

ONE MORE TURÁN NUMBER AND RAMSEY NUMBER FOR THE LOOSE 3-UNIFORM PATH OF LENGTH THREE

JOANNA POLCYN

Adam Mickiewicz University
Poznań, Poland

e-mail: joaska@amu.edu.pl

Abstract

Let P denote a 3-uniform hypergraph consisting of 7 vertices a, b, c, d, e, f, g and 3 edges $\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, g\}$. It is known that the r -color Ramsey number for P is $R(P; r) = r + 6$ for $r \leq 9$. The proof of this result relies on a careful analysis of the Turán numbers for P . In this paper, we refine this analysis further and compute the fifth order Turán number for P , for all n . Using this number for $n = 16$, we confirm the formula $R(P; 10) = 16$.

Keywords: Ramsey numbers, Turán numbers.

2010 Mathematics Subject Classification: 05D10.

1. INTRODUCTION

For the sake of brevity, 3-uniform hypergraphs will be called here *3-graphs*. Given a family of 3-graphs \mathcal{F} , we say that a 3-graph H is \mathcal{F} -free if for all $F \in \mathcal{F}$ we have $H \not\supseteq F$.

For a family of 3-graphs \mathcal{F} and an integer $n \geq 1$, the *Turán number of the 1st order*, that is, the ordinary Turán number, is defined as

$$\text{ex}(n; \mathcal{F}) = \text{ex}^{(1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every n -vertex \mathcal{F} -free 3-graph with $\text{ex}^{(1)}(n; \mathcal{F})$ edges is called *1-extremal for \mathcal{F}* . We denote by $\text{Ex}^{(1)}(n; \mathcal{F})$ the family of all, pairwise non-isomorphic, n -vertex 3-graphs which are 1-extremal for \mathcal{F} . Further, for an integer $s \geq 1$, the *Turán number of the $(s + 1)$ -st order* is defined as

$$\begin{aligned} \text{ex}^{(s+1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and} \\ \forall H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \not\supseteq H'\}, \end{aligned}$$

if such a 3-graph H exists. Note that if $\text{ex}^{(s+1)}(n; \mathcal{F})$ exists then, by definition,

$$(1) \quad \text{ex}^{(s+1)}(n; \mathcal{F}) < \text{ex}^{(s)}(n; \mathcal{F}).$$

An n -vertex \mathcal{F} -free 3-graph H is called $(s+1)$ -*extremal* for \mathcal{F} if $|E(H)| = \text{ex}^{(s+1)}(n; \mathcal{F})$ and for every $H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}^{(s)}(n; \mathcal{F})$, $H \not\subseteq H'$; we denote by $\text{Ex}^{(s+1)}(n; \mathcal{F})$ the family of n -vertex 3-graphs which are $(s+1)$ -extremal for \mathcal{F} . In the case when $\mathcal{F} = \{F\}$, we will write F instead of $\{F\}$.

A *loose 3-uniform path of length 3* is a 3-graph P consisting of 7 vertices, say, a, b, c, d, e, f, g , and 3 edges $\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, g\}$. The *Ramsey number* $R(P; r)$ is the least integer n such that every r -coloring of the edges of the complete 3-graph K_n results in a monochromatic copy of P . Gyárfás and Raeisi [6] proved, among many other results, that $R(P; 2) = 8$. (This result was later extended to loose paths of arbitrary lengths, but still $r = 2$, in [13].) Then Jackowska [9] showed that $R(P; 3) = 9$ and $r + 6 \leq R(P; r)$ for all $r \geq 3$. In turn, in [10, 11] and [15], the Turán numbers of the first four orders, $\text{ex}^{(i)}(n; P)$, $i = 1, 2, 3, 4$, have been determined for all feasible values of n . Using these numbers, in [11] and [15], we were able to compute the Ramsey numbers $R(P; r)$ for $4 \leq r \leq 9$.

Theorem 1 [6, 9, 11, 15]. *For all $r \leq 9$, $R(P; r) = r + 6$.*

In this paper we determine, for all $n \geq 7$, the Turán numbers for P of the fifth order, $\text{ex}^{(5)}(n; P)$. This allows us to compute one more Ramsey number.

Theorem 2. $R(P; 10) = 16$.

It seems that in order to make a further progress in computing the Ramsey numbers $R(P; r)$, $r \geq 11$, one would need to determine still higher order Turán numbers $\text{ex}^{(s)}(n; P)$, at least for some small values of n .

Throughout, we denote by S_n the 3-graph on n vertices and with $\binom{n-1}{2}$ edges, in which one vertex, referred to as the *center*, forms edges with all pairs of the remaining vertices. Every sub-3-graph of S_n without isolated vertices is called a *star*, while S_n itself is called the *full star*. We denote by C the *triangle*, that is, a 3-graph with six vertices a, b, c, d, e, f and three edges $\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, a\}$. Finally, M stands for a pair of disjoint edges. For a given 3-graph H and a vertex $v \in V(G)$ we denote by $\deg_H(v)$ the number of edges in H containing v .

In the next section we state some known and new results on Turán numbers for P , including Theorem 11 which provides a complete formula for $\text{ex}^{(5)}(n; P)$. We also define conditional Turán numbers and quote from [11] and [14] some useful lemmas about the conditional Turán numbers with respect to P , C , M . Then, in Section 3, we prove Theorem 2, while the remaining sections are devoted to proving Theorem 11.

2. TURÁN NUMBERS

We restrict ourselves exclusively to the case $k = 3$ only. A celebrated result of Erdős, Ko, and Rado [2] asserts, in the case of $k = 3$, that for $n \geq 6$, $\text{ex}^{(1)}(n; M) = \binom{n-1}{2}$. Moreover, for $n \geq 7$, $\text{Ex}^{(1)}(n; M) = \{S_n\}$. We will need the higher order versions of this Turán number, together with its extremal families. The second of these numbers has been found by Hilton and Milner, [8] (see [4] and [14] for a simple proof). For a given set of vertices V , with $|V| = n \geq 7$, let us define two special 3-graphs. Let $x, y, z, v \in V$ be four different vertices of V . We set

$$G_1(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : v \in h, h \cap \{x, y, z\} \neq \emptyset \right\},$$

$$G_2(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : |h \cap \{x, y, z\}| = 2 \right\}.$$

Note that for $i \in \{1, 2\}$, $M \not\subset G_i(n)$ and $|G_i(n)| = 3n - 8$.

Theorem 3 [8]. *For $n \geq 7$, $\text{ex}^{(2)}(n; M) = 3n - 8$ and $\text{Ex}^{(2)}(n; M) = \{G_1(n), G_2(n)\}$.*

Later, we will use the fact that $C \subset G_i(n) \not\supset P$, $i = 1, 2$.

Recently, the third order Turán number for M has been established for general k by Han and Kohayakawa in [7]. Let $G_3(n)$ be the 3-graph on n vertices, with distinguished vertices x, y_1, y_2, z_1, z_2 whose edge set consists of all edges spanned by x, y_1, y_2, z_1, z_2 except for $\{y_1, y_2, z_i\}$, $i = 1, 2$, and all edges of the form $\{x, z_i, v\}$, $i = 1, 2$, where $v \notin \{x, y_1, y_2, z_1, z_2\}$.

Theorem 4 [7]. *For $n \geq 7$, $\text{ex}^{(3)}(n; M) = 2n - 2$ and $\text{Ex}^{(3)}(n; M) = \{G_3(n)\}$.*

For $k = 3$ we were able to take the next step and determine the next Turán number for M .

Theorem 5 [14]. *For $n \geq 7$, $\text{ex}^{(4)}(n; M) = n + 4$.*

The number $\binom{n-1}{2}$ serves as the Turán number for two other 3-graphs, C and P . The Turán number $\text{ex}^{(1)}(n; C)$ has been determined in [3] for $n \geq 75$ and later for all n in [1].

Theorem 6 [1]. *For $n \geq 6$, $\text{ex}^{(1)}(n; C) = \binom{n-1}{2}$. Moreover, for $n \geq 8$, it holds $\text{Ex}^{(1)}(n; C) = \{S_n\}$.*

In [10], we filled an omission of [5] and [12] and calculated $\text{ex}^{(1)}(n; P)$ for all n . Given two 3-graphs F_1 and F_2 , by $F_1 \cup F_2$ denote a vertex-disjoint union of F_1 and F_2 . If $F_1 = F_2 = F$ we will sometimes write $2F$ instead of $F \cup F$.

