Discussiones Mathematicae
Graph Theory 37 (2017) 273–293
doi:10.7151/dmgt.1934

COMPUTING THE METRIC DIMENSION OF A GRAPH FROM PRIMARY SUBGRAPHS

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Abstract

Let G be a connected graph. Given an ordered set $W = \{w_1, \ldots, w_k\} \subseteq V(G)$ and a vertex $u \in V(G)$, the representation of u with respect to W is the ordered k-tuple $(d(u, w_1), d(u, w_2), \ldots, d(u, w_k))$, where $d(u, w_i)$ denotes the distance between u and w_i. The set W is a metric generator for G if every two different vertices of G have distinct representations. A minimum cardinality metric generator is called a *metric basis* of G and its cardinality is called the *metric dimension* of G. It is well known that the problem of finding the metric dimension of a graph is NP-hard. In this paper we obtain closed formulae for the metric dimension of graphs with cut vertices. The main results are applied to specific constructions including rooted product graphs, corona product graphs, block graphs and chains of graphs.

Keywords: metric dimension, metric basis, primary subgraphs, rooted product graphs, corona product graphs.

2010 Mathematics Subject Classification: 05C12, 05C76.

1. Introduction

Graph structures may be used to model computer networks. Servers, hosts or hubs in a network can be represented as vertices in a graph and edges could represent connections between them. Each vertex in a graph is a possible location for an intruder (fault in a computer network, spoiled device) and this fact motivates the necessity of uniquely recognize each vertex of a graph, i.e., the possible location of an intruder in a network. This necessity gave rise to the notion of locating sets and locating number of graphs, introduced by Slater in [26, 27]. Harary and Melter [17] also introduced independently the same concept, but using the terms resolving sets and metric dimension instead of locating sets and locating number, respectively. Moreover, in a more recent article by Sebö and Tannier [25], the terminology of metric generators and metric dimension for the concepts mentioned above, began to be used. In this article we follow the terminology and notation of Sebö and Tannier [25], which come from the general context of metric spaces, as the concept of metric dimension of a general metric space first appeared in 1953 in [2], but it attracted a little attention, except for the case of graphs.

A *generator* of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S [2]. Given a simple and connected graph G, we consider the metric $d_G : V(G) \times V(G) \to \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers and $d_G(x, y)$ is the length of a shortest path between x and y. The pair $(V(G), d_G)$ is readily seen to be a metric space. A vertex $v \in V(G)$ is said to *distinguish* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V(G)$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A metric generator S is *minimal* if no proper subset $S' \subsetneq S$ is a metric generator for G. A minimal metric generator of minimum cardinality is called a *metric basis* and its cardinality, the *metric dimension* of G, is denoted by $\dim(G)$. Moreover, a minimal metric generator of maximum cardinality is called an *upper metric basis* and its cardinality, the *upper metric dimension* of G, is denoted by $\dim^+(G)$. For instance, for complete graphs of order n, $\dim^+(K_n) = \dim(K_n) = n - 1$; for star graphs of order $r + 1 \geq 3$, $\dim^+(K_{1,r}) = \dim(K_{1,r}) = r - 1$; for cycle graphs of order n, $\dim^+(C_n) = \dim(C_n) = 2$; and for path graphs of order $n \geq 3$, $\dim^+(P_n) = 2 > \dim(P_n) = 1$. The concepts of upper metric generator and upper metric dimension were introduced first in [5].

On the other hand, studies about operations on graphs, particularly products of graphs, are being frequently presented and published in the last few decades. The metric dimension of Cartesian product graphs, lexicographic product graphs, strong product graphs, hierarchical product graphs and corona product graphs was studied in [3, 20, 22, 23, 10] and [28], respectively. Furthermore, it was shown in [15] that the problem of finding the metric dimension of a graph is NP-hard even when restricted to planar graphs [8], while other algorithmic or computational results have been presented in [9, 11, 18]. These facts suggest obtaining closed formulae for the metric dimension of special nontrivial families of graphs, or bounding the value of this invariant as tight as possible, or reducing the problem of computing the metric dimension of a graph to computing the value of other parameters in some subgraphs of the original graph. This last possibility regards the case of product graphs or, more general, those graphs obtained throughout some "operations" with other graphs, frequently called factor graphs or primary subgraphs. That is, one can reduce the required computation on the product graph to some similar computation in the factors or primary subgraphs.

Consider now a connected graph G constructed from a family of pairwise disjoint (nontrivial) connected graphs $G_1, \ldots, G_k$ in the following way. Select one vertex of G_1, one vertex of G_2, and identify these two vertices. Afterwards continue this procedure inductively. More precisely, let $G_1, \ldots, G_i$ be already used in the construction, where $i \in \{2, \ldots, k-1\}$. Select one vertex in the already constructed graph (particularly this vertex may be one of the already selected vertices) and one vertex of G_{i+1}, and then identify these two vertices. Figure 1 illustrates a geometrical representation of an example of a graph obtained in this manner. The concept above was introduced in [6], where the authors used it to compute the Hosoya polynomials of a graph. Moreover, this construction was used in [7] to study the terminal Hosoya polynomial of composite graphs and in [21] to compute the local metric dimension of graphs with cut vertices.

We say, as in [6], that G is obtained by *point-attaching* from $G_1, \ldots, G_k$ and that G_i's are the *primary subgraphs* of G. Furthermore, the vertices of G obtained by identifying two vertices of different primary subgraphs are the *attachment vertices* of G. Notice that the attachment vertices are cut vertices in the graph G. We denote by $A(G)$ the set of attachment vertices of G and by $A(G_i)$ the set of attachment vertices of G belonging to $V(G_i)$, i.e., $A(G_i) = A(G) \cap V(G_i)$. Observe that any graph constructed by point-attaching from a family of connected graphs has a tree-like structure, where the primary subgraphs are its building stones. Moreover, for any $x, y \in V(G_i)$ it holds $d_G(x, y) = d_{G_i}(x, y)$.

Examples of graphs obtained by point-attaching are block graphs, cactus graphs, corona product graphs, rooted product graphs, bouquets of graphs, circuits of graphs, chains of graphs, etc.

We say that a primary subgraph G_i is a *primary end-subgraph* whenever $|A(G_i)| = 1$ and it is a *primary internal subgraph* whenever $|A(G_i)| \geq 2$. For instance, G_2, G_3, G_6, G_7, G_9, G_{10} and G_{11} are primary end-subgraphs of the graph G illustrated in Figure 1, while G_1, G_4, G_5 and G_8 are primary internal subgraphs. In this case, $A(G_1) = \{a, b, c\}$, $A(G_2) = \{a\}$, $A(G_3) = \{b\}$ and so on. Clearly, any graph obtained by point attaching contains at least two primary end-subgraphs.

