UNION OF DISTANCE MAGIC GRAPHS

SYLWIA CICHACZ\textsuperscript{1} AND MATEUSZ NIKODEM\textsuperscript{2}

AGH University of Science and Technology

e-mail: cichacz@agh.edu.pl
nikodem@agh.edu.pl

Abstract

A distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection $\ell$ from $V$ to the set $\{1, \ldots, n\}$ such that the weight $w(x) = \sum_{y \in N_G(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element $\mu$, called the magic constant. In this paper, we study unions of distance magic graphs as well as some properties of such graphs.

Keywords: distance magic labeling, magic constant, sigma labeling, graph labeling, union of graphs, lexicographic product, direct product, Kronecker product, Kotzig array.

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1. Definitions

All graphs $G = (V, E)$ are finite undirected simple graphs. For standard graph theoretic notation and definitions we refer to Diestel [10]. For a graph $G$, we use $V(G)$ for the vertex set and $E(G)$ for the edge set of $G$. The open neighborhood $N(x)$ (or more precisely $N_G(x)$, when needed) of a vertex $x$ is the set of all vertices adjacent to $x$, and the degree $d(x)$ of $x$ is $|N(x)|$, i.e., the size of the neighborhood of $x$. By $N[x]$ (or $N_G[x]$) we denote the closed neighborhood $N(x) \cup \{x\}$ of $x$. By $C_n$ we denote a cycle on $n$ vertices.

Different kinds of labelings have been an important part of graph theory for years. See a dynamic survey [14] which covers the field. The subject of our investigation is the distance magic labeling. A distance magic labeling of a graph $G$ of order $n$ is a bijection $\ell : V \to \{1, 2, \ldots, n\}$ such that there exists a positive

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integer \( \mu \) such that the weight \( w(v) = \sum_{u \in N(v)} \ell(u) = \mu \) for all \( v \in V \), where \( N(v) \) is the open neighborhood of \( v \). The constant \( \mu \) is called the magic constant of the labeling \( \ell \). Any graph which admits a distance magic labeling is called a distance magic graph. Closed distance magic graphs are a variation of distance magic graphs, where the sums are taken over the closed neighborhoods \( N_G(x) \) instead of the open ones \( N_G(x) \), see [3, 4].

The concept of distance magic labeling has been motivated by the equalized incomplete tournaments (see [11, 12]). Finding an \( r \)-regular distance magic labeling is equivalent to finding equalized incomplete tournament \( \text{EIT}(n, r) \) [12].

In an equalized incomplete tournament \( \text{EIT}(n, r) \) of \( n \) teams with \( r \) rounds, every team plays exactly \( r \) other teams and the total strength of the opponents that team \( i \) plays is \( k \). Thus, it is easy to notice that finding an \( \text{EIT}(n, r) \) is the same as finding a distance magic labeling of some \( r \)-regular graph on \( n \) vertices.

From the point of view of this application it is interesting to find disconnected \( r \)-regular distance magic graphs (tournaments which could be played simultaneously in different locations). Therefore in the paper we show examples of distance magic graphs \( G \) such that the union of \( t \) disjoint copies of \( G \), denoted \( tG \), is distance magic as well.

We recall four graph products (see [16]). All four, the Cartesian product \( G \square H \), lexicographic product \( G \circ H \), direct product \( G \times H \) and the strong product \( G \boxtimes H \) are graphs with the vertex set \( V(G) \times V(H) \). Two vertices \((g, h)\) and \((g', h')\) are adjacent in:

- \( G \square H \) if \( g = g' \) and \( h \) is adjacent to \( h' \) in \( H \), or \( h = h' \) and \( g \) is adjacent to \( g' \) in \( G \),
- \( G \times H \) if \( g \) is adjacent to \( g' \) in \( G \) and \( h \) is adjacent to \( h' \) in \( H \),
- \( G \boxtimes H \) if \( g = g' \) and \( h \) is adjacent to \( h' \) in \( H \), or \( h = h' \) and \( g \) is adjacent to \( g' \) in \( G \), or \( g \) is adjacent to \( g' \) in \( G \) and \( h \) is adjacent to \( h' \) in \( H \),
- \( G \circ H \) if either \( g \) is adjacent to \( g' \) in \( G \) or \( g = g' \) and \( h \) is adjacent to \( h' \) in \( H \).

