A CONSTRUCTIVE EXTENSION OF THE CHARACTERIZATION ON POTENTIALLY $K_{s,t}$-BIGRAPHIC PAIRS

JI-YUN GUO AND JIAN-HUA YIN

Department of Mathematics
College of Information Science and Technology
Hainan University, Haikou 570228, P.R. China

e-mail: yinjh@hainu.edu.cn

Abstract

Let $K_{s,t}$ be the complete bipartite graph with partite sets of size $s$ and $t$. Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $c_1 \geq c_2 \geq \cdots \geq c_n$. We say that $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic (resp. $A_{s,t}$-bigraphic) if there is a simple bipartite graph $G$ with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$ and $G$ contains $K_{s,t}$ as a subgraph (resp. the induced subgraph of $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ in $G$ is a $K_{s,t}$). In this paper, we give a characterization of $L$ that is potentially $A_{s,t}$-bigraphic. As a corollary, we also obtain a characterization of $L$ that is potentially $K_{s,t}$-bigraphic if $b_1 \geq b_2 \geq \cdots \geq b_m$ and $d_1 \geq d_2 \geq \cdots \geq d_n$. This is a constructive extension of the characterization on potentially $K_{s,t}$-bigraphic pairs due to Yin and Huang (Discrete Math. 312 (2012) 1241–1243).

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1. Introduction

Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be two nonincreasing sequences of nonnegative integers. The pair $S = (A; B)$ is said to be bigraphic if there exists a

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\[2\] Corresponding author.
simple bipartite graph $G$ with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $d_G(x_i) = a_i$ for $1 \leq i \leq m$ and $d_G(y_i) = b_i$ for $1 \leq i \leq n$. In this case, $G$ is referred to as a realization of $S$. The following well-known theorem due to Gale [2] and Ryser [4] independently gave a characterization of $S$ that is bigraphic.

**Theorem 1** [2, 4]. $S = (A; B)$ is bigraphic if and only if $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$ and

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{n} \min\{k, b_i\} \text{ for all } k \text{ with } 1 \leq k \leq m.$$ 

The pair $S = (A; B)$ is said to be potentially $K_{s,t}$-bigraphic if there is a realization of $S$ containing $K_{s,t}$ as a subgraph. Yin and Huang [6] presented a characterization of $S$ that is potentially $K_{s,t}$-bigraphic.

**Theorem 2** [6]. $S = (A; B)$ is potentially $K_{s,t}$-bigraphic if and only if $a_s \geq t$, $b_t \geq s$, $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$ and

$$\sum_{i=1}^{p} a_i + \sum_{i=s+1}^{s+q} a_i \leq \sum_{i=1}^{t} \min\{p, i \geq s - p\} + \sum_{i=t+1}^{n} \min\{q, b_i\}$$

for all $p$ and $q$ with $0 \leq p \leq s$ and $0 \leq q \leq m - s$.

Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $c_1 \geq c_2 \geq \cdots \geq c_n$. We say that $L = (L_1; L_2)$ is bigraphic if there exists a simple bipartite graph $G$ with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$ and $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$. In this case, $G$ is referred to as a realization of $L$. Garg et al. [3] obtained a characterization of $L$ that is bigraphic.

**Theorem 3** [3]. $L = (L_1; L_2)$ is bigraphic if and only if

$$\sum_{i=1}^{k} a_i \leq \sum_{j=1}^{n} \min\{k, d_j\} \text{ for all } k \text{ with } 1 \leq k \leq m$$

and

$$\sum_{i=1}^{k} c_i \leq \sum_{j=1}^{m} \min\{k, b_j\} \text{ for all } k \text{ with } 1 \leq k \leq n.$$ 

Theorem 3 reduces to Theorem 1 when $a_i = b_i$ for $1 \leq i \leq m$ and $c_i = d_i$ for $1 \leq i \leq n$. We say that $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic (resp. $A_{s,t}$-bigraphic) if there is a simple bipartite graph $G$ with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$. 


for $1 \leq i \leq n$ and $G$ contains $K_{s,t}$ as a subgraph (resp. the induced subgraph of $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ in $G$ is a $K_{s,t}$).