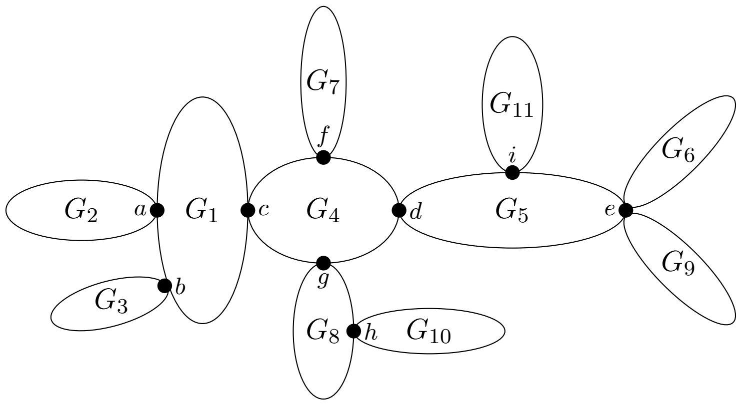

Figure 1. Sketch of a graph G constructed by point-attaching from the primary subgraphs $G_1, \ldots, G_{11}$.

In this paper we obtain closed formulae for the metric dimension of graphs obtained by point-attaching. The main result is applied to specific constructions including rooted product graphs, corona product graphs, block graphs and chain graphs. To begin with, we need to introduce some additional notation and terminology. Given a simple graph G, the neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$ and the eccentricity by $\epsilon_G(v)$. The diameter of G is denoted by $D(G)$, and given a set $S \subset V(G)$, the subgraph of G induced by S is denoted by $\langle S \rangle$. A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$. For example, even cycles and hypercubes are 2-antipodal graphs. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2. Main Results

We begin our exposition with a lower bound on the metric dimension of graphs from primary subgraphs in the general case. That is, when there is no rule for the construction of the graphs by point-attaching. Such constructions are of course depending on the attachment vertices of the primary subgraphs and, therefore, relatively complicate to deal with. In this sense, we shall use an extra parameter specifically related to the metric dimension of graphs from primary subgraphs, which we define below.

Let G be a graph obtained by point-attaching from $G_1, \ldots, G_k$. An *attaching metric generator* for a primary subgraph G_i is a set $W \subseteq V(G_i)$ such that $W \cup A(G_i)$ is a metric generator for G_i. A minimum cardinality attaching metric generator is called an *attaching metric basis* and its cardinality, the *attaching metric dimension* of G_i, is denoted by $\dim^*(G_i)$. For instance, assume that

$A(G_i) = \{v\}$. If v does not belong to any metric basis of G_i, then $\dim^*(G_i) = \dim(G_i)$, and if v belongs to a metric basis of G_i, then $\dim^*(G_i) = \dim(G_i) - 1$. In particular, for a path graph, a cycle graph or a complete graph of order n we have the following.

$$\dim^*(P_n) = \begin{cases} 1, & \text{if } P_n \text{ has exactly one attachment vertex which has degree 2;} \\ 0, & \text{otherwise.} \end{cases}$$

$$\dim^*(C_n) = \begin{cases} 1, & \text{if } C_n \text{ has exactly one attachment vertex or } (C_n \text{ has exactly} \\ & \text{two attachment vertices which are antipodal and } n \text{ is even);} \\ 0, & \text{otherwise.} \end{cases}$$

$$\dim^*(K_n) = \begin{cases} 0, & \text{if every vertex of } K_n \text{ is an attachment vertex,} \\ n - |A(K_n)| - 1, & \text{otherwise.} \end{cases}$$

We are now able to state the following lower bound.

Proposition 1. *For any graph G obtained by point-attaching from a family of connected graphs $G_1, \dots, G_k$,*

$$\dim(G) \geq \sum_{i=1}^{k} \dim^*(G_i).$$

Proof. Let M be a metric basis of G and let $M_i = M \cap (V(G_i) \setminus A(G_i))$, where $i \in \{1, \dots, k\}$. We claim that $M_i \cup A(G_i)$ is a metric generator for G_i. Let u and v be two different vertices of G_i. If u and v are not distinguished by any vertex in M_i, then they are distinguished by some vertex $y \in M_j \cup (M \cap A(G_j))$ for some $j \neq i$. Let $x \in A(G_i)$ be such that $d_G(y, x) = \min_{w \in V(G_i)}\{d_G(y, w)\}$. Hence, $d_G(u, y) = d_G(u, x) + d_G(x, y)$ and $d_G(v, y) = d_G(v, x) + d_G(x, y)$. Since $d_G(u, y) \neq d_G(v, y)$, we have that $d_G(u, x) \neq d_G(v, x)$. So, $M_i \cup A(G_i)$ is a metric generator for G_i and, as a consequence, M_i is an attaching metric generator for G_i. Therefore, $|M_i| \geq \dim^*(G_i)$ and it follows that

$$\dim(G) = |M| = \sum_{i=1}^{k} |M_i| + \left| M \cap \left(\bigcup_{i=1}^{k} A(G_i) \right) \right| \geq \sum_{i=1}^{k} |M_i| \geq \sum_{i=1}^{k} \dim^*(G_i).$$ ■

In order to show that the bound above is tight, we introduce some restrictions on the structure of the graphs obtained from primary subgraphs. Given a graph G constructed by point-attaching, we define the following properties of a primary subgraph G_i.

Property $\mathcal{P}_1$: For any $a \in A(G_i)$ and $z \in V(G_i) \setminus A(G_i)$ there exists $b \in A(G_i)$ such that $d_{G_i}(a, b) \geq d_{G_i}(z, b)$.

Property $\mathcal{P}_2$: $A(G_i) = \{v\}$ and one of the following conditions holds:
G_i is not a path, or
G_i is a path and v is not a leaf.

Notice that property $\mathcal{P}_1$ is satisfied by a wide family of connected graphs. For instance, when a primary internal subgraph G_i satisfies one of the following conditions.

- $A(G_i) = V(G_i)$.
- $D(G_i) = 2$ and $A(G_i)$ is any independent set for G_i.
- $\epsilon_{G_i}(x) = \epsilon_{G_i}(y) = d_{G_i}(x, y)$ for any pair of different vertices $x, y \in A(G_i)$. In particular, complete nontrivial graphs are included in this case.
- G_i is 2-antipodal and $A(G_i)$ is a set such that if $u \in A(G_i)$, then its antipodal vertex also belongs to $A(G_i)$.