The graph \( G \circ H \) is also called the composition and denoted by \( G[H] \) (see [17]). The product \( G \times H \) is also known as Kronecker product, tensor product, categorical product and graph conjunction. The direct product is commutative, associative, and it has several applications, for instance it may be used as a model for concurrency in multiprocessor systems [19]. Some other applications can be found in [18].

Some product related graphs, which are distance magic or closed distance magic can be found in [1–5, 9, 21, 22].

**Theorem 1.1** [21]. Let \( r \geq 1 \), \( n \geq 3 \), \( G \) be an \( r \)-regular graph and \( C_n \) be the cycle of length \( n \). Then the graph \( G \circ C_n \) admits a distance magic labeling if and only if \( n = 4 \).
Theorem 1.2 [2]. Let $G$ be an arbitrary regular graph. Then $G \times C_4$ is distance magic.

Theorem 1.3 [22]. The Cartesian product $C_n \square C_m$ is distance magic if and only if $n \equiv m \equiv 2 \pmod{4}$ and $n = m$.

Theorem 1.4 [2]. A graph $C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$, or $m \equiv n \equiv 0 \pmod{4}$.

Theorem 1.5 [3]. A graph $C_m \boxtimes C_n$ is distance magic if and only if at least one of the following conditions holds:

1. $m \equiv 3 \pmod{6}$ and $n \equiv 3 \pmod{6}$.
2. $\{m, n\} = \{3, x\}$ and $x$ is an odd number.

Let $K(n; r)$ denote the complete $r$-partite graph $K(n, n, \ldots, n)$.

Theorem 1.6 [8]. The Cartesian product $K(n; r) \square C_4$ is distance magic if and only if $n > 2$, $r > 1$ and $n$ is even.

The $d$-dimensional hypercube is denoted $Q_d$ where the vertices are binary $d$-tuples and two vertices are adjacent if and only if the $d$-tuples differ precisely in one position.

Theorem 1.7 [15]. A hypercube $Q_d$ has a distance magic labeling if and only if $d \equiv 2 \pmod{4}$.

The circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is the graph on the vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$ with edges $(x_i, x_{i+s_j})$ for $i = 0, \ldots, n-1$, $j = 1, \ldots, k$ where $i + s_j$ is taken modulo $n$.

Theorem 1.8 [7]. Let $p \geq 2$ and $n = p^2 - 1$ when $p$ is odd and $n = 2(p^2 - 1)$ when $p$ is even. Then $C_n(1, p)$ is a distance magic graph.

Theorem 1.9 [6]. If $p > 1$ is odd, then $C_{2p(p+1)}(1, 2, \ldots, p)$ is a distance magic graph.

By $tG$ we denote $t$ disjoint copies of a graph $G$. Here are some examples of disconnected distance magic graphs.

Theorem 1.10 [13, 20]. Let $nr$ be odd, $t$ be even, $r > 1$ and $t \geq 2$. Then $tK(n; r)$ is distance magic if and only if $r \equiv 3 \pmod{4}$.

Theorem 1.11 [20]. Let $m \geq 1$, $n \geq 2$ and $p \geq 3$. Then $mC_p \circ K_n$ has a distance magic labeling if and only if $n$ is even or $mnp$ is odd or $n$ is odd and $p \equiv 0 \pmod{4}$. 
Theorem 1.12 [9]. Let $m$ and $n$ be two positive even integers such that $m \leq n$. The graph $G = tK_{m,n}$ is distance magic if and only if the following conditions hold:

- $m + n \equiv 0 \pmod{4}$, and
- $1 = 2(2tn + 1)^2 - (2tm + 2tn + 1)^2$ or $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2} - 1}{2t}$.

Theorem 1.13 [3]. Given $n \geq 2$ and $t \geq 1$, the union $tK_n$ is closed distance magic if and only if $n(t + 1) \equiv 0 \pmod{2}$.

We say that an $r$-regular graph $G$ has a $p$-partition if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ (that is, $V(G) = V_1 \cup V_2 \cup \cdots \cup V_p$ where $V_i \cap V_j = \emptyset$ for $i \neq j$) such that for every $x \in V(G)$

$$|N(x) \cap V_1| = |N(x) \cap V_2| = \cdots = |N(x) \cap V_p|.$$ 

Analogously we say that an $r$-regular graph $G$ has a closed $p$-partition if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ such that for every $x \in V(G)$

$$|N[x] \cap V_1| = |N[x] \cap V_2| = \cdots = |N[x] \cap V_p|.$$ 

We show that if a distance magic graph $H$ has a 2-partition, then $tH$ is distance magic for every positive integer $t$. Moreover, for an $r$-regular graph $G$ the products $G \circ H$ and $G \times H$ are distance magic as well, and thus we generalize Theorems 1.1 and 1.2.