Theorem 7 [10]. $\text{ex}^{(1)}(n; P) =$

$$\begin{cases} \binom{n}{3} & \text{and } \text{Ex}^{(1)}(n; P) = \{K_n\} & \text{for } n \leq 6, \\ 20 & \text{and } \text{Ex}^{(1)}(n; P) = \{K_6 \cup K_1\} & \text{for } n = 7, \\ \binom{n-1}{2} & \text{and } \text{Ex}^{(1)}(n; P) = \{S_n\} & \text{for } n \geq 8. \end{cases}$$

In [11] we have completely determined the second order Turán number $\text{ex}^{(2)}(n; P)$, together with the corresponding 2-extremal 3-graphs. A *comet* $\text{Co}(n)$ is an n -vertex 3-graph consisting of the complete 3-graph K_4 and the full star S_{n-3} , sharing exactly one vertex which is the center of the star (see Figure 1). This vertex is called the *center* of the comet, while the set of the remaining three vertices of the K_4 is called its *head*.

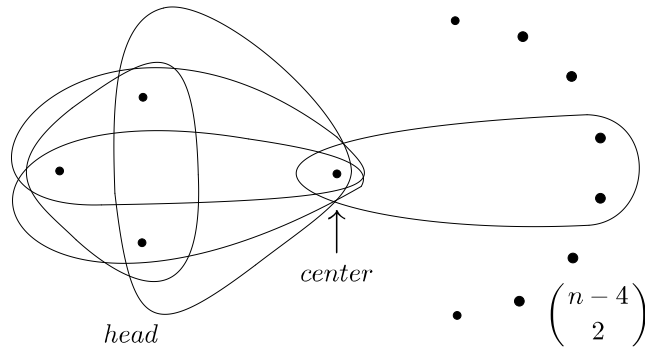


Figure 1. The comet $\text{Co}(n)$.

Theorem 8 [11]. $\text{ex}^{(2)}(n; P) =$

$$\begin{cases} 15 & \text{and } \text{Ex}^{(2)}(n; P) = \{S_7\} & \text{for } n = 7, \\ 20 + \binom{n-6}{3} & \text{and } \text{Ex}^{(2)}(n; P) = \{K_6 \cup K_{n-6}\} & \text{for } 8 \leq n \leq 12, \\ 40 & \text{and } \text{Ex}^{(2)}(n; P) = \{2K_6 \cup K_1, \text{Co}(13)\} & \text{for } n = 13, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}^{(2)}(n; P) = \{\text{Co}(n)\} & \text{for } n \geq 14. \end{cases}$$

In [11] ($n = 12$) and in [15] (for all n), we calculated the third order Turán number for P .

Theorem 9 [11, 15]. $\text{ex}^{(3)}(n; P) =$

$$\begin{cases} 3n - 8 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n)\} & \text{for } 7 \leq n \leq 10, \\ 25 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n), \text{Co}(n)\} & \text{for } n = 11, \\ 32 & \text{and } \text{Ex}^{(3)}(n; P) = \{\text{Co}(n)\} & \text{for } n = 12, \\ 20 + \binom{n-7}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_6 \cup S_{n-6}\} & \text{for } 13 \leq n \leq 14, \\ 4 + \binom{n-5}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_4 \cup S_{n-4}\} & \text{for } n \geq 15. \end{cases}$$

Surprisingly, as an immediate consequence we obtained also an exact formula for the 4th Turán number for P . We define a *rocket* $\text{Ro}(n)$ to be the 3-graph obtained from the star S_{n-4} with center x by adding to it 4 more vertices, say, a, b, c, d , and three edges: $\{x, a, b\}, \{a, b, c\}, \{a, b, d\}$. Let K_5^{+t} be the 3-graph obtained from K_5 by fixing two of its vertices, say a, b , and adding t more vertices v_1, v_2, \dots, v_t and t edges $\{a, b, v_i\}, i = 1, 2, \dots, t$.

Theorem 10 [15]. $\text{ex}^{(4)}(n; P) =$

$$\left\{ \begin{array}{lll} 12 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n), K_5^{+2}\} & \text{for } n = 7, \\ 2n - 2 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n)\} & \text{for } 8 \leq n \leq 9, \\ 20 & \text{and } \text{Ex}^{(4)}(n; P) = \{K_5 \cup K_5\} & \text{for } n = 10, \\ 20 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n)\} & \text{for } n = 11, \\ 28 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_1(n), G_2(n)\} & \text{for } n = 12, \\ 33 & \text{and } \text{Ex}^{(4)}(n; P) = \{K_6 \cup G_1(n), K_6 \cup G_2(n)\} & \text{for } n = 13, \\ 40 & \text{and } \text{Ex}^{(4)}(n; P) = \{2K_6 \cup 2K_1, K_4 \cup S_{10}\} & \text{for } n = 14, \\ 48 & \text{and } \text{Ex}^{(4)}(n; P) = \{\text{Ro}(n), K_6 \cup S_9\} & \text{for } n = 15, \\ 3 + \binom{n-5}{2} & \text{and } \text{Ex}^{(4)}(n; P) = \{\text{Ro}(n)\} & \text{for } n \geq 16. \end{array} \right.$$

The main Turán-type result of this paper provides a complete formula for the fifth order Turán number for P .

Theorem 11. $\text{ex}^{(5)}(n; P) =$

$$\left\{ \begin{array}{lll} 11 & \text{and } \text{Ex}^{(5)}(n; P) = \text{Ex}^{(4)}(7; M) & \text{for } n = 7, \\ 13 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_5^{+3}\} & \text{for } n = 8, \\ 14 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_5^{+4}, K_5 \cup K_4\} \cup \text{Ex}(9; \{P, C\} | M) & \text{for } n = 9, \\ 19 & \text{and } \text{Ex}^{(5)}(n; P) = \{\text{Co}(10)\} & \text{for } n = 10, \\ 19 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_4 \cup S_7\} & \text{for } n = 11, \\ 25 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_7, K_4 \cup S_8\} & \text{for } n = 12, \\ 32 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_4 \cup S_9, K_6 \cup K_5^{+2}, K_6 \cup G_3(7)\} & \text{for } n = 13, \\ 39 & \text{and } \text{Ex}^{(5)}(n; P) = \{\text{Ro}(14)\} & \text{for } n = 14, \\ 46 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_{10}\} & \text{for } n = 15, \\ 56 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_6 \cup S_{10}\} & \text{for } n = 16, \\ 65 & \text{and } \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_{12}, K_6 \cup S_{11}\} & \text{for } n = 17, \\ 10 + \binom{n-6}{2} & \text{and } \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_{n-5}\} & \text{for } n \geq 18. \end{array} \right.$$

To determine Turán numbers, it is sometimes useful to rely on Theorem 3 and divide all 3-graphs into those which contain M and those which do not. To this end, it is convenient to define conditional Turán numbers (see [10, 11]). For a family of 3-graphs \mathcal{F} , an \mathcal{F} -free 3-graph G , and an integer $n \geq |V(G)|$, the *conditional Turán number* is defined as

$$\text{ex}(n; \mathcal{F} | G) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and } H \supseteq G\}.$$

Every n -vertex \mathcal{F} -free 3-graph with $\text{ex}(n; \mathcal{F}|G)$ edges and such that $H \supseteq G$ is called G -extremal for \mathcal{F} . We denote by $\text{Ex}(n; \mathcal{F}|G)$ the family of all n -vertex 3-graphs which are G -extremal for \mathcal{F} . (If $\mathcal{F} = \{F\}$, we simply write F instead of $\{F\}$.)

To illustrate the above mentioned technique, observe that for $n \geq 7$

$$\begin{aligned} \text{ex}^{(2)}(n; P) &= \max\{\text{ex}(n; P|M), \text{ex}^{(2)}(n; M)\} \stackrel{\text{Thm.3}}{=} \max\{\text{ex}(n; P|M), 3n - 8\} \\ &= \text{ex}(n; P|M), \end{aligned}$$

the last equality holds for sufficiently large n (see [11] for details).

In the proof of Theorem 11 we will use the following five lemmas, all proved in [11] and [14]. For the first two we need one more piece of notation. If, in the above definition, we restrict ourselves to connected 3-graphs only (connected in the weakest, obvious sense), then the corresponding conditional Turán number and the extremal family are denoted by $\text{ex}_{\text{conn}}(n; \mathcal{F}|G)$ and $\text{Ex}_{\text{conn}}(n; \mathcal{F}|G)$, respectively.