It was shown in [4] that $\dim(H) = 1$ if and only if H is a path. Also, $\{v\}$ is a metric basis of a path graph if and only if v is a leaf. Hence, if G_i satisfies $\mathcal{P}_2$, then $\dim^*(G_i) \geq 1$.

Theorem 2. *Let G be a graph obtained by point-attaching from a family of connected graphs $G_1, \ldots, G_k$, $k \geq 3$, such that every primary internal subgraph satisfies $\mathcal{P}_1$, every primary end-subgraph satisfies $\mathcal{P}_2$, and $A(G_i) \cap A(G_j) = \emptyset$ for any pair G_i, G_j of primary end-subgraphs. Then*

$$\dim(G) = \sum_{i=1}^{k} \dim^*(G_i).$$

Proof. By Proposition 1, $\dim(G) \geq \sum_{i=1}^{k} \dim^*(G_i)$. It remains to prove that $\dim(G) \leq \sum_{i=1}^{k} \dim^*(G_i)$.

Let S_i be an attaching metric basis of G_i, $i \in \{1, \ldots, k\}$. We shall show that $S = \bigcup_{i=1}^{k} S_i$ is a metric generator for G. To this end, we consider the following cases for two different vertices $x, y \in V(G)$.

Case 1. $x, y \in V(G_i)$. Since $S_i \cup A(G_i)$ is a metric generator for G_i, there exists $u \in S_i \cup A(G_i)$ such that $d_{G_i}(x, u) \neq d_{G_i}(y, u)$. If $u \in S_i$, then we are done. Now, if $u \in A(G_i)$, then there exists a primary end-subgraph G_j, $j \neq i$, such that for any $w \in S_j$, $d_G(u, w) = \min_{v \in V(G_i)}\{d_G(v, w)\}$. Notice that since G_j satisfies $\mathcal{P}_2$, $S_j \neq \emptyset$. Hence,

$$d_G(x, w) = d_G(x, u) + d_G(u, w) \neq d_G(y, u) + d_G(u, w) = d_G(y, w).$$

Case 2. $x \in V(G_i)$ and $y \in V(G_j)$, where $i \neq j$. Let $a \in V(G_i)$ and $b \in V(G_j)$ be the attachment vertices such that $d_G(x, y) = d_G(x, a) + d_G(a, b) +$

$d_G(b, y)$. Note that if G_i and G_j have a common attachment vertex, then $a = b$. If $y = b = a$ or $x = a = b$, then we proceed as in Case 1, so we assume that x and y do not belong to the same primary subgraph, i.e., $y \neq a$ and $x \neq b$.

Subcase 2.1. $|A(G_i)| \geq 2$ or $|A(G_j)| \geq 2$. Without loss of generality, we assume that $|A(G_i)| \geq 2$. Since G_i satisfies $\mathcal{P}_1$, there exists $c \in A(G_i) \setminus \{a\}$, such that $d_{G_i}(a, c) \geq d_{G_i}(x, c)$. Now, let G_l, $l \neq i$, be a primary end-subgraph such that for any $t \in S_l$, $d_G(c, t) = \min_{v \in V(G_i)}\{d_G(v, t)\}$ ($S_l \neq \emptyset$, as G_l satisfies $\mathcal{P}_2$). Then for any $t \in S_l$,

$$\begin{aligned} d_G(x, t) &= d_G(x, c) + d_G(c, t) \leq d_G(a, c) + d_G(c, t) \\ &< d_G(y, a) + d_G(a, c) + d_G(c, t) = d_G(y, t). \end{aligned}$$

Subcase 2.2. $|A(G_i)| = |A(G_j)| = 1$. Clearly G_i and G_j are primary end-subgraphs and since they satisfy $\mathcal{P}_2$, it follows that S_i and S_j are not empty. Hence, let $p \in S_i$ and $q \in S_j$. If x, y are distinguished by p or q, then we are done. On the contrary, suppose that neither p nor q distinguish the vertices x and y. So, we have that

$$d_G(x, p) = d_G(y, p) = d_G(y, b) + d_G(b, a) + d_G(a, p) \tag{1}$$

and

$$d_G(y, q) = d_G(x, q) = d_G(x, a) + d_G(a, b) + d_G(b, q). \tag{2}$$

Observe that since $A(G_i) \cap A(G_j) = \emptyset$, we have $a \neq b$. Moreover,

$$d_G(x, p) \leq d_G(x, a) + d_G(a, p) \tag{3}$$

and

$$d_G(y, q) \leq d_G(y, b) + d_G(b, q). \tag{4}$$

From (1) and (3) we obtain

$$d_G(y, b) + d_G(b, a) + d_G(a, p) \leq d_G(x, a) + d_G(a, p), \tag{5}$$

and from (2) and (4)

$$d_G(x, a) + d_G(a, b) + d_G(b, q) \leq d_G(y, b) + d_G(b, q). \tag{6}$$

Finally, by adding (5) and (6) we have the following inequality

$$2 \cdot d_G(a, b) \leq 0,$$

which is a contradiction.

According to the two cases above, $\dim(G) \leq \sum_{i=1}^{k} \dim^*(G_i)$. ∎

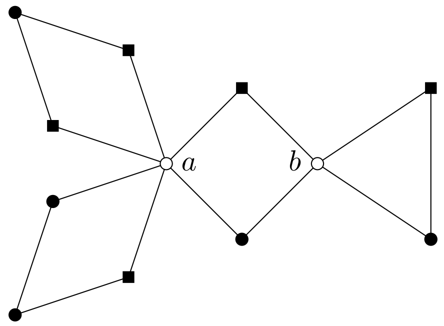

Figure 2. A graph G obtained by point-attaching from $G_1 \cong G_2 \cong G_3 \cong C_4$ and $G_4 \cong C_3$. In this case $A(G_1) = \{a, b\}$, $A(G_2) = A(G_3) = \{a\}$, $A(G_4) = \{b\}$, G_1 satisfies $\mathcal{P}_1$ while G_2, G_3 and G_4 satisfy $\mathcal{P}_2$. Square-shaped vertices form a metric basis of G.

Figure 2 shows an example of a graph that violates condition $A(G_i) \cap A(G_j) = \emptyset$ of Theorem 2, where also $\dim(G) \neq \sum_{i=1}^{k} \dim^*(G_i)$. In this case $A(G_2) \cap A(G_3) = \{a\}$ and $\dim(G) = 5 > 4 = \sum_{i=1}^{4} \dim^*(G_i)$.

The next sections are devoted to derive some consequences of Theorem 2. That is, we give closed formulae for the metric dimension of some specific families of graphs in terms of some parameters of its primary subgraphs, when the point-attaching process can be described as a graph composition scheme or when the primary subgraphs satisfy some specific property.