2. Distance Magic Graphs

Lemma 2.1. Let $G$ be an $r$-regular graph of order $n$ with a 2-partition (closed 2-partition). If $G$ is a distance magic (closed distance magic) graph, then $tG$ is a distance magic (closed distance magic) graph for any positive integer $t$.

**Proof.** Let $\ell$ be a distance magic (closed distance magic) labeling of $G$ with the magic constant $\mu$. In each copy $G^1, G^2, \ldots, G^t$ of $G$ we apply the partition defined above such that $V_1^j \cup V_2^j$ is the partition of the $j$-th copy $G^j$ of $G$. Define

$$\ell'(x) = \begin{cases} 
\ell(x) + (j - 1)n, & \text{if } x \in V_1^j, \\
\ell(x) + (t - j)n, & \text{if } x \in V_2^j.
\end{cases}$$

Obviously, $\ell'$ is a distance magic (closed distance magic) labeling of the graph $tG$ with the magic constant $\mu' = \mu + (t - 1)n(r + 1)/2$ (closed magic constant $\mu' = \mu + (t - 1)n(r + 1)/2$). □
We will now use Kotzig arrays as a tool. A Kotzig array was defined in [23] to be a \( j \times k \) matrix, each row being a permutation of \( \{0, 1, \ldots, k - 1\} \) and each column having a constant sum.

**Lemma 2.2** [23]. A Kotzig array of size \( j \times k \) exists whenever \( j > 1 \) and \( j(k-1) \) is even.

The following lemma shows that even if an \( r \)-regular distance magic graph \( G \) has no 2-partition, the union \( tG \) can be distance magic.

**Lemma 2.3.** Let \( p \geq 2 \) and \( G \) be an \( r \)-regular graph of order \( n \) having a \( p \)-partition (closed \( p \)-partition). If \( G \) is a distance magic (closed distance magic) graph, then for \( t \geq 0 \) where \( p(t-1) \) is even the graph \( tG \) is also distance magic (closed distance magic).

**Proof.** Let \( \ell \) be a distance magic (closed distance magic) labeling of \( G \) with the magic constant \( \mu \). In each copy \( G^1, G^2, \ldots, G^t \) of \( G \) we apply the partition defined above such that \( V^1 \cup V^2 \cup \cdots \cup V^t \) is the partition of \( j \)-th copy \( G^j \) of \( G \).

Let \( A = (a_{i,j}) \) be a Kotzig array of size \( p \times t \). Define

\[
\ell'(x) = \ell(x) + na_{n_i,j}, \quad x \in V^j.
\]

Obviously, \( \ell' \) is the distance magic (closed distance magic) labeling of the graph \( tG \) with a magic constant \( \mu' = \mu + (t-1)n(r+1)/2 \) (closed magic constant \( \mu' = \mu + (t-1)n(r+1)/2 \)).

We will now present some examples of graphs that have the desired 2-partition.

**Observation 1.** If
1. \( G = C_n \square C_m \) for \( n = m \) and \( n \equiv m \equiv 2 \) (mod 4),
2. \( G = C_n \times C_m \) for \( n = 4 \) or \( m = 4 \) or \( m \equiv n \equiv 0 \) (mod 4),
3. \( G = K(n;r) \square C_4 \) for \( n > 2 \), \( r > 1 \) and \( n \) even,
4. \( G = Q_d \) for \( d \equiv 2 \) (mod 4),
5. \( G = C_{p^2-1}(1,p) \) for \( p \) odd,
6. \( G = C_{2(p^2-1)}(1,p) \) for \( p \) even,
7. \( G = C_{2(p+1)}(1,2,\ldots,p) \) for \( p \) odd,

then \( G \) has a 2-partition.