The purpose of this paper is to investigate a characterization of $L$ that is potentially $K_{s,t}$-bigraphic. We first give a characterization of $L$ that is potentially $A_{s,t}$-bigraphic as follows.

**Theorem 4.** Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $c_1 \geq c_2 \geq \cdots \geq c_n$. If $a_s \geq t$ and $c_t \geq s$, then $L = (L_1; L_2)$ is potentially $A_{s,t}$-bigraphic if and only if

$$\begin{align*}
(1) \quad & \sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i \leq \sum_{i=1}^{t} \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_i\} \\
& \text{for all } p_1 \text{ and } q_1 \text{ with } 0 \leq p_1 \leq s \text{ and } 0 \leq q_1 \leq m - s \text{ and }
\end{align*}$$

$$\begin{align*}
(2) \quad & \sum_{i=1}^{p_2} c_i + \sum_{i=t+1}^{t+q_2} c_i \leq \sum_{i=1}^{s} \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^{m} \min\{p_2 + q_2, b_i\} \\
& \text{for all } p_2 \text{ and } q_2 \text{ with } 0 \leq p_2 \leq t \text{ and } 0 \leq q_2 \leq n - t.
\end{align*}$$

If $s = t = 0$, then $p_1 = p_2 = 0$ and Theorem 4 reduces to Theorem 3. If we further assume that $b_1 \geq b_2 \geq \cdots \geq b_m$ and $d_1 \geq d_2 \geq \cdots \geq d_n$, then we can prove the following theorem.

**Theorem 5.** Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $c_1 \geq c_2 \geq \cdots \geq c_n$. If $b_1 \geq b_2 \geq \cdots \geq b_m$ and $d_1 \geq d_2 \geq \cdots \geq d_n$, then $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic if and only if it is potentially $A_{s,t}$-bigraphic.

Combining Theorem 4 with Theorem 5, we have the following corollary.

**Corollary 6.** Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $c_1 \geq c_2 \geq \cdots \geq c_n$. If $a_s \geq t$, $c_t \geq s$, $b_1 \geq b_2 \geq \cdots \geq b_m$ and $d_1 \geq d_2 \geq \cdots \geq d_n$, then $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic if and only if (1) holds for all $p_1$ and $q_1$ with $0 \leq p_1 \leq s$ and $0 \leq q_1 \leq m - s$ and (2) holds for all $p_2$ and $q_2$ with $0 \leq p_2 \leq t$ and $0 \leq q_2 \leq n - t$.

Corollary 6 reduces to Theorem 2 when $a_i = b_i$ for $1 \leq i \leq m$ and $c_i = d_i$ for $1 \leq i \leq n$. 
2. Proofs of Theorems 4 and 5

The proof technique of Theorem 4 was developed earlier by Tripathi, Venugopalan and West [5].

**Proof of Theorem 4.** For the necessity, we suppose that \( G \) is a realization of \( L = (L_1; L_2) \) with partite sets \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_n\} \) such that \( a_i \leq d_G(x_i) \leq b_i \) for \( 1 \leq i \leq m \), \( c_i \leq d_G(y_i) \leq d_i \) for \( 1 \leq i \leq n \) and the induced subgraph of \( \{x_1, \ldots, x_s, y_1, \ldots, y_t\} \) in \( G \) is a \( K_{s,t} \). For \( p_1 \) and \( q_1 \), with \( 0 \leq p_1 \leq s \) and \( 0 \leq q_1 \leq m - s \), it is easy to see that \( \sum_{i=1}^{t} \min\{p_1 + q_1, d_G(y_i) - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_G(y_i)\} \) is the maximum contribution to \( \sum_{i=1}^{s} a_i + \sum_{i=s+1}^{s+q_1} a_i + \sum_{i=s+1}^{s+q_1} d_G(x_i) \). Thus,