Lemma 12 [11]. *For $n \geq 7$,*

$$\text{ex}_{\text{conn}}(n; P|C) = 3n - 8 \text{ and } \text{Ex}_{\text{conn}}(n; P|C) = \{G_1(n), G_2(n)\}.$$

Lemma 12 as stated in [11] does not provide the family $\text{Ex}_{\text{conn}}(n; P|C)$. However, it is clear from its proof that the extremal 3-graphs are the same as in Theorem 3. We will also need another lemma, which is not stated explicitly in [11], but it immediate results from the proof of the previous one.

Lemma 13 [11]. *For $n \geq 7$,*

$$\text{ex}_{\text{conn}}(n; P|\{C, M\}) = n + 5 \text{ and } \text{Ex}_{\text{conn}}(n; P|\{C, M\}) = \left\{ K_5^{+(n-5)} \right\}.$$

Moreover, if H is an n -vertex connected P -free 3-graph such that $C \subset H$ and $M \subset H$, then $H \subseteq K_5^{+(n-5)}$.

Lemma 14 [11].

$$\text{ex}(n; \{P, C\}|M) = \begin{cases} 2n - 4 & \text{for } 6 \leq n \leq 9, \\ 20 & \text{for } n = 10, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}(n; \{P, C\}|M) = \{\text{Co}(n)\} \text{ for } n \geq 11. \end{cases}$$

Lemma 15 [11]. *For $n \geq 6$,*

$$\text{ex}(n; \{P, C, P_2 \cup K_3\}|M) = 2n - 4,$$

where P_2 is a pair of edges sharing one vertex.

Lemma 16 [14]. *For $n \geq 6$,*

$$\text{ex}^{(2)}(n; \{M, C\}) = \max\{10, n\}.$$

3. PROOF OF THEOREM 2

As mentioned in the Introduction, Jackowska has shown in [9] that $R(P; r) \geq r+6$ for all $r \geq 1$. We are going to show that $R(P; 10) \leq 16$.

We will show that every 10-coloring of K_{16} yields a monochromatic copy of P . The idea of the proof is to gradually reduce the number of vertices and colors (by one in each step), until we reach a coloring which yields a monochromatic copy of P .

Let us consider an arbitrary 10-coloring of K_{16} , $K_{16} = \bigcup_{i=1}^{10} G_i$, and assume that for each $i \in [10]$, $P \not\subseteq G_i$. Since $|K_{16}| = 560$, the average number of edges per color is 56, and therefore, by Theorems 7–11, either for each $i \in [10]$, $G_i = K_6 \cup S_{10}$, or there exists a color, say G_{10} , contained in one of the 3-graphs: S_{16} , $\text{Co}(16)$, $K_4 \cup S_{12}$, $\text{Ro}(16)$. We will show that the latter case must occur. Indeed, for each vertex $v \in V(K_{16})$ we have $\deg_{K_{16}}(v) = \binom{15}{2} = 105$ whereas for $v \in V(K_6 \cup S_{10})$, $\deg_{K_6 \cup S_{10}}(v) \in \{10, 36, 8\}$ depending on whether v is a vertex of K_6 , the center of the star S_{10} or another vertex of the star. Since we are not able to obtain an odd number as a sum of even numbers, we can not decompose K_{16} into edge-disjoint copies of $K_6 \cup S_{10}$. Let us turn back to G_{10} . No matter in which of the four 3-graph G_{10} is contained, we remove the center of the star (or comet, or rocket) together with up to four more edges of G_{10} , so that we get rid of color 10 completely (note that some other colors can also be affected by this deletion).

As a result, we obtain a 3-graph H_{15} on 15 vertices, colored with 9 colors, $H_{15} = \bigcup_{i=1}^9 G_i$, with $|H(15)| \geq 451$ (with some abuse of notation we will keep denoting the subgraphs of G_i obtained in each step again by G_i). The average number of edges per color is at least 50.1, and therefore there exists a color, say G_9 , with $|G_9| \geq 51$. This time we use Theorems 7–9 to conclude that either $G_9 \subset S_{15}$ or $G_9 \subset \text{Co}(15)$. In either case we remove the center and, in case of the comet, one more edge being its head.

We get a 3-graph $H(14)$ on 14 vertices with $|H(14)| \geq 359$, colored by 8 colors, $H(14) = \bigcup_{i=1}^8 G_i$. The average number of edges per color is at least 44.9, and hence there exists a color, say G_8 , with $|G_8| \geq 45$. Similarly as in the previous step we reduce the picture to a 3-graph $H(13)$ on 13 vertices with $|H(13)| \geq 280$, colored by 7 colors, $H(13) = \bigcup_{i=1}^7 G_i$.

This time the average number of edges per color is at least 40, and therefore, by Theorems 7 and 8, either each color is a copy of $\text{Co}(13)$ or $K_6 \cup K_6 \cup K_1$, or there exists a color, say G_7 , contained in the full star S_{13} . We will show in the similar way as before that $H(13)$ can not be decomposed into edge-disjoint copies of $\text{Co}(13)$ and $K_6 \cup K_6 \cup K_1$, and therefore the latter case must occur. Indeed, let us assume that no color is contained in the full star S_{13} . First notice that there is not enough space for two edge-disjoint copies of $K_6 \cup K_6 \cup K_1$ in K_{13} and

therefore also in $H(13)$. Fix one copy of $K_6 \cup K_6 \cup K_1$ in K_{13} . By the pigeon-hole principle, any other copy of K_6 must share at least three vertices with one of the fixed copies of K_6 and therefore they are not edge-disjoint. Now observe that since during our procedure we have lost at most 6 edges of K_{13} , for each vertex $v \in V(H(13))$ we have $\deg_{H(13)}(v) \geq \binom{12}{2} - 6 = 60$ and also for each vertex of a comet $\text{Co}(13)$ which is not its center we have $\deg_{\text{Co}(13)}(v) \leq 8$. If $H(13)$ is decomposed into seven copies of $\text{Co}(13)$ or six copies of $\text{Co}(13)$ and one copy of $2K_6 \cup K_1$, then there must exist a vertex $v \in V(H(13))$ which is not a center of any of these comets and therefore $\deg_{H(13)}(v) \leq 10 + 6 \cdot 8 = 58 < 60$, a contradiction. Consequently, we have $G_7 \subseteq S_{13}$ and, by removing the center of this star, we obtain a 6-coloring of a 3-graph $H(12)$ on 12 vertices with $|H(12)| \geq 214$.

To proceed, let us assume for a while, that none of the colors G_i , $i \in [6]$, is a star. Then, by Theorems 7–9, each color with more than 32 edges is a subset of $K_6 \cup K_6$. The average number of edges per color is at least 35.6, and hence there exists a color, say G_6 , with $G_6 \subset K_6 \cup K_6$. We remove all edges of this copy of $K_6 \cup K_6$, getting a bipartite 3-graph $H'(12)$ with a bipartition $V(H'(12)) = W \cup U$, $|W| = |U| = 6$, and with $|H'(12)| \geq 174$ edges colored by 5 colors, $H'(12) = \bigcup_{i=1}^5 G_i$. Note that every subgraph of $K_6 \cup K_6$ contained in $H'(12)$ (and consequently each color class of $H'(12)$) has at most 36 edges. Since $2 \cdot 36 + 3 \cdot 33 = 171 < 174$, at least 3 color classes have at least 34 edges, and thus each of them must be subsets of $K_6 \cup K_6$. Now observe that if two color classes, say G_1 and G_2 , have at least 34 edges each, then they are disjoint unions of two copies of K_6 , one on the vertex set $U'_i \cup W'_i$, the other one on $U''_i \cup W''_i$, with four missing edges U'_i, U''_i, W'_i, W''_i , where $U = U'_i \cup U''_i$, $W = W'_i \cup W''_i$, $i = 1, 2$, and $\{U'_1, U''_1\} = \{U'_2, U''_2\}$ (see Figure 2).

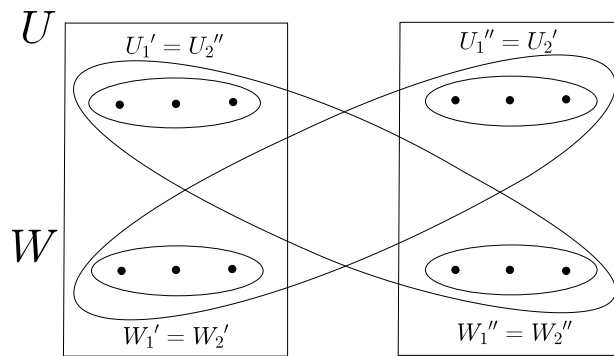


Figure 2. The partition of the set of vertices of $H'(12)$, G_1 and G_2 .