3. An Extremal Case

As above, let G be a graph obtained by point-attaching from $G_1, \ldots, G_k$. In this section we study the case where every minimal metric generator for a primary subgraph is minimum, i.e., the case where $\dim(G_i) = \dim^+(G_i)$. Let $\mathcal{B}(G_i)$ be the set of metric bases of G_i and let

$$\tau_i = \max_{B_j \in \mathcal{B}(G_i)} \{|A(G_i) \cap B_j|\}.$$

That is, τ_i quantifies the maximum number of attachment vertices of G belonging simultaneously to a metric basis of G_i.

Corollary 3. *Let G be a graph obtained by point-attaching from a family of connected graphs $G_1, \ldots, G_k$, $k \geq 3$, such that every primary internal subgraph satisfies $\mathcal{P}_1$, every primary end-subgraph satisfies $\mathcal{P}_2$, for any pair G_i, G_j of primary end-subgraphs $A(G_i) \cap A(G_j) = \emptyset$ and $\dim(G_l) = \dim^+(G_l)$, whenever $A(G_l) \neq V(G_l)$. Then*

$$\dim(G) = \sum_{i=1}^{k} (\dim(G_i) - \tau_i).$$

Proof. It is readily seen that for any primary subgraph G_i of G such that $\dim(G_i) = \dim^+(G_i)$, we have $\dim^*(G_i) = \dim(G_i) - \tau_i$. Therefore, the result is a direct consequence of Theorem 2. ■

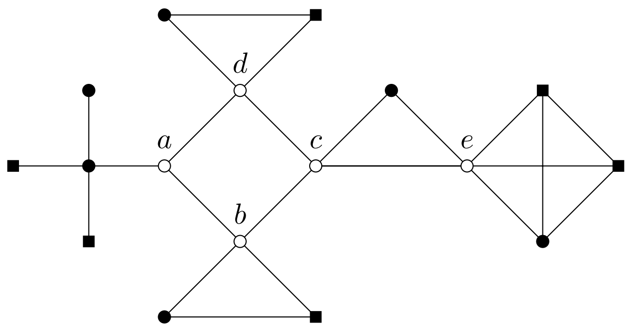

Figure 3. A graph G obtained by point-attaching from $G_1 \cong K_{1,4}$, $G_2 \cong C_4$, $G_3 \cong G_4 \cong G_5 \cong K_3$ and $G_6 \cong K_4$. In this case $A(G_1) = \{a\}$, $A(G_2) = \{a, b, c, d\}$, $A(G_3) = \{b\}$, $A(G_4) = \{c, e\}$, $A(G_5) = \{d\}$ and $A(G_6) = \{e\}$. Square-shaped vertices form a metric basis of G.

For the graph G shown in Figure 3 we have $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 1$, $\tau_4 = 2$, $\tau_5 = 1$ and $\tau_6 = 1$. In this case Corollary 3 leads to $\dim(G) = 6$.

A *block graph* is a graph in which every biconnected component (block) is a clique. Note that any block graph is obtained by point-attaching from a family of complete graphs. For any complete graph of order $n \geq 2$, $\dim(K_n) = n - 1 = \dim^+(K_n)$. Then the following remark is a particular case of Corollary 3.

Remark 4. Let G be a block graph obtained from a family of complete graphs $\{K_{r_1}, \ldots, K_{r_k}\}$, $k \geq 3$, such that any primary end-subgraph is different from K_2 and any two primary end-subgraphs have no common attachment vertex. Then

$$\dim(G) = \sum_{|A(K_{r_i})| < r_i} (r_i - |A(K_{r_i})| - 1).$$

4. Rooted Product Graphs

We continue in this section with an interesting particular case of graphs obtained by point-attaching: the rooted product of graphs. We must recall that some results on the metric dimension of rooted product graphs were already presented in [10] where the authors studied some variation on metric dimension which they call *rooted metric dimension*. As a consequence of their study, some closed formulae for the metric dimension of rooted product graphs were deduced. Nevertheless, several aspects on this topic were remaining from this work and also, the generalized version of rooted product graphs was not studied. We next give more results about that.

A *rooted graph* is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The special vertex is called the *root* of the graph. Let G be a labeled graph on n vertices and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of rooted graphs. The *rooted product graph* $G(\mathcal{H})$ is the graph obtained by identifying the root of H_i with the ith vertex of G [16]. Clearly, any rooted product graph $G[\mathcal{H}]$ is a graph obtained by point-attaching from the primary internal subgraph G, where $A(G) = V(G)$, and the family $\mathcal{H}$ consists of primary end-subgraphs having its attachment vertices in its roots. From Theorem 2 we deduce our next result.

Corollary 5. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed of rooted graphs satisfying $\mathcal{P}_2$, with roots $v_1, \ldots, v_n$, respectively. Then*

$$\dim(G(\mathcal{H})) = \sum_{H_i \in \mathcal{H}_1} \dim(H_i) + \sum_{H_i \in \mathcal{H}_2} (\dim(H_i) - 1),$$

where $H_i \in \mathcal{H}_1$ if v_i does not belong to any metric basis of H_i and $H_j \in \mathcal{H}_2$ if v_j belongs to a metric basis of H_j.

We consider now the case of a family of vertex transitive graphs $\mathcal{H}$. We recall that G is a *vertex transitive graph* if every pair of vertices is equivalent under some element of its automorphism group or, equivalently, a vertex-transitive graph is a graph whose automorphism group is transitive. Let $Aut(H)$ be the automorphism group of H. If $x, y \in V(H)$ and $\pi \in Aut(H)$, then $d(x, y) = d(\pi(x), \pi(y))$. So, if S is a metric basis of a connected graph H and $\pi \in Aut(H)$, then $\pi(S)$ is a metric basis of H. Thus, every vertex in a vertex transitive graph belongs to a metric basis and by using Corollary 5 we have the following.

Remark 6. Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of vertex transitive graphs of order greater than two. For any connected graph G of order $n \geq 2$,

$$\dim(G(\mathcal{H})) = \sum_{i=1}^{n} (\dim(H_i) - 1).$$

In particular, if $\mathcal{H} = \{K_{r_1}, \ldots, K_{r_n}\}$, then

$$\dim(G(\mathcal{H})) = \sum_{i=1}^{n} (r_i - 2),$$

and if $\mathcal{H} = \{C_{r_1}, \ldots, C_{r_n}\}$, then

$$\dim(G(\mathcal{H})) = n.$$

A particular case of rooted product graphs is when $\mathcal{H}$ consists of n isomorphic rooted graphs [24] (this was the case studied in [10]). More formally, assuming that $V(G) = \{u_1, \ldots, u_n\}$ and that the root vertex of H is v, we define the rooted product graph $G \circ_v H$, where $V(G \circ_v H) = V(G) \times V(H)$ and

$$E(G \circ_v H) = \bigcup_{i=1}^{n} \{(u_i, b)(u_i, y) :\ by \in E(H)\} \cup \{(u_i, v)(u_j, v) :\ u_i u_j \in E(G)\}.$$

Figure 4 shows two examples of rooted product graphs. We remark that this product was recently renamed as *hierarchical product* in [1].