\[
\sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i \leq \sum_{i=1}^{p_1} d_G(x_i) + \sum_{i=s+1}^{s+q_1} d_G(x_i)
\]

\[
\leq \sum_{i=1}^{t} \min\{p_1 + q_1, d_G(y_i) - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_G(y_i)\}
\]

\[
\leq \sum_{i=1}^{p_1} \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_i\},
\]

that is, (1) holds for \( p_1 \) and \( q_1 \). Similarly, we can prove that (2) holds for \( p_2 \) and \( q_2 \) with \( 0 \leq p_2 \leq t \) and \( 0 \leq q_2 \leq n - t \).

For the sufficiency, we assume that (1) holds for \( p_1 \) and \( q_1 \) with \( 0 \leq p_1 \leq s \) and \( 0 \leq q_1 \leq m - s \) and (2) holds for \( p_2 \) and \( q_2 \) with \( 0 \leq p_2 \leq t \) and \( 0 \leq q_2 \leq n - t \). A subrealization of \( L = (L_1; L_2) \) is a bipartite graph \( G \) with partite sets \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_n\} \) such that \( d_G(x_i) \leq b_i \) for \( 1 \leq i \leq m \) and \( d_G(y_i) \leq d_i \) for \( 1 \leq i \leq n \). If \( a_i \leq d_G(x_i) \leq b_i \) for \( 1 \leq i \leq m \) and \( c_i \leq d_G(y_i) \leq d_i \) for \( 1 \leq i \leq n \), then \( G \) is a realization of \( L \). We will construct a realization of \( L \) through successive subrealizations. The initial subrealization is \( K_{s,t} \cup K_{m-s} \cup K_{n-t} \), where \( K_r \) is the complement of \( K_r \). \( K_{s,t} \) has partite sets \( \{x_1, \ldots, x_s\} \) and \( \{y_1, \ldots, y_t\} \), \( V(K_{m-s}) = \{x_{s+1}, \ldots, x_m\} \) and \( V(K_{n-t}) = \{y_{t+1}, \ldots, y_n\} \).

In each successive subrealization, let \( p_1 \) be the largest index such that \( d(x_i) = a_i \) for \( 1 \leq i < p_1 \) and \( d(x_p) < a_p \) and \( q_1 \) be the largest index such that \( d(x_i) = a_i \) for \( s + 1 \leq i < s + q_1 \) and \( d(x_{s+q_1}) < a_{s+q_1} \). When \( p_1 \leq s \) or \( q_1 \leq m - s \), we can obtain a new subrealization containing the initial subrealization and having smaller deficiency \( (a_{p_1} - d(x_p)) + (a_{s+q_1} - d(x_{s+q_1})) \) at \( x_p \) and \( x_{s+q_1} \) while not changing the degree of any vertex \( x_i \) with \( i \in \{1, \ldots, p_1 - 1, s + 1, \ldots, s + q_1 - 1\} \).

Let \( X_1 = \{x_{p_1+1}, \ldots, x_s\} \) and \( X_2 = \{x_{s+q_1+1}, \ldots, x_m\} \). We maintain the condition that \( \{x_1, \ldots, x_s\} \) and \( \{y_1, \ldots, y_t\} \) form a \( K_{s,t} \), there is no edge between \( \{y_1, \ldots, y_t\} \) and \( X_2 \) and there is no edge between \( \{y_{t+1}, \ldots, y_n\} \) and \( X_1 \cup X_2 \),
which certainly hold initially. For convenience, we write \(v_i \leftrightarrow v_j\) for \("v_i is adjacent to v_j\) and \(v_i \not\leftrightarrow v_j\) for \("v_i is not adjacent to v_j\)."

**Case 0.** Suppose \(x_{p_1} \not\leftrightarrow y_k\) for some \(k > t\) such that \(d(y_k) < d_k\). Add \(x_{p_1}y_k\).