Indeed, otherwise, if $1 \leq |U'_1 \cap U'_2| \leq 2$, then G_1 and G_2 would share at least six edges, and thus $|G_1| + |G_2| \leq 36 + 36 - 6 < 2 \cdot 34$. This simply means that

one of the partitions, of U or of W , must be swapped. But this is impossible for three color classes. Consequently, at least one color, say G_6 , is a star. We remove the center of this star to get a 5-coloring of a 3-graph $H(11)$ on 11 vertices with $|H(11)| \geq 159$.

By repeating this argument three more times, we finally arrive at a 2-coloring of a 3-graph $H(8) = G_1 \cup G_2$, with $|H(8)| \geq 50$ which, by Theorem 7, should contain a copy of P , a contradiction.

4. PROOF OF THEOREM 11

Let us define $\mathcal{H}_n = \text{Ex}^{(1)}(n; P) \cup \text{Ex}^{(2)}(n; P) \cup \text{Ex}^{(3)}(n; P) \cup \text{Ex}^{(4)}(n; P)$. To prove Theorem 11 we need to find, for each $n \geq 7$, a P -free n -vertex 3-graph H with the biggest possible number of edges such that, whenever $G \in \mathcal{H}_n$, then $H \not\subseteq G$. Moreover, we will show that $|H| = h_n$, where h_n is the number of edges, given by the formula to be proved.

First note that for each $n \geq 7$, all candidates for being 5-extremal 3-graphs do qualify, that is, are P -free, are not contained in any of the 3-graphs from \mathcal{H}_n , and have h_n edges. To finish the proof, we will show that each P -free n -vertex 3-graph H , not contained in any of 3-graph from \mathcal{H}_n , satisfy $|H| < h_n$ unless it is one of the candidates for being 5-extremal 3-graph itself.

For the latter task, we distinguish two cases: when H is connected and disconnected. The entire proof is inductive, in the sense that here and there we apply the very Theorem 11 for smaller instances of n , once they have been confirmed.

For all $n \geq 7$, let H be P -free n -vertex 3-graph such that for each $G \in \mathcal{H}_n$, $H \not\subseteq G$. Moreover, let H be different from all candidates for being 5-extremal 3-graphs with the same number of vertices. We will show that $|H| < h_n$.

4.1. Connected case

We start with the connected case. First let us assume that $M \not\subseteq H$ and consider consecutive intersecting families. Recall that for all $n \geq 7$, $H \not\subseteq S_n$, for $7 \leq n \leq 12$, $H \not\subseteq G_1(n)$ and $H \not\subseteq G_2(n)$, for $7 \leq n \leq 9$ and $n = 11$, $H \not\subseteq G_3(n)$, and finally, for $n = 7$, H is not equal to any of 4-extremal 3-graphs for M . Therefore, by Theorems 3, 4 and 5, we get that for all $n \geq 7$,

$$|H| < h_n.$$

Consequently, we will be assuming by the end of the proof that $M \subset H$. If additionally $C \subset H$, then by Lemma 13, $H \subseteq K_5^{+(n-5)}$ and hence $|H| \leq |K_5^{+(n-5)}| = n + 5$. Therefore, for $n \geq 10$, $|H| < h_n$. If $n = 7$, as $K_5^{+2} \in \mathcal{H}_7$, we have $H \not\subseteq K_5^{+2}$ and thus we may exclude this case. Lastly, for $8 \leq n \leq 9$, by the definition of H ,

$H \neq K_5^{+(n-5)}$ and hence $|H| < h_n$. Therefore, in the rest of the proof we will be assuming that $C \not\subseteq H$.

Finally, let H be connected $\{P, C\}$ -free 3-graph containing M . Then by Lemma 14, for $7 \leq n \leq 8$, $|H| \leq 2n - 4 < h_n$ and for $n = 9$, since $H \notin \text{Ex}(9, \{P, C\}|M)$, we have $|H| < 14 = h_9$.

For $10 \leq n \leq 11$ we need two more facts, which we state here without the proof. Namely, $\text{ex}_{\text{conn}}(10; \{P, C\}|M) = 19$ and $\text{Ex}_{\text{conn}}(10; \{P, C\}|M) = \{\text{Co}(10)\}$. Since, by the definition of H , $H \neq \text{Co}(10)$, this implies that $|H| < 19 = h_{10}$. Whereas for $n = 11$ we have $\text{ex}_{\text{conn}}^{(2)}(11; \{P, C\}|M) = 18$, and therefore, as $H \not\subseteq \text{Co}(11)$, we get $|H| \leq \text{ex}_{\text{conn}}^{(2)}(11; \{P, C\}|M) = 18 < 19 = h_{11}$.

Recall that for all $n \geq 11$, $H \not\subseteq \text{Co}(n)$. Moreover, for $12 \leq n \leq 13$, since $|\text{Ro}(n)| < h_n$, we may assume that $H \not\subseteq \text{Ro}(n)$. Further, for $n = 14$, by the definition of H we have $H \neq \text{Ro}(14)$ and thus, if $H \subset \text{Ro}(14)$, then $|H| < |\text{Ro}(14)| = h_n$. Finally, for all $n \geq 15$ we have $H \not\subseteq \text{Ro}(n)$. Therefore, since for all $n \geq 12$ we have

$$h_n \geq \binom{n-6}{2} + 10,$$

to complete the proof of the connected case it is enough to prove the following Lemma.

Lemma 17. *If H is a connected, n -vertex, $n \geq 12$, $\{P, C\}$ -free 3-graph containing M such that $H \not\subseteq \text{Co}(n)$ and $H \not\subseteq \text{Ro}(n)$, then $|H| < \binom{n-6}{2} + 10$.*

We devote an entire Section 5 to prove Lemma 17.

4.2. Disconnected case

Now let H be disconnected and let $m = m(H)$ be the number of vertices in the smallest component of H . We have $m \neq 2$, since no component of a 3-graph may have two vertices. We now break the proof into several cases.

Let us express H as a vertex disjoint union of two 3-graphs:

$$H = H' \cup H'', \quad |V(H')| = m, \quad |V(H'')| = n - m.$$

Then, clearly, both H' and H'' are P -free, and thus

$$(2) \quad |H| \leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n - m; P).$$

Below, to bound $|H|$, we use the Turán numbers for P of the 1st, 2nd, 3rd, 4th and 5th order and utilize, respectively, Theorems 7, 8, 9, 10 and 11 (by induction).

Let v be an isolated vertex ($m = 1$). Since for $n = 7$ and any 3-graph H'' , $K_1 \cup H'' \subseteq K_1 \cup K_6 \in \mathcal{H}_7$, we may assume that $n \geq 8$. For $8 \leq n \leq 11$,

as H cannot be a sub-3-graph of $S_n, K_6 \cup K_{n-6}, G_1(n)$ or $G_2(n)$, H'' is not a sub-3-graph of $S_{n-1}, K_6 \cup K_{n-7}, G_1(n-1)$ and $G_2(n-1)$. Consequently, for $n = 8, 10$,

$$|H| = |H''| \leq \text{ex}^{(4)}(n-1; P) < h_n.$$

For $n = 9$ additionally we have $H'' \not\subseteq G_3(8)$ and therefore

$$|H| \leq \text{ex}^{(5)}(8; P) = 13 < 14 = h_9,$$

whereas for $n = 11$, $H'' \not\subseteq K_5 \cup K_5$ and $H'' \not\subseteq \text{Co}(10)$. Consequently,

$$|H| = |H''| < \text{ex}^{(5)}(10; P) = 19 = h_{11}.$$

For $n \geq 12$, since $H = K_1 \cup H''$ is not a sub-3-graph of any of the 3-graphs in \mathcal{H}_n , we have $H'' \not\subseteq S_{n-1}$ and $H'' \not\subseteq \text{Co}(n-1)$. Moreover, for $n = 12, 13$, $H'' \not\subseteq K_6 \cup K_{n-7}$, for $n = 12$, $H'' \not\subseteq G_1(n-1)$ and $H'' \not\subseteq G_2(n-1)$, for $n = 14$, $H'' \not\subseteq 2K_6 \cup K_1$, for $n = 14, 15$, $H'' \not\subseteq K_6 \cup S_{n-7}$ and finally, for $n \geq 15$, $H'' \not\subseteq K_4 \cup S_{n-5}$. Consequently,

$$|H| = |H''| \leq \text{ex}^{(4)}(n-1; P) < h_n.$$

For $m = 3$ and $n = 7, 8$, by (2) we get

$$|H| \leq \text{ex}^{(1)}(3; P) + \text{ex}^{(1)}(n-3; P) = 1 + \text{ex}^{(1)}(n-3; P) < h_n.$$

Since each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 3$ and $|V(H'')| = 6$ is a sub-3-graph of $K_3 \cup K_6 \in \mathcal{H}_9$, we may assume that $n \neq 9$. For $n = 10$ we have $K_3 \cup K_6 \cup K_1 \subset K_4 \cup K_6 \in \mathcal{H}_{10}$. Consequently, $H'' \not\subseteq K_6 \cup K_1$ and thus $|H''| \leq \text{ex}^{(2)}(7; P) = 15$. Hence $|H| \leq 1 + 15 = 16 < 19 = h_{10}$.

