THE DICHROMATIC NUMBER OF INFINITE FAMILIES OF CIRCULANT TOURNAMENTS

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Abstract

The dichromatic number \( dc(D) \) of a digraph \( D \) is defined to be the minimum number of colors such that the vertices of \( D \) can be colored in such a way that every chromatic class induces an acyclic subdigraph in \( D \). The cyclic circulant tournament is denoted by \( T = \overrightarrow{C}_{2n+1}(1, 2, \ldots, n) \), where \( V(T) = \mathbb{Z}_{2n+1} \) and for every jump \( j \in \{1, 2, \ldots, n\} \) there exist the arcs \((a, a+j)\) for every \( a \in \mathbb{Z}_{2n+1} \). Consider the circulant tournament \( \overrightarrow{C}_{2n+1} \langle k \rangle \) obtained from the cyclic tournament by reversing one of its jumps, that is, \( \overrightarrow{C}_{2n+1} \langle k \rangle \) has the same arc set as \( \overrightarrow{C}_{2n+1}(1, 2, \ldots, n) \) except for \( j = k \) in which case, the arcs are \((a, a-k)\) for every \( a \in \mathbb{Z}_{2n+1} \). In this paper, we prove that \( dc(\overrightarrow{C}_{2n+1} \langle k \rangle) \in \{2, 3, 4\} \) for every \( k \in \{1, 2, \ldots, n\} \). Moreover, we classify which circulant tournaments \( \overrightarrow{C}_{2n+1} \langle k \rangle \) are vertex-critical \( r \)-dichromatic for every \( k \in \{1, 2, \ldots, n\} \) and \( r \in \{2, 3, 4\} \). Some previous results by Neumann-Lara are generalized.

Keywords: tournament, dichromatic number, vertex-critical \( r \)-dichromatic tournament.

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1. Introduction

A tournament \( T \) is an orientation of the complete graph. If \( T \) contains no directed cycles, \( T \) is called transitive and denoted by \( TT_k \), where \( k \in \mathbb{N} \) is its order.

The definition of the dichromatic number of a digraph and the first important results were introduced by Neumann-Lara in 1982 [9]. Independently, Jacob and Meyniel defined the same notion in 1983, see [6]. In 1977, Erdős visited Mexico
and began to work on the dichromatic number of a graph (see the definition below) with Neumann-Lara. The results of this collaboration were summarized in a survey by Erdős in 1979 (see [3] for details).

Other results concerning this topic can be found in a paper by Erdős, Gimbel and Kratsch [4]. According to this paper, the \textit{dichromatic number of a digraph} $D$, denoted by $dc(D)$, is “the minimum number of parts the vertex set of $D$ must be partitioned into, so that each part induces an acyclic digraph.” Equivalently, the dichromatic number of $D$ is the minimum number of colors such that the vertices of $D$ can be colored in such a way that every chromatic class induces an acyclic subdigraph in $D$. The main result of paper [4] for tournaments is the following: every tournament with $n$ vertices can be colored with $O(n/\log n)$ and there exists tournaments (for example, random tournaments) having dichromatic number $\Omega(n/\log n)$ (see Theorem 5 of the aforementioned paper). There are more interesting asymptotic results in [5] by Harutyunyan. In particular, Theorem 2.3.8 states that if $T$ is a tournament of order $n$, then $dc(T) \leq \frac{n}{\log n}(1 + o(1))$.

The \textit{dichromatic number of a graph} $G$ was defined by Erdős and Neumann-Lara in [3] as

$$dc(G) = \max \{dc(\vec{G}) : \vec{G} \text{ is an orientation of } G\}.$$ 

Determining the dichromatic number of a general (di)graph is a very hard problem. Exact values of this parameter are only known for some special classes of digraphs, particularly in a few cases of circulant tournaments (see [1, 7, 9, 12, 10, 11] and [13]). In this paper, we prove that

(i) $dc(\vec{C}_{2n+1}(1)) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } 4 \leq n \leq 7, \\ 4 & \text{if } n \geq 8, \end{cases}$

(Section 3, Corollary to Theorem 11),

(ii) $dc(\vec{C}_{2n+1}(2)) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8, \end{cases}$

(Section 3, Corollary to Theorem 17),

(iii) if $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$, then $dc(\vec{C}_{2n+1}(k)) = 4$ for $n \geq 7$ (Section 4, Theorem 22),