**Case 1.** Suppose \(x_{s+q_1} \not\leftrightarrow y_k\) for some \(k\) such that \(d(y_k) < d_k\). Add \(x_{s+q_1}y_k\).

**Case 2.** Suppose \(d(y_k) \neq \min\{p_1 + q_1, d_k\}\) for some \(k\) with \(k \geq t + 1\). In a subrealization, \(d(y_k) \leq d_k\). Since there is no edge between \(\{y_{t+1}, \ldots, y_n\}\) and \(X_1 \cup X_2\), \(d(y_k) \leq p_1 + q_1\). Hence, \(d(y_k) < \min\{p_1 + q_1, d_k\}\). Case 0 and Case 1 apply, unless \(x_{p_1} \leftrightarrow y_k\) and \(x_{s+q_1} \leftrightarrow y_k\). Since \(d(y_k) < p_1 + q_1\), there exists \(i \in \{1, \ldots, p_1 - 1, s + 1, \ldots, s + q_1 - 1\}\) such that \(x_i \not\leftrightarrow y_k\). By \(d(x_i) > d(s+q_1)\), there exists \(e \in N(x_i) \setminus N(x_{p_1})\), then replace \(ux_i\) by \(\{x_iy_k, ux_{s+q_1}\}\). If \(i \in \{s + 1, \ldots, s + q_1 - 1\}\), by \(d(x_i) > d(s+q_1)\), there exists \(u \in N(x_i) \setminus N(x_{s+q_1})\), then replace \(ux_i\) by \(\{x_iy_k, ux_{s+q_1}\}\).

**Case 3.** Suppose \(d(y_k) - s + p_1 \neq \min\{p_1 + q_1, d_k - s + p_1\}\) for some \(k\) with \(k \leq t\). In a subrealization, \(d(y_k) - s + p_1 \leq d_k - s + p_1\). Since there is no edge between \(\{y_1, \ldots, y_t\}\) and \(X_2\), \(d(y_k) - s + p_1 \leq p_1 + q_1\). Hence \(d(y_k) - s + p_1 < \min\{p_1 + q_1, d_k - s + p_1\}\). Case 1 applies unless \(x_{s+q_1} \leftrightarrow y_k\). Since \(d(y_k) - s + p_1 < p_1 + q_1\) and \(x_i \leftrightarrow y_k\) for \(1 \leq i \leq p_1\), there exists \(i \in \{s + 1, \ldots, s + q_1 - 1\}\) such that \(x_i \not\leftrightarrow y_k\). By \(d(x_i) > d(s+q_1)\), there exists \(u \in N(x_i) \setminus N(x_{s+q_1})\), then replace \(ux_i\) by \(\{x_iy_k, ux_{s+q_1}\}\).

If none of Cases 0–3 applies, then \(d(y_k) = \min\{p_1 + q_1, d_k\}\) for \(k \geq t + 1\) and \(d(y_k) - s + p_1 = \min\{p_1 + q_1, d_k - s + p_1\}\) for \(k \leq t\). Since \(\{x_1, \ldots, x_t\}\) and \(\{y_1, \ldots, y_t\}\) form a \(K_{s,t}\), there is no edge between \(\{y_1, \ldots, y_t\}\) and \(X_2\) and there is no edge between \(\{y_{t+1}, \ldots, y_n\}\) and \(X_1 \cup X_2\), we have that

\[
\sum_{i=1}^{p_1} d(x_i) + \sum_{i=1}^{q_1} d(x_{s+i}) = \sum_{i=1}^{t} \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_i\}.
\]