Further, for all $n \geq 11$, since $K_3 \cup S_{n-3} \subseteq \text{Co}(n) \in \mathcal{H}_n$, we have $H'' \not\subseteq S_{n-3}$. Therefore for $n \geq 12$,

$$|H| \leq 1 + \text{ex}^{(2)}(n-3; P) < h_n,$$

whereas, for $n = 11$ additionally we have $H \not\subseteq K_3 \cup K_6 \cup K_2 \subset K_6 \cup K_5 \in \mathcal{H}_{11}$. Thus $H'' \not\subseteq K_6 \cup K_2$ and consequently,

$$|H| \leq 1 + \text{ex}^{(3)}(8; P) = 17 < 19 = h_{11}.$$

For $m = 4$ and $n = 8$ by (2) we have

$$|H| \leq \text{ex}^{(1)}(4; P) + \text{ex}^{(1)}(4; P) = 4 + 4 = 8 < h_8.$$

For $n = 9$, by the definition of H , $H \neq K_4 \cup K_5$ and therefore $|H| < |K_4 \cup K_5| = 14 = h_9$. Similarly like before, we may skip the case $n = 10$, because each

disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 4$ and $|V(H'')| = 6$ is a sub-3-graph of $K_4 \cup K_6 \in \mathcal{H}_{10}$. For $n = 11$, since $K_4 \cup K_6 \cup K_1 \subset K_5 \cup K_6 \in \mathcal{H}_{11}$, we have $H'' \not\subset K_6 \cup K_1$ and therefore $|H''| \leq \text{ex}^{(2)}(7; P) = 15$ with the equality only for $H'' = S_7$. But, by the definition of H , $H \neq K_4 \cup S_7$, and hence

$$|H| < |K_4 \cup S_7| = 19 = h_{11}.$$

Further, for $n = 12, 13$, since $\text{Ex}^{(1)}(n - 4; P) = \{S_{n-4}\}$ and $H \neq H_4 \cup S_{n-4}$, we have $|H| < |H_4 \cup S_{n-4}| = h_n$. Finally, for $n \geq 14$, since $K_4 \cup S_{n-4} \in \mathcal{H}_n$ we get $H'' \not\subset S_{n-4}$ and consequently,

$$|H| \leq \text{ex}^{(1)}(4; P) + \text{ex}^{(2)}(n - 4; P) < h_n.$$

Now let $m = 5$. Notice that each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 5$ and $5 \leq |V(H'')| \leq 6$ is a sub-3-graph of $K_5 \cup K_5 \in \mathcal{H}_{10}$ and $K_5 \cup K_6 \in \mathcal{H}_{11}$, respectively. Therefore we may consider only $n \geq 12$. For $n = 12$, since $K_5 \cup K_6 \cup K_1 \subset K_6 \cup K_6 \in \mathcal{H}_{12}$, we have $|H''| \leq \text{ex}^{(2)}(7; P) = 15$ with the equality only for $H'' = S_7$. But, by the definition of H , $H \neq K_5 \cup S_7$ and hence $|H| < |K_5 \cup S_7| = 25 = h_{12}$. Finally, for $n \geq 13$, by (2),

$$|H| \leq \text{ex}^{(1)}(5; P) + \text{ex}^{(1)}(n - 5; P) = 10 + \binom{n - 6}{2} \leq h_n,$$

where the equality is achieved only by the candidates for 5-extremal 3-graphs with the proper number of vertices.

For $m = 6$ we have $n \geq 12$, but as each disconnected 3-graph $H' \cup H''$ with $|V(H')| = |V(H'')| = 6$ is a sub-3-graph of $K_6 \cup K_6 \in \mathcal{H}_{12}$, we may consider only $n \geq 13$. Recall that $\{2K_6 \cup K_1, K_6 \cup S_7, K_6 \cup G_1(7), K_6 \cup G_2(7)\} \subset \mathcal{H}_{13}$ and therefore, for $n = 13$, H'' is not contained in any of the 3-graphs $K_6 \cup K_1, S_7, G_1(7), G_2(7)$. Consequently, $|H''| \leq \text{ex}^{(4)}(7; P) = 12$ with the equality only for $H'' = G_3(7)$ and $H'' = K_5^{+2}$. But, by the definition of H , $H \neq K_6 \cup K_5^{+2}$ and $H \neq K_6 \cup G_3(7)$ and thus

$$|H| < |K_6 \cup K_5^{+2}| = |K_6 \cup G_3(7)| = h_{13}.$$

For the same reason, if $n = 14$, then $H'' \not\subset S_8$ and $H'' \not\subset K_6 \cup K_2$. Consequently,

$$|H| = |H'| + |H''| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(3)}(8; P) = 20 + 16 < 39 = h_{14},$$

whereas for $n = 15$, we have $H'' \not\subset S_9$ and hence

$$|H| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(2)}(9; P) = 20 + 21 < 46 = h_{15}.$$

Further, for $n = 16, 17$, by the definition of H , $H \neq K_6 \cup S_{n-6}$. Consequently, as $\text{Ex}(n - 6; P) = \{S_{n-6}\}$, we get

$$|H| < |K_6 \cup S_{n-6}| = h_n.$$

Finally, for $n \geq 18$, by (2),

$$|H| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(1)}(n - 6; P) = 20 + \binom{n - 7}{2} < \binom{n - 6}{2} + 10 = h_n.$$

If $m = 7$, then $n \geq 14$. For $n = 14$, since $H \not\subseteq 2K_6 \cup 2K_1 \in \mathcal{H}_{14}$, at least one of the components of H is not a sub-3-graph of $K_6 \cup K_1$ and therefore has at most $\text{ex}^{(2)}(7; P) = 15$ edges. Consequently,

$$|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(2)}(7; P) = 20 + 15 < 39 = h_{14}.$$

To bound the number of edges of H for $n \geq 15$ we use (2) to get

$$|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(1)}(n - 7; P) = 20 + \binom{n - 8}{2} < \binom{n - 6}{2} + 10 \leq h_n.$$

Finally, for $m \geq 8$ we have $n \geq 16$ and, by (2),

$$\begin{aligned} |H| &\leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n - m; P) = \binom{m - 1}{2} + \binom{n - m - 1}{2} \\ &\leq \binom{7}{2} + \binom{n - 9}{2} < \binom{n - 6}{2} + 10 \leq h_n. \end{aligned}$$

5. THE PROOF OF LEMMA 17

Recall that H is a connected, n -vertex, $n \geq 12$, $\{P, C\}$ -free 3-graph such that $M \subset H$, $H \not\subseteq \text{Co}(n)$ and $H \not\subseteq \text{Ro}(n)$. We need to show that

$$|H| < \binom{n - 6}{2} + 10.$$

Since for $n \geq 11$, by Lemma 15

$$\text{ex}(\{n; P, C, P_2 \cup K_3\} | M) = 2n - 4 < \binom{n - 6}{2} + 10,$$

we may assume that $P_2 \cup K_3 \subset H$. Let us denote a copy of P_2 from $P_2 \cup K_3$ in H by Q and the vertex of degree two in Q by x . We let $U = V(Q)$, $V = V(H)$ and $W = V \setminus U$. Moreover, let W_0 be the set of vertices of degree zero in $H[W]$ and $W_1 = W \setminus W_0$ (see Figure 3). Note that, by definition, $H[W] = H[W_1]$ and $|W_1| \geq 3$.