**Proof.** 1. Let \( V(C_m \square C_n) = \{v_{i,j} : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\} \), where \( N(v_{i,j}) = \{v_{i-1,j}, v_{i+1,j}, v_{i,j-1}, v_{i,j+1}\} \) and the addition in the first suffix is taken modulo \( m \) and in the second suffix modulo \( n \). Let \( V_1 = \{v_{i,j} : i = 0, 1, \ldots, m - 1, j = 0, 2, \ldots, n - 2\}, V_2 = \{v_{i,j} : i = 0, 1, \ldots, m - 1, j = 1, 3, \ldots, n - 1\} \). Notice that for any \( v \in G \) we obtain \( |N(v) \cap V_1| = |N(v) \cap V_2| = 2 \).
2. Let $V(C_m \times C_n) = \{v_{i,j} : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$, where $N(v_{i,j}) = \{v_{i-1,j-1}, \ldots\}$ for $u \in V(H)$, $u_j \in V(H_j)$, $j = 1, 2, \ldots, t$. Obviously, $\ell'$ is a bijection. Moreover, for any $(g, h) \in V(G \circ H)$ we have

\[ |N(v) \cap V_1| = |N(v) \cap V_2| = 2. \]

3. Let $V(K(n;r)) = \{v_i^r : i = 1, \ldots, n, j = 1, \ldots, r\}$, $C_4 = xuywx$, and $H = K(n;r) \square C_4$. Let $V_1 = \{(v_i^r, x), (v_i^r, u), (v_i^{n/2+r}, y), (v_i^{n/2}, w)\}$, where $i = 1, 2, \ldots, n/2, j = 1, 2, \ldots, r$, $V_2 = \{(v_i^{n/2+r}, x), (v_i^{n/2}, u), (v_i^r, y), (v_i, w)\}$, where $i = 1, 2, \ldots, n/2, j = 1, 2, \ldots, r$. Obviously for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = n(r - 1)/2 + 1.$

4. Let us define the set of vertices of $Q_n$ as the set of binary strings of length $n$, that is, $V = \{(a_1, a_2, \ldots, a_n) : a_i \in \{0, 1\}\}$. Two vertices are adjacent if and only if the corresponding strings differ in exactly one position. Then $V_1 = \{(a_1, \ldots, a_n)\}$, where $a_1 + \cdots + a_{n/2}$ is even}, $V_2 = \{(a_1, \ldots, a_n)\}$, where $a_1 + \cdots + a_{n/2}$ is odd. Notice that each vertex has $n/2$ neighbours in $V_1$ and $n/2$ in $V_2$.

5. Let $V(G) = \{x_0, x_1, \ldots, x_{p^2-2}\}$, where $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and the addition in the suffix is taken modulo $n$. Let $V_1 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 0, 1, \ldots, p-1\}$, $V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 2, \ldots, p-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = 2.$

6. Let $V(G) = \{x_0, x_1, \ldots, x_{2p^2-3}\}$, where $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and the addition in the suffix is taken modulo $n$. Let $V_1 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 0, 1, \ldots, 2p-1\}$, $V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 2, \ldots, 2p-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = 2.$

7. Let $V(G) = \{x_0, x_1, \ldots, x_{2p(p+1)-1}\}$, where $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and the addition in the suffix is taken modulo $n$. Let $V_1 = \{x_{i+j(p+1)} : i = 0, 1, \ldots, p, j = 0, 1, \ldots, p-2\}$, $V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, 2p-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = p.$

Below we show some interesting properties of distance magic unions of graphs.

**Theorem 2.4.** If $G$ is an $r$-regular graph of order $t$ and $H$ is $p$-regular such that $tH$ is distance magic, then the product $G \circ H$ is distance magic.

**Proof.** Let $\ell$ be a distance magic labeling of the graph $tH = H_1 \cup H_2 \cup \cdots \cup H_t$ with a magic constant $\mu$. For any $u \in V(H)$ let $u_j$ be the corresponding vertex belonging to $V(H_j)$, $j = 1, 2, \ldots, t$. Let $V(G) = \{1, 2, \ldots, t\}$. Notice that for any $i = 1, 2, \ldots, t$ we have $\sum_{v \in V(H_i)} \ell(v) = \frac{|H_j| \mu}{p}$.

Define the labeling $\ell'$ of $G \circ H$ as $\ell'(j, u) = \ell(u_j)$ for $u \in V(H)$, $u_j \in V(H_j)$, $j = 1, 2, \ldots, t$. Obviously, $\ell'$ is a bijection. Moreover, for any $(g, h) \in V(G \circ H)$
we obtain
\[
\sum_{(j,u) \in N_{G \circ H}(g,h)} \ell'(j,u) = \sum_{j \in N_G(g)} \sum_{u \in V(H)} \ell'(j,u) + \sum_{u \in N_H(h)} \ell'(g,u) = r \sum_{u \in V(H)} \ell(u) + \sum_{u \in N_H(h)} \ell(u) = (2r + 1)\mu,
\]
for any \((g, h) \in V(G \circ H)\).