(iv) if $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lfloor \frac{2}{3}n \right\rfloor$, then $dc(\vec{C}_{2n+1}(k)) = 4$ (Section 5, Theorem 26), and

(v) $dc(\vec{C}_{2n+1}(k)) = 3$ for $k = \left\lfloor \frac{2}{3}n \right\rfloor + 1, \ldots, n$, where $n \geq 3$ (Section 5, Theorem 29).
Our results generalize some theorems obtained by Neumann-Lara. At the end of Section 5, we characterize the vertex-critical \( r \)-dichromatic circulant tournaments \( \overrightarrow{C}_{2n+1}(k) \) for every \( k \in \{1, 2, \ldots, n\} \) and \( r \in \{2, 3, 4\} \), see Theorem 32, the main theorem of this paper.

2. Preliminaries

Let \( \mathbb{Z}_m \) be the cyclic group of integers modulo \( m \), where \( m \in \mathbb{N} \) and \( J \) is a nonempty subset of \( \mathbb{Z}_m \setminus \{0\} \) such that \( w \in J \) if and only if \( -w \notin J \) for every \( w \in \mathbb{Z}_m \). The circulant digraph \( \overrightarrow{C}_m(J) \) is defined by \( V(\overrightarrow{C}_m(J)) = \mathbb{Z}_m \) and

\[
A(\overrightarrow{C}_m(J)) = \{(i, j) : i, j \in \mathbb{Z}_m \text{ and } j - i \in J\}.
\]

Notice that \( \overrightarrow{C}_{2n+1}(J) \) is a circulant (or rotational) tournament if and only if \( |J| = n \). We recall that circulant tournaments are regular and their automorphism group is vertex-transitive. We define

\[
\overrightarrow{C}_{2n+1}(1, 2, \ldots, n) := \overrightarrow{C}_{2n+1}(\emptyset) \quad \text{and} \quad \overrightarrow{C}_{2n+1}(1, \ldots, k - 1, -k, k + 1, \ldots, n) = \overrightarrow{C}_{2n+1}(1, \ldots, k - 1, k + 1, \ldots, n, 2n + 1 - k) := \overrightarrow{C}_{2n+1}(k).
\]

Observe that the circulant \( \overrightarrow{C}_m(1) = \overrightarrow{C}_m \) is the directed cycle of length \( m \). If \( V(\overrightarrow{C}_m) = \{a_1, a_2, \ldots, a_m\} \), we denote \( \overrightarrow{C}_m = (a_1, a_2, \ldots, a_m, a_1) \). The tournament \( \overrightarrow{C}_{2n+1}(\emptyset) \) is called the cyclic tournament. It is straightforward to check that there is only one cyclic tournament on \( n \) vertices up to isomorphism for every \( n \in \mathbb{N} \). The isomorphism between digraphs \( G \) and \( H \) is denoted by \( G \cong H \). A digraph \( D \) is called \( r \)-dichromatic if \( dc(D) = r \). It is vertex-critical \( r \)-dichromatic if \( dc(D) = r \) and \( dc(D - v) < r \) for every \( v \in V(D) \). For general terminology, see [2]. In what follows, we will need the following results of [11] and [13].

**Theorem 1** ([13], Theorem 1). If \( T_{2n+1} \) is a regular tournament on \( 2n + 1 \) vertices, then \( dc(T_{2n+1}) = 2 \) if and only if \( T_{2n+1} \cong \overrightarrow{C}_{2n+1}(\emptyset) \).

**Theorem 2** ([13], Theorem 2). \( \overrightarrow{C}_{2n+1}(n) \) is a vertex-critical 3-dichromatic circulant tournament for \( n \geq 3 \).

**Theorem 3** ([11]). \( \overrightarrow{C}_{6m+1}(2m) \) is a vertex-critical 4-dichromatic circulant tournament for \( m \geq 2 \).