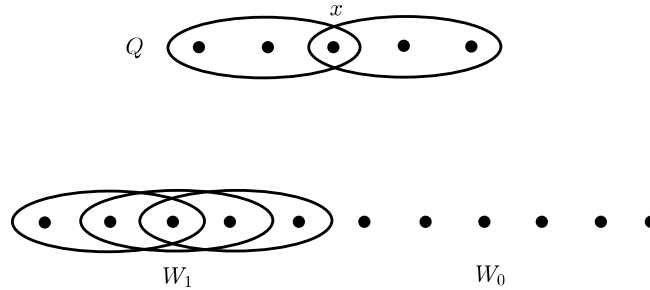


Figure 3. Set-up for the proof of Lemma 17.

We also split the set of edges of H . First, notice that since H is P -free, there is no edge with one vertex in each U , W_0 , and W_1 . We define $H_i = \{h \in H : h \cap U \neq \emptyset, h \cap W_i \neq \emptyset\}$, where $i = 0, 1$. Then, clearly,

$$(3) \quad H = H[U] \cup H[W] \cup H_0 \cup H_1,$$

with all four parts edge-disjoint. Since by definition $H[U] \cup H_0 = H[U \cup W_0]$, sometimes we will use the following equality

$$(4) \quad H = H[U \cup W_0] \cup H_1 \cup H[W].$$

Recall that H is C -free, and therefore one can use Theorem 6 to get the bounds, for $|W_0| \geq 1$

$$(5) \quad |H[U \cup W_0]| \leq \binom{|U \cup W_0| - 1}{2} = \binom{|W_0| + 4}{2}$$

and for $|W_1| \geq 6$,

$$(6) \quad |H[W]| \leq \binom{|W_1| - 1}{2}.$$

Notice that for each edge $h \in H_0 \cup H_1$ with $|h \cap U| = 1$ we have $h \cap U = \{x\}$, because otherwise h together with Q would form a copy of P in H . We let

$$F^0 = \{h \in H_0 \cup H_1 : h \cap U = \{x\}\}.$$

Also, to avoid a copy of C in H , if for $h \in H_0 \cup H_1$ we have $|h \cap U| = 2$, then the pair $h \cap U$ is contained in an edge of Q . For $k = 1, 2$, we define

$$F^k = \{h \in H_0 \cup H_1 : |h \cap U \setminus \{x\}| = k\}.$$

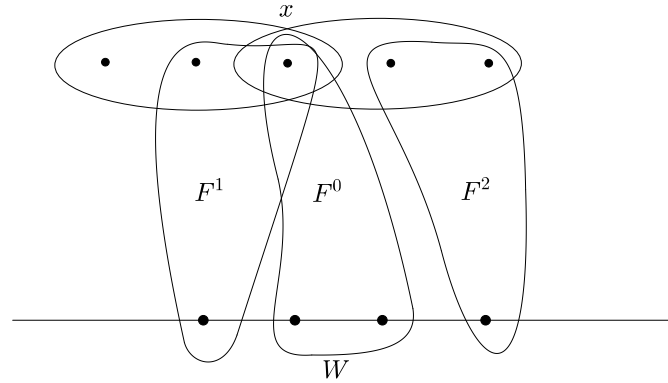


Figure 4. Three types of edges in $H_0 \cup H_1$.

Clearly, $H_0 \cup H_1 = F^0 \cup F^1 \cup F^2$ (see Figure 4). Further, for $i = 0, 1$ and $k = 0, 1, 2$, we set

$$F_i^k = F^k \cap H_i.$$

It is easy to see that, as H is P -free, $F_1^1 = \emptyset$ and therefore,

$$(7) \quad H_1 = F_1^0 \cup F_1^2.$$

Moreover, for all $v \in W$ we have

$$(8) \quad F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset,$$

and, by the definition of F^1 and F^2 ,

$$(9) \quad |F^1(v)| \leq 4 \quad \text{and} \quad |F^2(v)| \leq 2,$$

where for a given subset of edges $G \subseteq H$ and for a vertex $v \in V(H)$ we set $G(v) = \{h \in G : v \in h\}$.

In the whole proof we will be using the fact that for all edges $e \in F^0$, the pair $e \cap W_1$ is *nonseparable* in $H[W]$, that is, every edge of $H[W]$ must contain both these vertices or none. Consequently, for each $v \in W_0$, $|F^0(v)| \leq |W_0| - 1$ and thus, by (8) and (9),

$$(10) \quad |H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leq 4 + \max\{2, |W_0| - 1\}.$$

Moreover, if $F_1^0 \neq \emptyset$, then there exists at least one nonseparable pair in W_1 , and therefore one can show the following fact.

Fact 1. If $F_1^0 \neq \emptyset$, then $|H[W]| \leq \binom{|W_1|-2}{3} + |W_1| - 2$. Moreover, if in addition $H[W_1] \subseteq S_{|W_1|}$, then $|H[W]| \leq \binom{|W_1|-3}{2} + 1$.

To prove another fact let us define an auxiliary graph G for nonseparable pairs on the set of vertices W_1 , $G = \{e \setminus \{x\} : e \in F_1^0\}$. Then each component of G has size at most 3. This gives the proof of the following inequality. For any $W'_1 \subseteq W_1$ we let $F_1^0[W'_1] \subseteq F_1^0$ to be the set of edges $h \in F_1^0$ such that $h \cap W_1 \subseteq W'_1$. Then,

$$(11) \quad |F_1^0[W'_1]| \leq |W'_1|.$$

Observe also that, because H is connected, $H_1 \neq \emptyset$. Consequently, since the presence of any edge of H_1 forbids at least 4 edges of $H[U]$,

$$(12) \quad |H[U]| \leq 6.$$

Moreover, in [11] the authors have proved the following bounds on the number of edges in H_1 .

$$(13) \quad \text{For } |W_1| \geq 4, \quad |F_1^2| \leq 2|W_1| - 4.$$

$$(14) \quad \text{For } |W_1| \geq 3, \quad |H_1| \leq 2|W_1| - 3.$$

As a consequence of these inequalities one can prove the following.

$$(15) \quad \text{For } |W_1| \geq 7, \quad |H[U]| + |H_1| \leq 2|W_1| - 1.$$

Indeed, if $|H_1| \leq |W_1|$, then (15) results from (12) and the inequality $|W_1| - 1 \geq 7 - 1 = 6$. Otherwise, by (11), (7) and (8), there exists a vertex $v \in W_1$ such that $|F_1^2(v)| = 2$. As expected, assume $|H_1| > |W_1|$, and there does not exist the desired vertex v , i.e., for any vertex $v \in W_1$, $|F_1^2(v)| \leq 1$. Further, let $W'_1 \subseteq W_1$ be the set of vertices v such that $F_1^2(v) = \emptyset$, and let $W''_1 = W_1 \setminus W'_1$. Then by (7), (8), (11) and the definition of W'_1 , we have

$$|H_1| = |F_1^0| + |F_1^2| = |F_1^0[W'_1]| + \sum_{v \in W''_1} |F_1^2(v)| \leq |W'_1| + |W''_1| = |W_1|,$$

a contradiction. As H is $\{P, C\}$ -free, by the definition of $F_1^2(v)$, this implies that $|H[U]| = 2$ and (15) follows from (14).

We also need the following fact proven in [15].

Fact 2 [15]. If $F_1^2 \neq \emptyset$, then

$$(16) \quad |H[U \cup W_0]| \leq \begin{cases} 8 & \text{for } |W_0| = 1, \\ 3|W_0| + 7 & \text{for } 2 \leq |W_0| \leq 4, \\ \binom{|W_0|+2}{2} + 1 & \text{for } |W_0| \geq 5. \end{cases}$$

We split the whole proof of Lemma 17 into a few short parts, Facts 3–7.

Fact 3. For $n \geq 13$, if $W_0 = \emptyset$ and $H_1 \neq \emptyset$, then $|H| < 10 + \binom{n-6}{2}$.