**Theorem 2.5.** If \(G\) is an \(r\)-regular graph of order \(t\) and \(H\) is \(p\)-regular such that \(tH\) is closed distance magic, then the product \(G \circ H\) is closed distance magic.

Notice that the assumption that \(H\) is a regular graph is not necessary, as shown in the observation below.

**Observation 2.** Let \(G\) be an \(r\)-regular graph of order \(t\). If \(m\) and \(n\) are two positive even integers such \(m + n \equiv 0 \pmod{4}\) and either \(2(2tn + 1)^2 - (2tm + 2tn + 1)^2 = 1\) or \(m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2} - 1}{2t}\), then the product \(G \circ K_{m,n}\) is distance magic.

**Proof.** The graph \(tK_{m,n}\) is distance magic by Theorem 1.12. Let \(\ell\) be a distance magic labeling of the graph \(tK_{m,n} = K_{m,n}^1 \cup K_{m,n}^2 \cup \cdots \cup K_{m,n}^t\) with the magic constant \(\mu\). For any \(u \in V(K_{m,n})\) let \(u_j\) be the corresponding vertex belonging to \(V(K_{m,n}^j)\), \(j = 1, 2, \ldots, t\). Let \(V(G) = \{1, 2, \ldots, t\}\). We have \(\sum_{v \in V(K_{m,n}^i)} \ell(v) = 2\mu\) for any \(i = 1, 2, \ldots, t\). Define the labeling \(\ell'\) of \(G \circ H\) as \(\ell'(j,u) = \ell(u_j)\) for \(u \in V(K_{m,n}), u_j \in V(K_{m,n}^j), j = 1, 2, \ldots, t\). As in the proof of Theorem 2.4 we have

\[
\sum_{(j,u) \in N_{G \circ K_{m,n}}((g,h))} \ell'(j,u) = \sum_{j \in N_G(g)} \sum_{u \in V(H)} \ell'(j,u) + \sum_{u \in N_H(h)} \ell'(g,u) = r |H|\mu + \mu = (r|H| + p)\mu.
\]

Using the same technique we can prove an analogous theorem for closed distance magic labeling.

**Theorem 2.6.** If \(G\) is an \(r\)-regular graph of order \(t\) and \(H\) is such that \(tH\) is distance magic, then the product \(G \times H\) is distance magic.

**Proof.** Let \(\ell\) be a distance magic labeling of the graph \(tH = H_1 \cup H_2 \cup \cdots \cup H_t\) with the magic constant \(\mu\). For any \(u \in V(H)\) let \(u_j\) be the corresponding vertex...
belonging to $V(H_j)$, $j = 1, 2, \ldots, t$. Let $V(G) = \{1, 2, \ldots, t\}$. Set the labeling $\ell'$ of $G \times H$ as $\ell'(j, u) = \ell(u_j)$ for $u \in V(H)$, $u_j \in V(H_j)$, $j = 1, 2, \ldots, t$. Therefore

$$w(g, h) = \sum_{(j, u) \in N_G(g) \times N_H(h)} \ell'(j, u) = \sum_{j \in N_G(g)} \sum_{u_j \in N_{H_j}(h_j)} \ell'(j, u) = \sum_{j \in N_G(g)} \mu = r\mu,$$

for any $(g, h) \in V(G \times H)$.

Now we present a theorem, which is a corollary of Lemma 2.1 and Theorems 2.4 and 2.6.

**Theorem 2.7.** If $G$ is an $r$-regular graph and $H$ is a $p$-regular distance magic graph with a 2-partition, then the products $G \circ H$ and $G \times H$ are both distance magic.

Notice that even if $G$ and $H$ are both regular distance magic graphs with 2-partitions, then the product $G \square H$ is not necessarily distance magic (for instance $G = H = C_4$).

Below are presented some families of disconnected distance magic graphs.

**Theorem 2.8.** If

1. $H = C_n \boxtimes C_m$ for $n = m$ and $m \equiv n \equiv 2 \pmod{4}$,
2. $H = C_n \times C_m$ for $n = 4$ or $m = 4$, or $m \equiv n \equiv 0 \pmod{4}$,
3. $H = K(n; r) \sqcap C_4$ for $n > 2$, $r > 1$ and $n$ even,
4. $H = Q_d$ for $d \equiv 2 \pmod{4}$,
5. $H = C_{p^2-1}(1, p)$ for $p$ odd,
6. $H = C_{2(p^2-1)}(1, p)$ for $p$ even,
7. $H = C_{2p(p+1)}(1, 2, \ldots, p)$ for $p$ odd,

then $tH$ is distance magic. Moreover, if $G$ is an $r$-regular graph, then the products $G \circ H$ and $G \times H$ are distance magic as well.