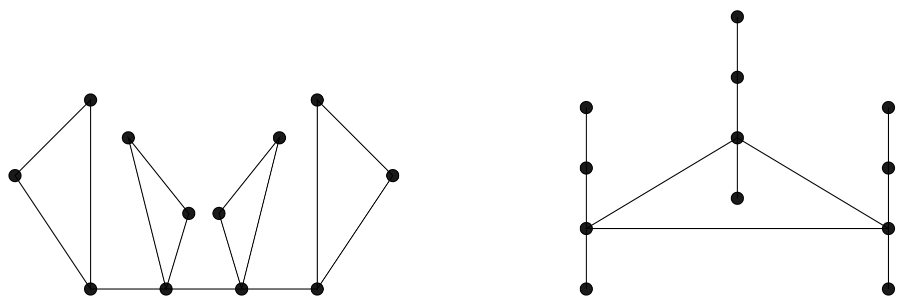

Figure 4. Rooted products $P_4 \circ C_3$ and $C_3 \circ_v P_4$, where v has degree two.

Notice that for the particular case of rooted product graphs $G \circ_v H$, Corollary 5 becomes the next propositions.

Proposition 7 [10]**.** *Let H be a connected graph and let v be a vertex of H. If v does not belong to any metric basis of H, then for any connected graph G of order n,*

$$\dim(G \circ_v H) = n \cdot \dim(H).$$

Proposition 8 [10]**.** *Let H be a connected graph different from a path and let v be a vertex of H. If v belongs to a metric basis of H, then for any connected graph G of order $n \geq 2$,*

$$\dim(G \circ_v H) = n \cdot (\dim(H) - 1).$$

Propositions 7 and 8 give rise to the problem of determining necessary and/or sufficient conditions for a vertex $v \in V(H)$ to belong to a metric basis of H. For instance, it is easy to see that a vertex v of a path P belongs to a metric basis of P if and only if v is a leaf of P. In connection with this fact, by using Proposition 7, we have the following result.

Corollary 9. *Let H be a connected graph and let $v \in V(H)$ be a vertex not belonging to any metric basis of H. For any connected graph G of order n, $\dim(G \circ_v H) = n$ if and only if H is a path graph and the root of H is not a leaf.*

We observe that in Proposition 7 the graph H can be a path whenever the root is not a leaf. However, in Proposition 8 paths are not allowed. This makes interesting the case of rooted product graphs $G \circ_v H$ when the graph H is a path and v is a leaf. For that case, the following lower bound is known.

Proposition 10 [10]**.** *Let P be a path graph and let v be a leaf of P. For any connected graph G of order $n \geq 2$,*

$$\dim(G \circ_v P) \geq \dim(G).$$

To obtain an upper bound we need some extra terminology and notation. A *dominating set* for a graph G is a set $S \subseteq V(G)$ such that every vertex not in S is adjacent to at least one member of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The following well-known upper bound on the domination number of a graph is useful to prove Proposition 12.

Theorem 11 (Ore, 1962)**.** *If a graph G of order n has no isolated vertices, then*

$$\gamma(G) \leq \frac{n}{2}.$$

Given $X \subset V(G)$ we denote by $I(X)$ the set of isolated vertices of $\langle V(G) \setminus X\rangle$. Also, for connected graphs we define $I(G) = \max_{S \in \mathcal{B}(G)}\{|I(S)|\}$, where $\mathcal{B}(G)$ is the set of all the metric bases of G.

Proposition 12. *Let P be a path graph and let v be a leaf of P. For any connected graph G of order $n \geq 2$,*

$$\dim(G \circ_v P) \leq \frac{\dim(G) + n - I(G)}{2}.$$

Proof. Let S be a metric basis of G such that $I(G) = |I(S)|$. Let S' be a dominating set for $\langle V(G) \setminus (S \cup I(S))\rangle$. We show that $(S \cup S') \times \{v'\}$ is a metric generator for $G \circ_v P$, where v' is the leaf of P which is different from v. Let $(x, y), (x', y')$ be two different vertices of $G \circ_v P$. We differentiate the following cases.

Case 1. $y = y'$. In this case $x \neq x'$. So, there exists $u \in S$ such that $d_G(x, u) \neq d_G(x', u)$. If $u = x$ or $u = x'$, say $u = x$, then we clearly have that

$$\begin{aligned} d_{G\circ_v P}((x', y'), (u, v')) &= d_{G\circ_v P}((x', y'), (x, y)) + d_{G\circ_v P}((x, y), (u, v')) \\ &> d_{G\circ_v P}((u, v'), (x, y)). \end{aligned}$$

Now, if $u \neq x$ and $u \neq x'$, then

$$\begin{aligned} d_{G\circ_v P}((u,v'),(x,y)) &= d_P(v',v) + d_G(u,x) + d_P(v,y) \\ &\neq d_P(v',v) + d_G(u,x') + d_P(v,y) \\ &= d_P(v',v) + d_G(u,x') + d_P(v,y') \\ &= d_{G\circ_v P}((u,v'),(x',y')). \end{aligned}$$

Case 2. $x = x'$. In this case (a,v') resolves the pair $(x,y),(x',y')$, for every $a \in S$.

Case 3. $x \neq x'$ and $y \neq y'$. Since, the pair $(x,y),(x',y')$ is resolved by (x,v') and also by (x',v'), we suppose $x,x' \notin S \cup S'$. With this assumption in mind we consider the following subcases.