Let \( T \) be a tournament and \( k, l \in \mathbb{N} \). We recall that a transitive subtournament \( TT_k \) of \( T \) is maximal if there does not exist a transitive subtournament \( TT_l \) of \( T \) \((k < l)\) such that \( TT_k \) is a subtournament of \( TT_l \).
Remark 4 ([8]). Up to isomorphism
(i) there exists a unique circulant tournament of order 5, that is, $\overrightarrow{C}_5(1, 2) = \overrightarrow{C}_5(\emptyset)$,
(ii) there exist two circulant tournaments of order 7 which are
$\overrightarrow{C}_7(1, 2, 3) = \overrightarrow{C}_7(\emptyset) \cong \overrightarrow{C}_7(1) \cong \overrightarrow{C}_7(2)$ and $\overrightarrow{C}_7(1, 2, 4) = \overrightarrow{C}_7(3)$,
(iii) there exist three circulant tournaments of order 9 which are
$\overrightarrow{C}_9(1, 2, 3, 4) = \overrightarrow{C}_9(\emptyset)$,
$\overrightarrow{C}_9(1, 2, 3, 5) = \overrightarrow{C}_9(4) \cong \overrightarrow{C}_9(1) \cong \overrightarrow{C}_9(3)$ and
$\overrightarrow{C}_9(1, 3, 4, 7) = \overrightarrow{C}_9(2)$,

where $\overrightarrow{C}_9(2) \cong \overrightarrow{C}_3[\overrightarrow{C}_3]$ is the composition of $\overrightarrow{C}_3$ and $\overrightarrow{C}_3$).

In the following three sections we determine the exact value of $dct(\overrightarrow{C}_{2n+1}(k))$
for every $n, k \in \mathbb{N}$. We subdivide the calculations into five cases (the cases are illustrated in Figure 1):
(i) $k = 1$,
(ii) $k = 2$,
(iii) $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$,
(iv) $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lfloor \frac{2}{3}n \right\rfloor$ and
(v) $\left\lfloor \frac{2}{3}n \right\rfloor + 1 \leq k \leq n$.

3. The Dichromatic Number of $\overrightarrow{C}_{2n+1}(1)$ and $\overrightarrow{C}_{2n+1}(2)$

To begin with, let us observe the following facts.

Remark 5.
(i) For every $j \in \mathbb{Z}_{2n+1}$, the set of vertices $\{j - 2, j - 1, j\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}(1)$ and $\overrightarrow{C}_{2n+1}(2)$, respectively,
(ii) $\overrightarrow{C}_{11}(1) \cong \overrightarrow{C}_{11}(4)$,
(iii) $\overrightarrow{C}_{13}(1) \cong \overrightarrow{C}_{13}(3)$.
Proposition 6. \( \text{dc}(\vec{C}_{11}(1)) = \text{dc}(\vec{C}_{13}(1)) = \text{dc}(\vec{C}_{15}(1)) = 3. \)

\textbf{Proof.} Consider \( \vec{C}_{11}(1) \). Since \( \vec{C}_{11}(1) \not\cong \vec{C}_{11}(\emptyset) \), by Theorem 1, \( \text{dc}(\vec{C}_{11}(1)) \geq 3. \) The partition of the vertices of \( \vec{C}_{11}(1) \) given by \( P_1 = \{0, 3, 2, 5, 10, 2n-1\} \) and \( P_3 = \{4, 6, 8\} \) implies that \( \text{dc}(\vec{C}_{11}(1)) = 3. \) Observe that \( P_1 \) and \( P_2 \) induce a \( TT_4 \), and \( P_3 \) induces a \( TT_3 \) in \( \vec{C}_{11}(1) \), respectively. By Remark 5(ii), we have that \( \text{dc}(\vec{C}_{11}(4)) = 3. \) Notice that the transitive subtournaments induced by \( P_1 \), \( P_2 \) and \( P_3 \) are maximal. If \( \langle P_1 \rangle \) was not a maximal transitive subtournament, then the only vertex that we can add is the vertex 10, but \( (2, 5, 10, 2) \cong \vec{C}_3 \). Then \( P_1 \) induces a maximal transitive subtournament. The arguments are similar for \( P_2 \) and \( P_3 \). Analogously we can prove the others cases \( n = 6 \) and \( n = 7. \)

The following lemmas will be useful tools in order to prove Theorem 11. Let \( a \) and \( b \) nonnegative integers such that \( 0 \leq a < b \leq n \). We define the interval \( [a, b] = \{a, a+1, \ldots, b\} \), \( X_0 = [0, n] \), \( X_3 = [0, 2n] \) and
\[
Y_0 = \{j \in X_0 : j \equiv 1 \mod 3\}.
\]

\textbf{Lemma 7.} The tournament \( \vec{C}_{2n+1}(1) \) contains a maximal transitive subtournament of order \( n + 1 - \left\lfloor \frac{n}{3} \right\rfloor \) if \( n \equiv 0 \mod 3 \) and of order \( n + 2 - \left\lceil \frac{n}{3} \right\rceil \) vertices if \( n \equiv 1 \mod 3 \).