By (1) and the observation that \(d(x_i) = a_i\) for \(1 \leq i \leq p_1 - 1\) and \(d(x_{s+i}) = a_{s+i}\) for \(1 \leq i \leq q_1 - 1\), we get that \(\sum_{i=1}^{p_1} a_i + \sum_{i=1}^{s+q_1} a_i = \sum_{i=1}^{p_1} d(x_i) + \sum_{i=1}^{q_1} d(x_{s+i})\), which implies that \(d(x_{p_1}) = a_{p_1}\) and \(d(x_{s+q_1}) = a_{s+q_1}\). Now we have shown that while \(p_1 \leq s\) or \(q_1 \leq m - s\), we obtain a new subrealization containing the initial subrealization and having \(d(x_{p_1}) = a_{p_1}\) and \(d(x_{s+q_1}) = a_{s+q_1}\) while not changing the degree of any vertex \(x_i\) with \(i \in \{1, \ldots, p_1 - 1, s + 1, \ldots, s + q_1 - 1\}\). Increase \(p_1\) by 1 and \(q_1\) by 1, and repeat the process from Case 0 to Case 3. Thus when \(p_1 = s\) and \(q_1 = m - s\), a subrealization \(G'\) containing the initial subrealization can be obtained so that \(d(x_i) = a_i\) for \(1 \leq i \leq m\) and \(d(y_i) \leq d_i\) for \(1 \leq i \leq n\).

We now regard \(G'\) as a new initial subrealization. In the following, for each successive subrealization, we define \(p_2\) to be the largest index such that \(d(y_{p_2}) \geq c_{i+1}\) for \(1 \leq i < p_2\) and \(d(y_{p_2}) < c_{p_2}\), and \(q_2\) to be the largest index such that \(d(y_{q_2}) \geq c_{i+1}\) for \(t+1 \leq i < t+q_2\) and \(d(y_{t+q_2}) < c_{t+q_2}\). While \(p_2 \leq t\) or \(q_2 \leq n-t\), we can obtain
a new subrealization having smaller deficiency \((c_{p_2} - d(y_{p_2})) + (c_{t+q_2} - d(y_{t+q_2}))\) at \(y_{p_2}\) and \(y_{t+q_2}\) while maintaining the conditions that \(\{x_1, \ldots, x_s\}\) and \(\{y_1, \ldots, y_t\}\) form a \(K_{s,t}\), \(d(y_i) \geq c_i\) for \(i \in \{1, \ldots, p_2-1, t+1, \ldots, t+q_2-1\}\) and \(a_i \leq d(x_i) \leq b_i\) for \(1 \leq i \leq m\). The process can only stop when the subrealization is a realization of \(L\).

**Case 4.** Suppose, for some \(j > s\), \(x_j \leftrightarrow y_k\) for some \(p_2 + 1 \leq k \leq t\) and \(x_j \not\leftrightarrow y_\ell\) for some \(\ell \leq p_2\). If \(\ell = p_2\), then replace \(y_kx_j\) by \(y_{p_2}x_j\). If \(\ell < p_2\), then replace \(\{y_kx_j, yv\}\) by \(\{y_{p_2}x_j, yv\}\), where \(v \in N(y_\ell) \setminus N(y_{p_2})\).

**Case 5.** Suppose, for some \(j \in \{1, \ldots, m\}\), \(x_j \leftrightarrow y_k\) for some \(k > t + q_2\) and \(x_j \not\leftrightarrow y_\ell\) for some \(1 + t \leq \ell \leq t + q_2\). If \(\ell = t + q_2\), then replace \(x_jy_k\) by \(x_jy_{t+q_2}\). If \(t + 1 \leq \ell < t + q_2\), then replace \(\{x_jy_k, yv\}\) by \(\{x_jy_{t+q_2}, yv\}\), where \(v \in N(y_\ell) \setminus N(y_{t+q_2})\).

**Case 6.** Suppose \(d(x_j) < b_j\) for some \(j > s\) and \(x_j \not\leftrightarrow y_\ell\) for some \(\ell \leq p_2\). If \(\ell = p_2\), then add \(x_jy_{p_2}\). If \(\ell < p_2\), then replace \(vy_\ell\) by \(\{vy_{p_2}, yvx_j\}\), where \(v \in N(y_\ell) \setminus N(y_{p_2})\).