Case 3.1. $x,x' \notin I(S)$. In such a case, there exist $a,a' \in S'$ such that $x \in N_G(a)$ and $x' \in N_G(a')$. Now, if $x' \in N_G(a)$, then

$$\begin{aligned} d_{G\circ_v P}((a,v'),(x,y)) &= d_P(v',v) + 1 + d_P(v,y) \neq d_P(v',v) + 1 + d_P(v,y') \\ &= d_{G\circ_v P}((a,v'),(x',y')). \end{aligned}$$

Analogously, if $x \in N_G(a')$, then we deduce that (a',v') resolves the pair of vertices $(x,y),(x',y')$. Finally, we suppose that $x \notin N_G(a')$ and $x' \notin N_G(a)$. Since $d_G(a,x') \geq 2$,

$$d_{G\circ_v P}((a,v'),(x,y)) = d_P(v',v) + 1 + d_P(v,y)$$

and

$$d_{G\circ_v P}((a,v'),(x',y')) = d_P(v',v) + d_G(a,x') + d_P(v,y'),$$

we deduce that if (a,v') does not resolve the pair $(x,y),(x',y')$, then $d_P(v,y') < d_P(v,y)$ and, as a consequence,

$$\begin{aligned} d_{G\circ_v P}((a',v'),(x',y')) &= d_P(v',v) + 1 + d_P(v,y') < d_P(v',v) + 1 + d_P(v,y) \\ &< d_P(v',v) + d_G(a',x) + d_P(v,y) = d_{G\circ_v P}((a',v'),(x,y)). \end{aligned}$$

Case 3.2. $x \in I(S)$ and $x' \notin I(S)$. In this case there exist $a \in S$ and $a' \in S'$ such that $x \in N_G(a)$ and $x' \in N_G(a')$. Now we proceed as in Case 3.1. If $x' \in N_G(a)$, then we obtain that (a,v') resolves the pair $(x,y),(x',y')$. Analogously, if $x' \notin N_G(a)$, then we deduce that either the pair $(x,y),(x',y')$ is resolved by (a,v') or it is resolved by (a',v').

Case 3.3. $x,x' \in I(S)$. In this case we take $a,a' \in S$ such that $x \in N_G(a)$ and $x' \in N_G(a')$ and we proceed as in Case 3.1.

Hence, $(S \cup S') \times \{v'\}$ is a metric generator for $G \circ_v P$. Moreover, by Theorem 11 we have $|S'| \leq \frac{n - \dim(G) - |I(S)|}{2}$. Therefore,

$$\dim(G \circ_v P) \leq |S| + |S'| \leq \dim(G) + \frac{n - \dim(G) - |I(S)|}{2} = \frac{\dim(G) + n - I(G)}{2}.$$

■

By Propositions 10 and 12 we obtain the following.

Proposition 13. *Let G be a connected graph of order $n \geq 2$ and let v be a leaf of a path graph P. If $I(G) = n - \dim(G)$, then $\dim(G \circ_v P) = \dim(G)$.*

The converse of Proposition 13 is false. For instance, $\dim(C_4 \circ_v P) = \dim(C_4) = 2$, while $I(C_4) = 0$.

Note that $I(K_n) = n - \dim(K_n) = 1$. Now we construct a family $\mathcal{F}$ of graphs where $I(G) = n - \dim(G)$, for every $G \in \mathcal{F}$. We begin with the star $S_{1,t}$, $t \geq 3$, with the center v and the set of leaves $X = \{x_1, x_2, \ldots, x_t\}$. Then to obtain a graph $G_t \in \mathcal{F}$ we add the set of vertices $Y = \{y_1, y_2, \ldots, y_t\}$ and edges $x_i y_j$ for every $i, j \in \{1, \ldots, t\}$ with $i \neq j$. Notice that for every $i, j \in \{1, \ldots, t\}$, $i \neq j$, it follows $d(v, x_i) = 1$, $d(v, y_i) = 2$, $d(x_i, x_j) = 2$, $d(y_i, y_j) = 2$, $d(x_i, y_j) = 1$ and $d(x_i, y_i) = 3$. Also, note that the graph $G_t \in \mathcal{F}$ can be obtained from the complete bipartite graph $K_{t,t+1}$ by removing a maximum matching. The graph G_4 is shown in Figure 5.

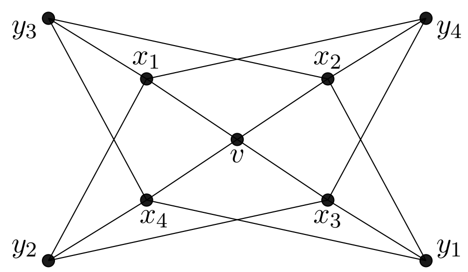

Figure 5. The graph G_4 satisfies $I(G_4) = n - \dim(G_4) = 5$. The set $X = \{x_1, x_2, x_3, x_4\}$ is a metric basis of G_4.

Proposition 14. *For any graph $G_t \in \mathcal{F}$ of order n, $I(G_t) = n - \dim(G_t)$.*

Proof. With the notation above we show that $X = \{x_1, x_2, \ldots, x_t\}$ is a metric basis of G_t. Since for every $i, j \in \{1, \ldots, t\}$, $i \neq j$, $d(x_i, y_j) = 1$, $d(x_i, y_i) = 3$ and $d(v, x_i) = 1$, we have that X is a metric generator of G_t and, as a consequence, $\dim(G_t) \leq t$. Let S be a set of vertices of G_t such that $|S| < t$. We shall prove that S is not a metric generator. To this end, we consider the following cases.

Case 1. $S \subsetneq X$. Let $x_j \notin S$. Since $d(x_l, v) = d(x_l, y_j) = 1$ for $l \neq j$, we have that S is not a metric generator.

Case 2. $S \subsetneq Y$. Let $y_j \notin S$. Since $d(v, y_l) = d(y_l, y_j) = 2$ for $l \neq j$, we have that S is not a metric generator.

Case 3. $S \subsetneq X \cup \{v\}$ and $v \in S$. So, there exist at least two vertices $x_i, x_j \notin S$, $i \neq j$. Notice that $d(x_i, v) = d(x_j, v) = 1$ and $d(x_i, x_l) = d(x_j, x_l) = 2$ for every $l \neq i, j$. Thus, S is not a metric generator.

Case 4. $S \subsetneq Y \cup \{v\}$ and $v \in S$. So, there exist at least two vertices $y_i, y_j \notin S$, $i \neq j$. Notice that $d(y_i, v) = d(y_j, v) = 2$ and $d(y_i, y_l) = d(y_j, y_l) = 2$ for every $l \neq i, j$. Thus, S is not a metric generator.

Case 5. $S \cap X \neq \emptyset$, $S \cap Y \neq \emptyset$ and $v \notin S$. Since $|S| < t$, we can assume that there exists $y_j \notin S$ such that also $x_j \notin S$. Hence we have that $d(y_j, x_l) = d(v, x_l) = 1$ for every $x_l \in S \cap X$ and $d(y_j, y_k) = d(v, y_k) = 2$ for every $y_k \in S$. Thus, S is not a metric generator.