**Proof.** We obtain that $tH$ is distance magic by Lemma 2.1, Observation 1 and Theorems 1.3, 1.4, 1.6, 1.7, 1.8 and 1.9, respectively. By Theorem 2.7 we obtain now that $G \circ H$ and $G \times H$ are distance magic. 

We conclude this section with an observation that can be obtained easily by applying Theorems 1.10, 1.11, 2.4 and 2.6.

**Observation 3.** If $G$ is an $r$-regular graph of order $t$ and

1. $H = K(n; p)$ for $n$ odd, $t \geq 2$ even and $p \equiv 3 \pmod{4}$,
2. \( H = C_p \circ \overline{K_n} \) for \( t \geq 1, \ n \geq 3 \) and \( p \geq 3, \) \( tnp \) odd or \( n \) odd and \( p \equiv 0 \) (mod 4),
then the products \( G \circ H \) and \( G \times H \) are distance magic.

3. Closed Distance Magic Graphs

We start with the following observations about closed distance magic graphs:

**Observation 4** [4]. Let \( u \) and \( v \) be vertices of a closed distance magic graph. Then \( |N(u) \cup N(v)| = 0 \) or \( |N(u) \cup N(v)| > 2. \)

**Observation 5** [3]. If \( G \) is an \( r \)-regular graph on \( n \) vertices having a closed distance magic labeling with a magic constant \( \mu' \), then \( \mu' = \frac{(r+1)(n+1)}{2} \).

We will present now two examples of graphs that have a closed 3-partition.

**Observation 6.** If
1. \( G = C_3 \), or
2. \( G = C_n \otimes C_m \) for \( n = 3 \) and \( m \) odd, or \( m \equiv n \equiv 3 \) (mod 6),
then \( G \) has the closed 3-partition.

**Proof.** 1. Let \( V(C_3) = \{v_0, v_1, v_2\} \). Let \( V_i = \{v_i\} \) for \( i = 0, 1, 2. \)
2. Let \( V(C_m \otimes C_n) = \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\} \), where \( N(v_{i,j}) = \{v_{i-1,j-1}, v_{i-1,j+1}, v_{i-1,j+1}, v_{i,j-1}, v_{i+1,j-1}, v_{i+1,j+1}, v_{i,j+1}, v_{i+1,j+1}\} \) and the addition in the first suffix is taken modulo \( m \) and in the second suffix modulo \( n \). Let \( V_p = \{v_{i,j} : i + j \equiv p \) (mod 3)\}. Notice that for any \( v \in G \) we obtain \( |N[v] \cap V_1| = |N[v] \cap V_2| = |N[v] \cap V_3| = \frac{mn}{3}. \)

**Theorem 3.1.** If
1. \( G = C_3 \), or
2. \( G = C_n \otimes C_m \) for \( n = 3 \) and \( m \) odd, or \( m, n \equiv 3 \) (mod 6),
then \( tG \) is closed distance magic if and only if \( t \) is odd.

**Proof.** Notice that if \( G = C_3 \) then it is closed distance magic. Note that \( G = C_n \otimes C_m \) for \( n = 3 \) and \( m \) odd, or \( m, n \equiv 3 \) (mod 6), is closed distance magic by Theorem 1.13. Since \( G \) has a closed 3-partition, then the graph \( tG \) is closed distance magic by Lemma 2.3 for odd \( t. \) Observe that \( G \) is an \( r \)-regular graph with \( r \) even. Suppose now that \( t \) is even. Then \( |V(tG)| \) is even as well and \( \frac{(r+1)(|V(tG)|+1)}{2} \) is not an integer. Therefore the graph \( G \) is not closed distance magic by Observation 5. \( \blacksquare \)
By Lemma 4 it is now obvious that $tC_n$ is closed distance magic if and only if $t$ is odd and $n = 3$. Moreover, by Theorem 2.4 we obtain immediately the following observation.

**Observation 7.** When $G$ is an $r$-regular graph with $r$ odd and

1. $H = C_3$, or
2. $H = C_n \boxtimes C_m$ for $n = 3$ and $m$ odd, or $m \equiv n \equiv 3 \pmod{6}$,

then the product $G \circ H$ is closed distance magic.

**References**


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