**Case 7.** Suppose \(d(x_j) < b_j\) for some \(j \in \{1, \ldots, m\}\) and \(x_j \not\leftrightarrow y_\ell\) for some \(t + 1 \leq \ell \leq t + q_2\). If \(\ell = t + q_2\), then add \(x_jy_{t+q_2}\). If \(t + 1 \leq \ell < t + q_2\), then replace \(vy_\ell\) by \(\{vy_{t+q_2}, yvx_j\}\), where \(v \in N(y_\ell) \setminus N(y_{t+q_2})\).

**Case 8.** Suppose, for some \(j > s\), \(x_j \leftrightarrow y_k\) for some \(p_2 + 1 \leq k \leq t\) and \(x_j \not\leftrightarrow y_\ell\) for some \(t + 1 \leq \ell \leq t + q_2\). If \(\ell = t + q_2\), then replace \(x_jy_k\) by \(x_jy_{t+q_2}\). If \(t + 1 \leq \ell < t + q_2\), then replace \(\{x_jy_k, yv\}\) by \(\{x_jy_{t+q_2}, yv\}\), where \(v \in N(y_\ell) \setminus N(y_{t+q_2})\).

**Case 9.** Suppose, for some \(j > s\), \(x_j \leftrightarrow y_k\) for some \(k > t + q_2\) and \(x_j \not\leftrightarrow y_\ell\) for some \(\ell \leq p_2\). If \(\ell = p_2\), then replace \(x_jy_k\) by \(x_jy_{p_2}\). If \(\ell < p_2\), then replace \(\{x_jy_k, vy\}\) by \(\{x_jy_{p_2}, vy\}\), where \(v \in N(y_\ell) \setminus N(y_{p_2})\).

**Case 10.** Suppose \(d(y_i) > c_i\) for some \(i \in \{1, \ldots, p_2-1, t+1, \ldots, t+q_2-1\}\). If \(i \in \{1, \ldots, p_2-1\}\), then replace \(vy_i\) by \(vy_{p_2}\), where \(v \in N(y_i) \setminus N(y_{p_2})\). If \(i \in \{t+1, \ldots, t+q_2-1\}\), then replace \(vy_i\) by \(vy_{t+q_2}\), where \(v \in N(y_i) \setminus N(y_{t+q_2})\).

If none of Cases 4–9 applies, we can prove the following claim.

**Claim.** Assume that none of Cases 4–9 applies. Then

(i) For each \(x_j \in \{x_1, \ldots, x_s\}\), \(\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\}\) and \(\min\{p_2 + q_2, b_j - t + p_2\}\) is the maximum contribution to \(\sum_{i=1}^{q_2} d(y_{i+1})\) from edges incident to \(x_j\).

(ii) For each \(x_j \in \{x_1, \ldots, x_m\}\), \(\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\}\) and \(\min\{p_2 + q_2, b_j\}\) is the maximum contribution to \(\sum_{i=1}^{q_2} d(y_{i+1})\) from edges incident to \(x_j\).

**Proof.** If \(x_j \in \{x_1, \ldots, x_s\}\), we consider the following two cases depending on whether \(x_j\) is adjacent to all the vertices in \(\{y_{t+1}, \ldots, y_{t+q_2}\}\) or not.
Suppose $x_j \leftrightarrow y_k$ for all $t+1 \leq k \leq t+q_2$. Since $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$ form a $K_{s,t}$, $x_j$ is adjacent to every vertex in $\{y_1, \ldots, y_t\}$. Thus, $p_2 + q_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$.

By $b_j - t + p_2 \geq d(x_j) - t + p_2 \geq p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\} = p_2 + q_2$ and $\min\{p_2 + q_2, b_j - t + p_2\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$.

Suppose $x_j \not\leftrightarrow y_k$ for some $t+1 \leq k \leq t+q_2$. Since Case 5 and Case 7 cannot apply, we have that $x_j \not\leftrightarrow y_k$ for all $\ell > t + q_2$ and $d(x_j) = b_j$. This implies that $b_j - t + p_2 = d(x_j) - t + p_2 < p_2 + q_2$, $\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\} = b_j - t + p_2$ and $b_j - t + p_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$.