\textbf{Proof.} Consider \( \vec{C}_{2n+1}(1) = \vec{C}_{2n+1}(2, 3, 4, 5, 6, \ldots, n, 2n) \). Recall that circulant tournaments are vertex-transitive, so it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that
\[
N^+(0) = \{2, 3, 4, 5, 6, \ldots, n, 2n\}.
\]

We have two cases.

Case 1. \( n \equiv 0 \mod 3 \). Let \( n = 3k \) where \( k \in \mathbb{N} \). Notice that the subset of vertices \( j \equiv 0, 2 \mod 3 \) belonging to the set \( X_0 \) induces a transitive subtournament
\[
H_0 = \langle X_0 \setminus Y_0 \rangle,
\]
by Remark 5(i). Observe that \(|Y_0| = \left\lfloor \frac{n}{3} \right\rfloor\) and \(|H_0| = n + 1 - \left\lfloor \frac{n}{3} \right\rfloor = 2k + 1\). It remains to prove that \(H_0\) is maximal. If \(H_0\) was not maximal, then the only vertex we can add is \(2n\) by Remark 5(i). Observe that in this case the set \(\{n, 2n, n - 4\}\) induces a \(\overrightarrow{C}_3\) in \(\overrightarrow{C}_{2n+1}(1)\), which implies that \(H_0 \cup \{2n\}\) cannot induce a maximal transitive subtournament.

**Case 2.** \(n \equiv 1\) mod 3. This case is analogous to Case 1. The maximal transitive subtournament is

\[ H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0 \rangle. \]

**Lemma 8.** Let \(\overrightarrow{C}_{2n+1}(1)\) be such that \(n \equiv 0, 1\) mod 3. Then the subtournaments induced by

\[ X_3 \setminus (X_0 \setminus Y_0) \] if \(n \equiv 0\) mod 3 and \[ X_3 \setminus ((X_0 \cup \{2n\}) \setminus Y_0) \] if \(n \equiv 1\) mod 3

contain a maximal transitive subtournament of \(n - \left\lfloor \frac{n}{3} \right\rfloor\) and \(n - \left\lfloor \frac{n}{3} \right\rfloor - 1\) vertices, respectively.

**Proof.** Suppose that \(n \equiv 0\) mod 3. By Lemma 7, \(\overrightarrow{C}_{2n+1}(1)\) contains the transitive subtournament \(H_0\). Consider \(X_1 = [n + 1, 2n]\) and define

\[ Y_1 = \{j \in X_1 : j \equiv 2\) mod 3\} and \(J_0 = \langle X_1 \setminus Y_1 \rangle.\]

By Remark 5(i), \(J_0\) is transitive. Notice that \(J_0\) has order \(n - \left\lfloor \frac{n}{3} \right\rfloor\). We prove that \(J_0\) is maximal in the same way as in the proof of Lemma 7. For a contradiction, if \(J_0\) was not maximal, then the only vertex we can add is vertex 1 by Remark 5(i). Observe that in this case, the set \(\{1, n + 1, n + 3\}\) induces a \(\overrightarrow{C}_3\) in \(\overrightarrow{C}_{2n+1}(1)\), which implies that \(J_0 \cup \{1\}\) cannot induce a maximal transitive subtournament.

When \(n \equiv 1\) mod 3, the arguments are similar. The maximal transitive subtournament \(J_1\) is given by

\[ J_1 = \langle X_1 \setminus (Y_2 \cup \{2n\}) \rangle, \]

where \(Y_2 = \{j \in X_1 : j \equiv 0\) mod 3\}.

**Lemma 9.** A maximal transitive subtournament contained in \(\overrightarrow{C}_{2n+1}(1)\) has \(n + 1 - \left\lfloor \frac{n}{3} \right\rfloor\) vertices if \(n \equiv 2\) mod 3.