Proof. Let us consider two cases, whether or not $H[W] \subseteq S_{n-5}$. First assume that $H[W]$ is a subset of the star S_{n-5} with the center $y \in W_1$. Further assume also that $F^2 \setminus F^2(y) \neq \emptyset$, say, there exists $v \in W_1$, $v \neq y$, such that $F^2(v) \neq \emptyset$. Let $h \in H[W_1]$ be the edge containing y and v (because $H[W] \subseteq S_{n-5}$). Since H is P -free, for any $h' \in H[W_1]$, $h' \neq h$, it holds that $|h' \cap h| \geq 2$ and $y \in h' \cap h$. Thus $|H[W_1]| \leq 1 + 2(|W_1| - 3)$. By (15), we have $|H| \leq 2|W_1| - 1 + (1 + 2(|W_1| - 3)) = 4n - 26 < \binom{n-6}{2} + 10$. Otherwise, $|F^2| = |F^2(y)| \leq 2$. Additionally, if $F_1^0 = \emptyset$, then by (3), (12), (7) and (6),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 2 + \binom{n-6}{2} = \binom{n-6}{2} + 8 < \binom{n-6}{2} + 10.$$

Otherwise, $F_1^0 \neq \emptyset$ and therefore by Fact 1, $|H[W]| \leq \binom{n-8}{2} + 1$. By (11), $|F_1^0| \leq |W_1| = n - 5$ and hence by (7), $|H_1| \leq n - 5 + 2 = n - 3$. Consequently, by (3) and (12),

$$\begin{aligned} |H| &= |H[U]| + |H_1| + |H[W]| \leq 6 + n - 3 + \binom{n-8}{2} + 1 \\ &= \binom{n-7}{2} + 12 < \binom{n-6}{2} + 10. \end{aligned}$$

Now we move to the case $H[W] \not\subseteq S_{n-5}$. Since for $n \geq 13$, by Theorem 7, $\text{Ex}^{(1)}(n - 5; P) = \{S_{n-5}\}$, we may bound the number of edges in $H[W]$ by $\text{ex}^{(2)}(n - 5; P)$. Moreover, by (15), $|H[U]| + |H_1| \leq 2(n - 5) - 1 = 2n - 11$. Consequently, by (3) and Theorem 8,

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 2 \cdot n - 11 + \text{ex}^{(2)}(n - 5; P) < \binom{n-6}{2} + 10,$$

where the last inequality is valid for $n \geq 15$. For $13 \leq n \leq 14$ we have to strengthen the bound of $H[W]$. Since W does not contain isolated vertices, we have $H[W] \not\subseteq K_6 \cup K_2$. Therefore by Theorems 7 and 8 we get $|H[W]| < \text{ex}^{(2)}(n - 5; P)$ and consequently, for $n = 14$, $|H| < \binom{n-6}{2} + 10$. In addition, for $n = 13$, we use the fact that for $i = 1, 2$, $C \subset G_i(8)$ and H is C -free, hence $H[W] \neq G_i(8)$. Then by Theorems 7, 8 and 9 we have $|H[W]| < \text{ex}^{(3)}(8; P) = 16$ and therefore

$$|H| < 2 \cdot 13 - 11 + 16 = 31 = \binom{13-6}{2} + 10. \quad \blacksquare$$

Fact 4. For $n \geq 13$, if $H_1 \neq \emptyset$, $H \not\subseteq \text{Co}(n)$ and $|W_1| = 3$, then $|H| < 10 + \binom{n-6}{2}$.

Proof. We have $|H[W]| = 1$, $|U \cup W_0| = n - 3$ and by (14), $|H_1| \leq 3$. Therefore, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$

Consequently, all we need to do is to bound the number of edges in $H[U \cup W_0]$. Since $H \not\subseteq \text{Co}(n)$, either $F_1^2 \neq \emptyset$ or $H[U \cup W_0] \not\subseteq S_{n-3}$. In the former case we use Fact 2 to get $|H[U \cup W_0]| \leq \binom{n-6}{2} + 1$ and therefore

$$|H| \leq \binom{n-6}{2} + 1 + 4 = \binom{n-6}{2} + 5 < \binom{n-6}{2} + 10.$$

Otherwise, $H[U \cup W_0] \not\subseteq S_{n-3}$, so by Theorem 7, $|H[U \cup W_0]| \leq ex^{(2)}(n-3; P)$. Consequently, by Theorem 8, for $13 \leq n \leq 15$, $|H[U \cup W_0]| \leq 20 + \binom{n-3-6}{3}$ and therefore,

$$|H| \leq 20 + \binom{n-9}{3} + 4 = \binom{n-9}{3} + 24 < \binom{n-6}{2} + 10.$$

Whereas for $n \geq 16$ we get $|H[U \cup W_0]| \leq 4 + \binom{n-3-4}{2}$, and hence

$$|H| \leq \binom{n-7}{2} + 4 + 4 = \binom{n-7}{2} + 8 < \binom{n-6}{2} + 10. \quad \blacksquare$$

Fact 5. For $n \geq 13$, if $H_1 \neq \emptyset$, $H \not\subseteq \text{Ro}(n)$ and $|W_1| = 4$, then $|H| < 10 + \binom{n-6}{2}$.

Proof. The proof goes along the lines of the previous one. We have $|H[W]| \leq \binom{4}{3} = 4$, $|U \cup W_0| = n - 4$ and by (14), $|H_1| \leq 5$. Therefore, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 5 + 4 = |H[U \cup W_0]| + 9.$$

Consequently, to finish the proof we need to bound $|H[U \cup W_0]|$. Assume that $F_1^2 = \emptyset$ and $H[U \cup W_0] \subseteq S_{n-4}$. Then, since $F_1^0 \neq \emptyset$, $|W_1| = 4$ and for $e \in F_1^0$ the pair $e \cap W_1$ is nonseparable in $H[W]$, we get $H \subseteq \text{Ro}(n)$, a contradiction. Therefore $F_1^2 \neq \emptyset$ or $H[U \cup W_0] \not\subseteq S_{n-4}$. In the former case we use Fact 2 to get for $n = 13$, $|H[U \cup W_0]| \leq 19$ and consequently,

$$|H| \leq 19 + 9 = 28 < 31 = 10 + \binom{13-6}{2}.$$

Whereas for $n \geq 14$, $|H[U \cup W_0]| \leq \binom{n-7}{2} + 1$ and hence,

$$|H| \leq \binom{n-7}{2} + 1 + 9 = \binom{n-7}{2} + 10 < \binom{n-6}{2} + 10.$$

Otherwise, $H[U \cup W_0] \not\subseteq S_{n-4}$ so we use Theorem 7 to get $|H[U \cup W_0]| \leq \text{ex}^{(2)}(n-4; P)$. Consequently, by Theorem 8, for $13 \leq n \leq 16$, $|H[U \cup W_0]| \leq 20 + \binom{n-4-6}{3}$ and hence

$$|H| \leq 20 + \binom{n-10}{3} + 9 = \binom{n-10}{3} + 29 < \binom{n-6}{2} + 10.$$

Whereas for $n \geq 17$ we have $|H[U \cup W_0]| \leq 4 + \binom{n-4-4}{2}$ and therefore,

$$|H| \leq 4 + \binom{n-8}{2} + 9 = \binom{n-8}{2} + 13 < \binom{n-6}{2} + 10. \quad \blacksquare$$

Fact 6. If $n = 12$, $H_1 \neq \emptyset$ and $H \not\subseteq \text{Co}(12)$, then $|H| < 10 + \binom{12-6}{2} = 25$.

Proof. Let us split the proof into five parts according to the size of the set W_1 . We start with $|W_1| = 3$. Then $|W_0| = 4$, $|U \cup W_0| = 9$, $|H[W]| = 1$ and by (14), $|H_1| \leq 3$. Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$

Further, as $H \not\subseteq \text{Co}(12)$, either $F_1^2 \neq \emptyset$ or $H[U \cup W_0] \not\subseteq S_{n-3}$. In the former case we use Fact 2 to get $|H[U \cup W_0]| \leq 19$. Otherwise, $H[U \cup W_0] \not\subseteq S_{n-3}$, and since $H[U \cup W_0] \neq K_6 \cup K_3$, by Theorems 7 and 8, $|H[U \cup W_0]| < 21$. In both cases $|H[U \cup W_0]| \leq 20$ and therefore

$$|H| \leq |H[U \cup W_0]| + 4 \leq 20 + 4 = 24 < 25.$$

For $|W_1| = 4$ we have $|W_0| = 3$, $|U \cup W_0| = 8$ and $|H[W]| \leq \binom{4}{3} = 4$. If $F_1^2 = \emptyset$, then $H_1 = F_1^0 \neq \emptyset$ and as for each $h \in F_1^0$ the pair $h \cap W_1$ is nonseparable, $|H_1| = 1$ and by Fact 1, $|H[W]| \leq 2$. Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{7}{2} + 1 + 2 = 24 < 25.$$

Otherwise, $F_1^2 \neq \emptyset$ and we can use Fact 2 to get $|H[U \cup W_0]| \leq 16$. For $F_1^0 \neq \emptyset$, $|H[W]| = 2$ and consequently, by (4) and (14),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 16 + 5 + 2 = 23 < 25.$$

