin{cases} 1 & \text{if } i < j, \\ 2 & \text{otherwise.} \end{cases}$$

We will see that ρ is a rainbow connected 2-colouring of D. Let i,j be vertices of D such that $i < j$, and let b be the base expansion of the vertices in D (observe that the base in D_1 is 4, for D_2 and D_5 is 2, and for D_3 and D_4 is 3).

Consider the vertices $k = b^i + ab^j$ and $k' = ab^i + b^j$, where

$$a = \begin{cases} 0 & \text{if } D = D_5, \\ 1 & \text{if } D = D_2, \\ 2 & \text{if } D \in \{D_1, D_3, D_4\}. \end{cases}$$

For the case when $D = D_5$, $k_2(i) = 1$, $k_2(j) = 0$, $k'_2(i) = 0$ and $k'_2(j) = 1$. Since $i, j < k, k'$, $\{(j,k), (k,i), (i,k'), (k',j)\} \subseteq A(D)$ and the both paths jki and $ik'j$ are rainbow. For the case when $D = D_2$ we see that $k = k'$ and $k_2(i) = k_2(j) = 1$, and therefore $\{(i,k), (k,i), (j,k), (k,j)\} \subseteq A(D)$. For the other cases $k_b(i) = 1$, $k_b(j) = 2$, $k'_b(i) = 2$ and $k'_b(j) = 1$ which implies that $\{(i,k), (k,j), (j,k'), (k',i)\} \subseteq A(D)$. Since $i, j < k, k'$, in the last four cases the both paths ikj and $jk'i$ are rainbow, hence ρ is a rainbow connected 2-colouring on D and therefore $\overrightarrow{rc}(D) = 2$. Since $\overrightarrow{rc}(D) = 2$ it follows that $\overrightarrow{src}(D) = 2$. ■

Finally, we turn our attention to graphs. The Rado Graph R has all the non-negative integers as vertices, and two distinct vertices a and b are adjacent if and only if $a_2(b) = 1$ or $b_2(a) = 1$.

Theorem 15. *The rainbow connection number of the Rado Graph is* 2.

Proof. Let $R = (V(R), E(R))$ be the Rado Graph and let ρ be a colouring defined for each $ab \in E(R)$ as

$$\rho(ab) = \begin{cases} 1 & \text{if } a < b \text{ and } (a+1)b \in E(R), \\ 2 & \text{otherwise.} \end{cases}$$

We will see that ρ is a rainbow connected 2-colouring of R. Let a, b be non-adjacent vertices of R such that $a < b$, and consider the vertex $k = 2^a + \cdots + 2^b$. Notice that $k_2(a) = k_2(b) = 1$ which implies that $\{ak, bk\} \subseteq E(R)$. Besides, $k_2(a+1) = 1$ and therefore $(a+1)k \in E(R)$ which implies that $\rho(ak) = 1$. Finally, observe that $k_2(b+1) = 0$ and, since $b < k$, $(b+1)_2(k) = 0$, thus $(b+1)k \notin E(R)$ and therefore $\rho(bk) = 2$. Hence akb is an ab-rainbow path and the result follows. ■

References

[1] J. Alva-Samos and J.J. Montellano-Ballesteros, *Rainbow connection in some digraphs*, Graphs Combin. **32** (2016) 2199–2209.
doi:10.1007/s00373-016-1723-x

[2] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications (Springer-Verlag, 2002).
doi:10.1007/978-1-4471-3886-0

[3] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer-Verlag, 2008).

[4] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[5] P. Dorbec, I. Schiermeyer, E. Sidorowicz and E. Sopena, *Rainbow connection in oriented graphs*, Discrete Appl. Math. **179** (2014) 69–78. doi:10.1016/j.dam.2014.07.018

[6] R. Holliday, C. Magnant and P.S. Nowbandegani, *Note on rainbow connection in oriented graphs with diameter* 2, Theory Appl. Graphs **1** (2014).

Received 29 November 2015
Revised 3 August 2016
Accepted 21 October 2016