If $x_j \in \{x_{s+1}, \ldots, x_m\}$, we consider the following two cases depending on whether $x_j$ is adjacent to all the vertices in $\{y_1, \ldots, y_{p_2}\}$ or not.

Suppose $x_j \leftrightarrow y_k$ for all $k \leq p_2$. If $x_j \leftrightarrow y_{\ell}$ for all $\ell \leq t + q_2$, then $p_2 + q_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$. By $b_j \geq d(x_j) \geq p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = p_2 + q_2$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$. Assume that $x_j \not\leftrightarrow y_\ell$ for some $t+1 \leq \ell \leq t+q_2$. Since Case 5, Case 7 and Case 8 cannot apply, we have that $x_j \not\leftrightarrow y_k$ for all $k > t + q_2$, $d(x_j) = b_j$ and $x_j \not\leftrightarrow y_\ell$ for all $p_2 + 1 \leq k \leq t$. Thus, $b_j$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$. By $b_j = d(x_j) < p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$.

Suppose $x_j \not\leftrightarrow y_k$ for some $k \leq p_2$. Since Case 4, Case 6 and Case 9 cannot apply, we have that $x_j \not\leftrightarrow y_\ell$ for all $p_2 + 1 \leq \ell \leq t$, $x_j \not\leftrightarrow y_k$ for all $\ell > t + q_2$ and $d(x_j) = b_j$. By $b_j = d(x_j) < p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_j$. The claim is proved. \hfill \Box

We now continue to proceed with the proof of theorem. By the previous claim, we have that $\sum_{i=1}^{s} \min\{p_2 + q_2, d(x_i) - t + p_2\} + \sum_{i=s+1}^{m} \min\{p_2 + q_2, d(x_i)\} = \sum_{i=1}^{s} \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^{m} \min\{p_2 + q_2, b_i\}$, and $\sum_{i=1}^{s} \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^{m} \min\{p_2 + q_2, b_i\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_1, \ldots, x_m$. If Case 10 cannot apply, then $d(y_i) = c_i$ for all $i \in \{1, \ldots, p_2 - 1, t + 1, \ldots, t + q_2 - 1\}$. Thus, we obtain that

$$
\sum_{i=1}^{s} c_i + \sum_{i=1}^{q_2-1} c_{t+i} + d(y_{p_2}) + d(y_{t+q_2}) = \sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})
$$

$$
= \sum_{j=1}^{m} \min\{p_2 + q_2, b_j - t + p_2\} + \sum_{j=s+1}^{m} \min\{p_2 + q_2, b_j\}.
$$
By (2), we further have that \( c_{p_2} + c_{t+q_2} \leq d(y_{p_2}) + d(y_{t+q_2}) \), which implies that \( d(y_{p_2}) = c_{p_2} \) and \( d(y_{t+q_2}) = c_{t+q_2} \). Now we have shown that while \( p_2 \leq t \) or \( q_2 \leq n-t \), we obtain a new subrealization having \( d(y_{p_2}) = c_{p_2} \) and \( d(y_{t+q_2}) = c_{t+q_2} \) while maintaining the conditions that \( \{x_1, \ldots, x_s\} \) and \( \{y_1, \ldots, y_t\} \) form a \( K_{s,t} \), \( d(y_i) \geq c_i \) for \( i \in \{1, \ldots, p_2 - 1, t + 1, \ldots, t + q_2 - 1\} \) and \( a_i \leq d(x_i) \leq b_i \) for \( 1 \leq i \leq m \). Increase \( p_2 \) by 1 and \( q_2 \) by 1, and repeat the process from Case 4 to Case 10. When \( p_2 = t \) and \( q_2 = n - t \), we finally get a realization of \( L \) containing \( K_{s,t} \) and satisfying \( a_i \leq d(x_i) \leq b_i \) for \( 1 \leq i \leq m \) and \( c_i \leq d(y_i) \leq d_i \) for \( 1 \leq i \leq n \), where \( V(K_{s,t}) = \{x_1, \ldots, x_s, y_1, \ldots, y_t\} \). In other words, \( L \) is potentially \( A_{s,t} \)-bigraphic. The proof of Theorem 4 is completed. ■

This constructive proof can be implemented as an algorithm to construct a realization of \( L \) containing \( K_{s,t} \). The following lemma due to Ferrara et al. [1] will be useful as we proceed with the proof of Theorem 5.