Case 6. $S \cap X \neq \emptyset$, $S \cap Y \neq \emptyset$ and $v \in S$. Since $|S \cap (X \cup Y)| \leq t - 2$, there exist $i, j \in \{1, \ldots, t\}$, $i \neq j$, such that $x_i, x_j, y_i, y_j \notin S$. Notice that $d(x_i, v) = d(x_j, v) = 1$, $d(x_i, x_l) = d(x_j, x_l) = 2$ for every $x_l \in S$ and $d(x_i, y_k) = d(x_j, y_k) = 1$ for every $y_k \in S$. Thus, S is not a metric generator.

As a consequence of the cases above, we have that there is no metric generator for G_t with cardinality less than $|X|$. Therefore, the set X is a metric basis of G_t and, as a consequence, $\dim(G_t) = t$. Finally, since G_t has order $n = 2t+1$ and the subgraph induced by $Y \cup \{v\}$ is empty, we obtain $I(G_t) = n - \dim(G_t) = t+1$. ∎

Although the graphs belonging to the family $\mathcal{F}$ are bipartite, Proposition 14 does not hold for the general case of bipartite graphs. In order to show a graph where $|n - \dim(G) - I(G)|$ is arbitrarily large, we define the graph G as follows. We take a connected graph H of order r (arbitrarily large) and r copies of the empty graph N_s of order $s \geq 2$ and then we construct G by adding an edge between each vertex of the ith copy of N_s and the ith vertex of H. In this case $I(G) = 0$ and, as shown in [28], $\dim(G) = r(s-1)$. Hence, $|n - \dim(G) - I(G)| = 2r$. Notice that if H is bipartite, then G is bipartite.

We continue observing the case when the roots of the paths in a rooted product graph $G \circ_v P$ are leaves, but now we consider the case when G is a tree. A vertex of degree at least 3 in a tree T is called a *major vertex* of T. Any leaf u of T is said to be a *terminal vertex* of a major vertex v of T if $d_T(u, v) < d_T(u, w)$ for every other major vertex w of T. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree. Let $n_1(T)$ denotes the number of leaves of T, and let $ex(T)$ denotes the number of exterior major vertices of T. We can now state the formula for the dimension of a tree [4].

Theorem 15 [4]**.** *If T is a tree that is not a path, then*

$$\dim(T) = n_1(T) - ex(T).$$

If v is a leaf of a path P and T is a tree of order $n \geq 3$, then $T \circ_v P$ is a tree, $n_1(T \circ_v P) = n$ and $ex(T \circ_v P) = n - n_1(T)$. Hence, as a consequence of Theorem 15 we deduce the following result.

Corollary 16. *Let P be a path graph of order at least two and let v be a leaf of P. For any tree T of order $n \geq 3$,*

$$\dim(T \circ_v P) = n_1(T).$$

The inequalities of Propositions 10 and 12 lead to the following problem. Given a path P and a leaf v of P, is there a graph G of order n such that $\dim(G) = a$ and $\dim(G \circ_v P) = b$, for every integers a, b, n with $2 \leq a < b \leq \frac{a+n}{2}$?

In order to give an answer to the question above, we construct a tree $T(a, b, n)$ in the following way. Let $S_{1,a}$ be a star graph with a leaves and let P' be a path graph of order $n - b + 1$. To obtain $T(a, b, n)$ we proceed as follows.

- Identify one leaf of P' with the center of the star $S_{1,a}$.
- Add one pendant vertex to $b - a - 1$ vertices of degree two of the path P'.

Since P' has $n - b - 1$ vertices of degree two, we have that $b - a - 1 \leq n - b - 1$. Thus, $b \leq \frac{a+n}{2}$. Also, $n_1(T(a, b, n)) = b$ and $ex(T(a, b, n) = b - a$. Thus, Theorem 15 leads to $\dim(T(a, b, n)) = a$ and, if v is a leaf of a path graph P, Corollary 16 leads to $\dim(T(a, b, n) \circ_v P) = b$, which gives answer to the question mentioned above.

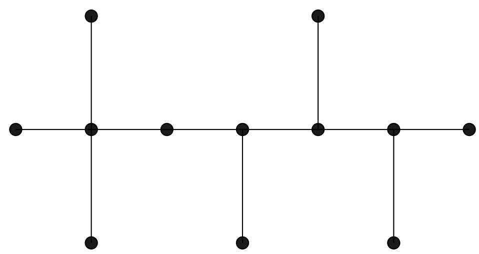

Figure 6. A tree $T(3, 7, 12)$.

Proposition 17. *Let P be a path graph and let v be a leaf of P. For any integers a, b, n with $2 \leq a < b \leq \frac{a+n}{2}$, there exists a graph G of order n such that $\dim(G) = a$ and $\dim(G \circ_v P) = b$.*

5. Corona Product Graphs

We consider now an interesting construction, which can be understood as a rooted product graph and, consequently, as a graph obtained by using the point-attaching process. The *corona product graph* $G \odot \mathcal{H}$ is defined as the graph obtained from a graph G of order n and a family of graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ by adding an edge between each vertex of H_i and the ith-vertex of G, [14]. Hence, $G \odot \mathcal{H}$ is a rooted product graph $G(K_1 + \mathcal{H})$ where $K_1 + \mathcal{H} = \{K_1 + H_1, K_1 + H_2, \ldots, K_1 + H_n\}$ and $K_1 + H_i$ is the join graph obtained from K_1 and H_i. By Corollary 5 we deduce the following result.

Remark 18. Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of nontrivial graphs. Then

$$\dim(G \odot \mathcal{H}) = \sum_{H_i \in \mathcal{H}_1} \dim(K_1 + H_i) + \sum_{H_i \in \mathcal{H}_2} (\dim(K_1 + H_i) - 1),$$

where $H_i \in \mathcal{H}_1$ if the vertex of K_1 does not belong to any metric basis of $K_1 + H_i$ and $H_j \in \mathcal{H}_2$ if the vertex of K_1 belongs to a metric basis of $K_1 + H_j$.

The metric dimension of corona product graphs $G \odot \mathcal{H}$, where $\mathcal{H}$ consists of n graphs isomorphic to a given graph H, was studied in [12, 13, 19, 28]. In this case we use the notation $G \odot H$ instead of $G \odot \mathcal{H}$.

We would emphasize the following particular case of the result above, which improve some results obtained in [28] and corrects a result[1] stated in [19].

Corollary 19. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. Then*

$$\dim(G \odot H) = \begin{cases} n \cdot (\dim(K_1 + H) - 1), & \text{if the vertex of } K_1 \text{ belongs to a metric basis of } K_1 + H; \\ n \cdot \dim(K_1 + H), & \text{otherwise.} \end{cases}$$

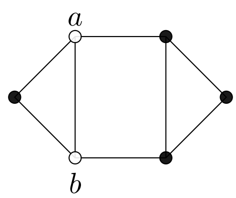

Figure 7. A graph H where $\dim(K_1 + H) = 3$. A metric basis of $K_1 + H$ is $\{v, a, b\}$, where v is the vertex of K_1.