Whereas for $F_1^0 = \emptyset$ we use (4), (7) and (13) to get

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 16 + 4 + 4 = 24 < 25.$$

Now let $|W_1| = 5$, $|W_0| = 2$, $|U \cup W_0| = 7$ and $|H[W]| \leq \binom{5}{3} = 10$. For $F_1^2 \neq \emptyset$, by Fact 2 we get $|H[U \cup W_0]| \leq 13$ and moreover $|H[W]| \leq 6$, because otherwise

we would not be able to avoid a path P in H . If additionally $P_2 \subseteq H[W]$, then again by $P \not\subseteq H$, $|H_1| = |F_1^0| + |F_1^2| \leq 2 + 2 = 4$. Hence, by (4)

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 13 + 4 + 6 = 23 < 25.$$

Otherwise, $P_2 \not\subseteq H[W]$ and consequently one can show that $|H[W]| \leq |W_1| - 2 = 3$. Therefore, by (4) and (14),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 13 + 7 + 3 = 23 < 25.$$

For $F_1^2 = \emptyset$ we have $F_1^0 \neq \emptyset$. Hence, since for each $h \in F_1^0$ the pair $h \cap W_1$ is nonseparable, $|H_1| = |F_1^0| \leq 2$ and by Fact 1, $|H[W]| \leq 4$. Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{7-1}{2} + 2 + 4 = 21 < 25.$$

We move to $|W_1| = 6$. Then $|W_0| = 1$, $|U \cup W_0| = 6$ and by (6), $|H[W]| \leq \binom{6-1}{2} = 10$. Let us again start with the case $F_1^2 \neq \emptyset$. By (16) we get $|H[U \cup W_0]| \leq 8$. If $P_2 \subseteq H[W]$, then since H is P -free, $|H_1| = |F_1^0| + |F_1^2| \leq 2 + 4 = 6$. Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 8 + 6 + 10 = 24 < 25.$$

Otherwise, $P_2 \not\subseteq H[W]$ and therefore one can show that $|H[W]| \leq |W_1| - 2 = 4$. By (14), $|H_1| \leq 9$ and consequently by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 8 + 9 + 4 = 21 < 25.$$

For $F_1^2 = \emptyset$ we have $F_1^0 \neq \emptyset$, thus by Fact 1, $|H[W]| \leq 8$ and by (11), $|H_1| = |F_1^0| \leq 6$. Therefore, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 10 + 6 + 8 = 24 < 25.$$

Finally, $|W_1| = 7$, $W_0 = \emptyset$ and by (6), $|H[W]| \leq \binom{7-1}{2} = 15$. First assume that $H[W]$ is a subset of the star S_7 with the center $y \in W_1$. Further assume also that $F^2 \setminus F^2(y) \neq \emptyset$, say, there exists $v \in W_1$, $v \neq y$, such that $F^2(v) \neq \emptyset$. Let $h \in H[W_1]$ be the edge containing y and v (because $H[W] \subseteq S_7$). Since H is P -free, for any $h' \in H[W_1]$, $h' \neq h$, it holds that $|h' \cap h| \geq 2$ and $y \in h' \cap h$. Thus $|H[W_1]| \leq 1 + 2(|W_1| - 3) = 9$. By (15), we have $|H| \leq 2|W_1| - 1 + 9 = 22 < 25$. Otherwise, $|F^2| = |F^2(y)| \leq 2$. If additionally $F_1^0 = \emptyset$, then by (3), (7) and (12),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 2 + 15 = 23 < 25.$$

Otherwise $F_1^0 \neq \emptyset$, thus $|H_1| = |F_1^0| + |F_1^2| \leq 3 + 2 = 5$ and by Fact 1, $|H[W]| \leq 7$. Therefore by (3) and (12),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 5 + 7 = 18 < 25.$$

The last case we have to consider is $H[W] \not\subseteq S_7$. If $M \subseteq H[W]$, then by Lemma 14, $|H[W]| \leq \text{ex}(7; \{P, C\} | M) = 10$. Otherwise, by Lemma 16, $|H[W]| \leq \text{ex}^{(2)}(7; \{M, C\}) = 10$. Hence by (3) and (15),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 13 + 10 = 23 < 25. \quad \blacksquare$$

Fact 7. For $n \geq 12$, if $|W_1| \geq 5$ and $H_1 \neq \emptyset$, then

$$(17) \quad |H| < \binom{n-6}{2} + 10.$$

Proof. The proof is by induction on n with the initial step $n = 12$ done in Fact 6. Let $n \geq 13$. For $W_0 = \emptyset$ the inequality (17) results from Fact 3. Otherwise, there exist a vertex $v \in W_0$. Notice that since $|W_1| \geq 5$, we have $|W_0| \leq n - 10$ and consequently, by (10), $|H(v)| \leq 4 + \max\{2, |W_0| - 1\} \leq 4 + n - 11 = n - 7$. Finally, by the induction assumption we get $|H - v| < \binom{n-7}{2} + 10$. Therefore,

$$|H| = |H(v)| + |H - v| < n - 7 + \binom{n-7}{2} + 10 = \binom{n-6}{2} + 10. \quad \blacksquare$$

Acknowledgments

I would like to thank both referees for reading the manuscript and for their remarks. I am especially grateful to Referee 4 for his/her valuable comments and suggestions leading to a great improvement in the presentation of the paper.

REFERENCES

- [1] R. Csákány and J. Kahn, *A homological approach to two problems on finite sets*, J. Algebraic Combin. **9** (1999) 141–149.
doi:10.1023/A:1018630111976
- [2] P. Erdős, C. Ko and R. Rado, *Intersection theorems for systems of finite sets*, Q. J. Math. **12** (1961) 313–320.
doi:10.1093/qmath/12.1.313
- [3] P. Frankl and Z. Füredi, *Exact solution of some Turán-type problems*, J. Combin. Theory Ser. A **45** (1987) 226–262.
doi:10.1016/0097-3165(87)90016-1
- [4] P. Frankl and Z. Füredi, *Non-trivial intersecting families*, J. Combin. Theory Ser. A **41** (1986) 150–153.
doi:10.1016/0097-3165(86)90121-4

- [5] Z. Füredi, T. Jiang and R. Seiver, *Exact solution of the hypergraph Turán problem for k -uniform linear paths*, *Combinatorica* **34** (2014) 299–322.
doi:10.1007/s00493-014-2838-4
- [6] A. Gyárfás and G. Raeisi, *The Ramsey number of loose triangles and quadrangles in hypergraphs*, *Electron. J. Combin.* **19** (2012) # R30.
- [7] J. Han and Y. Kohayakawa, *The maximum size of a non-trivial intersecting uniform family that is not a subfamily of the Hilton-Milner family*, *Proc. Amer. Math. Soc.* **145** (2017) 73–87.
doi:101090/proc/13221
- [8] A.J.W. Hilton, and E.C. Milner, *Some intersection theorems for systems of finite sets*, *Q. J. Math.* **18** (1967) 369–384.
doi:10.1093/qmath/18.1.369
- [9] E. Jackowska, *The 3-color Ramsey number for a 3-uniform loose path of length 3*, *Australas. J. Combin.* **63** (2015) 314–320.
- [10] E. Jackowska, J. Polcyn and A. Ruciński, *Turán numbers for linear 3-uniform paths of length 3*, *Electron. J. Combin.* **23(2)** (2016) #P2.30.
- [11] E. Jackowska, J. Polcyn and A. Ruciński, *Multicolor Ramsey numbers and restricted Turán numbers for the loose 3-uniform path of length three*, submitted.
- [12] A. Kostochka, D. Mubayi and J. Verstraëte, *Turán problems and shadows I: paths and cycles*, *J. Combin. Theory Ser. A* **129** (2015) 57–79
doi:10.1016/j.jcta.2014.09.005
- [13] G.R. Omid and M. Shahsiah, *Ramsey numbers of 3-uniform loose paths and loose cycles*, *J. Combin. Theory Ser. A* **121** (2014) 64–73.
doi:10.1016/j.jcta.2013.09.003
- [14] J. Polcyn and A. Ruciński, *A hierarchy of maximal intersecting triple systems*, *Opuscula Math.* **37(4)** (2017), to appear.
arXiv:1608.06114
- [15] J. Polcyn and A. Ruciński, *Refined Turán and Ramsey numbers for the loose 3-uniform path of length three*, *Discrete Math.* **340** (2017) 107–118.
doi:10.1016/j.disc.2016.08.006

Received 28-12-2015

Revised 9-05-2016

Accepted 9-05-2016