**Lemma 7** [1]. Let \( S \) be a bigraphic pair with realization \( G = (X \cup Y, E) \) having partite sets \( X \) and \( Y \). Let \( H = (X' \cup Y', E') \) be a subgraph of \( G \) such that \( X' \) and \( Y' \) are contained in \( X \) and \( Y \), respectively. Then there exists a realization \( G_1 = (X \cup Y, E_1) \) of \( S \) containing \( H \) as a subgraph such that \( X' \) and \( Y' \) lie on the vertices of highest degree in \( X \) and \( Y \), respectively.

**Proof of Theorem 5.** We only need to show that if \( L = (L_1; L_2) \) is potentially \( K_{s,t} \)-bigraphic, then it is potentially \( A_{s,t} \)-bigraphic. Let \( G \) be a simple bipartite graph with partite sets \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_n\} \) such that \( a_i \leq d_G(x_i) \leq b_i \) for \( 1 \leq i \leq m \), \( c_i \leq d_G(y_i) \leq d_i \) for \( 1 \leq i \leq n \) and \( G \) contains \( K_{s,t} = (X' \cup Y', E') \) as a subgraph. Denote \( d_{1i} = d_G(x_i) \) for \( 1 \leq i \leq m \) and \( d_{2i} = d_G(y_i) \) for \( 1 \leq i \leq n \). Let \( A = (d_{11}, \ldots, d_{1m}) \) and \( B = (d_{21}, \ldots, d_{2n}) \). By Lemma 7, \( (A; B) \) has a realization \( G_1 = (X \cup Y, E_1) \) satisfying \( d_{1i}(x_i) = d_{1i} \) for \( 1 \leq i \leq m \), \( d_{2i}(y_i) = d_{2i} \) for \( 1 \leq i \leq n \) and \( G_1 \) contains \( K_{s,t} \) so that \( X' \) and \( Y' \) lie on the vertices of highest degree in \( X \) and \( Y \), respectively. Let \( D = \{x_1, \ldots, x_s\} \setminus X' \), \( D' = \{x_{s+1}, \ldots, x_m\} \cap X' \), \( C = \{y_1, \ldots, y_t\} \setminus Y' \) and \( C' = \{y_{t+1}, \ldots, y_n\} \setminus Y' \). Then, it is easy to see that

\[
\max\{a_i|x_i \in D\} \leq \max\{d_{1i}|x_i \in D\} \leq d_{1j} \leq \min\{b_i|x_i \in D\}
\]

for each \( x_j \in D' \),

\[
\max\{a_i|x_i \in D'\} \leq d_{1j} \leq \min\{d_{1i}|x_i \in D'\} \leq \min\{b_i|x_i \in D'\}
\]

for each \( x_j \in D' \),

\[
\max\{c_i|y_i \in C\} \leq \max\{d_{2i}|y_i \in C\} \leq d_{2j} \leq \min\{d_i|y_i \in C\}
\]

for each \( y_j \in C' \),

\[
\max\{c_i|y_i \in C'\} \leq d_{2j} \leq \min\{d_{2i}|y_i \in C'\} \leq \min\{d_i|y_i \in C'\}
\]

for each \( y_j \in C' \).

Thus, we can see that \( (L_1; L_2) \) is potentially \( A_{s,t} \)-bigraphic by exchanging \( D \) with \( D' \) and \( C \) with \( C' \). ■

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