For instance, for the graph H shown in Figure 7 we have $\dim(K_1 + H) = 3$. A metric basis of $K_1 + H$ is $\{v, a, b\}$, where v is the vertex of K_1. Therefore, Corollary 19 leads to $\dim(G \odot H) = 2n$, for any graph G of order $n \geq 2$.

[1] Corollary 19 corrects Theorem 1 of [19], which states that if H does not have dominating vertices, then $\dim(G \odot H) = n \cdot \dim(K_1 + H)$. A counterexample is shown in Figure 7.

Now, according to Remark 18, a significant problem consists of determining necessary and/or sufficient conditions for the vertex of K_1 to belong to a metric basis of K_1+H. For instance, it was shown in [28] that if H is a graph of diameter $D(H) \geq 6$ or it is a cycle graph of order greater than 6, then the vertex of K_1 does not belong to any metric basis of K_1+H and so $\dim(G \odot H) = n \cdot \dim(K_1+H)$. In this direction we state the following result.

Lemma 20. *Let H be a graph of radius $r(H)$ and maximum degree $\Delta(H)$. If $r(H) \geq 4$ or $\dim(K_1+H) > \Delta(H)+1$, then the vertex of K_1 does not belong to any metric basis of K_1+H.*

Proof. Let B be a metric basis of K_1+H. We suppose that the vertex v of K_1 belongs to B. Since v is adjacent to every vertex of H, there must exist a vertex $u \in V(H) \setminus B$ such that $B \subset N(u)$, otherwise $B \setminus \{v\}$ is a metric generator for K_1+H.

Now, if $r(H) \geq 4$, then we take $u' \in V(H)$ such that $d_H(u,u') = 4$ and a shortest path $uu_1u_2u_3u'$. In such a case we have $d_{K_1+H}(b,u_3) = d_{K_1+H}(b,u') = 2$, for every $b \in B \setminus \{v\}$, which is a contradiction. Thus, in this case v does not belong to any metric basis of K_1+H. On the other hand, if $|B| > \Delta(H)+1$, then for any vertex $w \notin B$, there exists a vertex $w' \in B$ such that w and w' are not adjacent. Clearly $w' \neq v$ and we observe that $B \setminus \{v\}$ is a metric generator, which is again a contradiction. Therefore, we also have that $v \notin B$. ∎

The converse of Lemma 20 is not true. In Figure 8 we show a graph H of radius three where $\dim(K_1+H) = 4 < 5 = \Delta(H)+1$ and the vertex of K_1 does not belong to any metric basis of K_1+H.

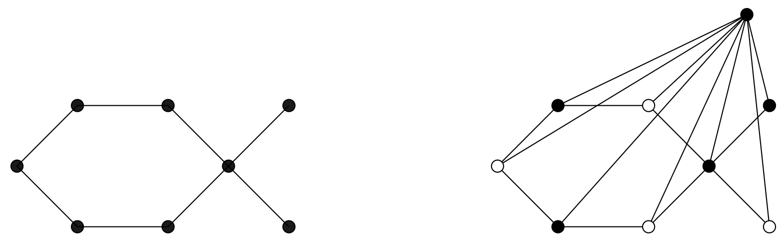

Figure 8. A graph H and the join graph K_1+H. White vertices form a metric basis of K_1+H.

Remark 18 and Lemma 20 lead to the next result.

Proposition 21. *Let G be a connected graph of order n and let H be a graph of radius $r(H)$ and maximum degree $\Delta(H)$. If $r(H) \geq 4$ or $\dim(K_1+H) > \Delta(H)+1$, then*

$$\dim(G \odot H) = n \cdot \dim(K_1+H).$$

It was shown in [28] that for any corona graph $G = G_1 \odot G_2$, such that G_1 is connected and both G_1 and G_2 are non-null graphs, it follows that the vertices of G_1 do not belong to any metric basis of G. Moreover, if G_1 has order $n_1 \geq 2$ and H has diameter $D(G_1) \leq 2$, then $\dim(G_1 \odot G_2) = n_1 \cdot \dim(G_2)$, [28].

Therefore, as a consequence of Corollary 19 we obtain the following result.

Remark 22. Let G and G_1 be two connected graphs of order n and $n_1 \geq 2$, respectively. Then for any $v \in V(G_1)$ and any graph G_2 of diameter one or two,

$$\dim(G \circ_v (G_1 \odot G_2)) = nn_1 \cdot \dim(G_2).$$

6. Chain of Graphs

Let $G_1, G_2, \ldots, G_k$ be a finite sequence of pairwise disjoint (nontrivial) connected graphs and let $x_i, y_i \in V(G_i)$. A *chain* G is a graph obtained by point-attaching from $G_1, G_2, \ldots, G_k$, where the vertex y_i is identified with the vertex x_{i+1} for $i \in \{1, \ldots, k-1\}$.

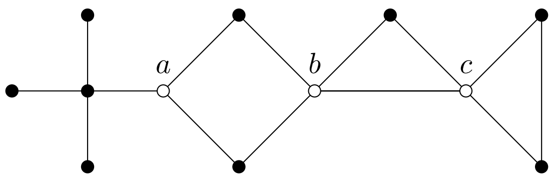

Figure 9. A chain graph where $G_1 \cong K_{1,4}$, $G_2 \cong C_4$, $G_3 \cong K_3 \cong G_4$, $A(G_1) = \{a\}$, $A(G_2) = \{a, b\}$, $A(G_3) = \{b, c\}$ and $A(G_4) = \{c\}$.

From Theorem 2 we deduce our next result.

Corollary 23. *Let G be a chain obtained by point-attaching from a family of connected graphs $G_1, \ldots, G_k$, $k \geq 3$, such that G_1 and G_k satisfy $\mathcal{P}_2$. If the attachment vertices of the primary subgraphs G_i are diametral in G_i, for $i \in \{2, \ldots, k-1\}$, then*

$$\dim(G) = \sum_{i=1}^{k} \dim^*(G_i).$$

For instance, for the chain graph G shown in Figure 9 we have $\dim(G) = 4$, as $\dim^*(G_1) = 2$, $\dim^*(G_2) = 1$, $\dim^*(G_3) = 0$ and $\dim^*(G_4) = 1$.

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Received 14 July 2015
Revised 30 April 2016
Accepted 30 April 2016