**Proof.** It is similar to the proof of Lemma 7. The maximal transitive subtournament is

\[ H_2 = \langle X_0 \setminus Y_0 \rangle. \]
Lemma 10. Let $\vec{C}_{2n+1}(1)$ be such that $n \equiv 2 \mod 3$. Then the subtournament induced by
\[ X_3 \setminus (X_0 \setminus Y_0) \]
contains a maximal transitive subtournament of order $n - \left\lceil \frac{n}{3} \right\rceil$.

**Proof.** It is similar to the proof of Lemma 8. In this case, every vertex $j \equiv 0, 1 \mod 3$ in $X_1$ induces a transitive subtournament
\[ J_2 = (X_1 \setminus Y_3), \]
where $Y_3 = \{ j \in X_1 : j \equiv 2 \mod 3 \}$.

**Theorem 11.** Let $n \in \mathbb{N}$. Then $dc(\vec{C}_{2n+1}(1)) = 4$ for every $n \geq 8$.

**Proof.** By Theorem 1, we have that $dc(\vec{C}_{2n+1}(1)) \geq 3$. In the first place, we prove that $dc(\vec{C}_{2n+1}(1)) \geq 4$. For a contradiction, suppose that $dc(\vec{C}_{2n+1}(1)) = 3$. Thus, $\vec{C}_{2n+1}(1)$ has a partition of its vertices inducing three transitive subtournaments. Suppose that $n \equiv 0 \mod 3$ (it is similar when $n \equiv 1, 2 \mod 3$). By Lemmas 7 and 8, two maximal disjoint transitive subtournaments in $\vec{C}_{2n+1}(1)$ are $H_0$ and $J_0$. Hence, the remaining vertex set $X_3 \setminus \{ V(H_0) \cup V(J_0) \}$,
\[ \{1, 4, 7, \ldots, n + 2, n + 5, \ldots \}, \]
induces the third transitive subtournament. Observe that the vertex set $\{ 1, 7, n + 2 \}$ induces a $\vec{C}_3$, this is a contradiction. Therefore, $dc(\vec{C}_{2n+1}(1)) \geq 4$. We show that $dc(\vec{C}_{2n+1}(1)) = 4$. By Lemmas 7 and 8, we have that the two maximal transitive subtournaments $H_0$ and $J_0$ have cardinality $n + 1 - \left\lceil \frac{n}{3} \right\rceil$ and $n - \left\lceil \frac{n}{3} \right\rceil$, respectively. Define a third subtournament
\[ K_0 = \langle \{1\} \cup Y_1 \rangle. \]
Notice that $|K_0| = \left\lceil \frac{n}{3} \right\rceil + 1$ and $K_0$ is transitive by the definition of $Y_1$. We will prove that $K_0$ is a maximal transitive subtournament in $\vec{C}_{2n+1}(1) \setminus \{ H_0 \cup J_0 \}$. If $K_0$ was not a maximal transitive subtournament, then we can add at least one vertex of $Y_0 \setminus \{1\}$. Notice that if $i \in Y_0 \setminus \{1\}$, we have that $(i, i + n - 2, i + n + 1, i) \cong \vec{C}_3$. Therefore, $K_0$ is a maximal transitive subtournament. Finally, the subtournament $L_0 = \langle Y_0 \setminus \{1\} \rangle$ is transitive by the definition of $Y_0$ and maximal. Thus, $dc(\vec{C}_{2n+1}(1)) = 4$. The proof is completely analogous for the cases when $n \equiv 1, 2 \mod 3$. The partition into transitive subtournaments is
\[ H_1 = (\langle X_0 \cup \{2n\} \rangle \setminus Y_0), \quad J_1 = (\langle X_1 \setminus Y_2 \cup \{2n\} \rangle, \quad K_1 = (\langle Y_2 \cup \{1\} \rangle, \quad L_1 = (\langle Y_0 \setminus \{1\} \rangle, \]
for $n \equiv 1 \mod 3$. For $n \equiv 2 \mod 3$, we have that
\[ H_2 = (\langle X_0 \setminus Y_0 \rangle, \quad J_2 = (\langle X_1 \setminus Y_3 \rangle, \quad K_2 = (\langle Y_3 \cup \{1\} \rangle, \quad L_2 = (\langle Y_0 \setminus \{1\} \rangle. \]

[The Dichromatic Number of Infinite Families of ... 227]
From Proposition 6, Theorems 1, 2 and 11 and Remark 4(iii), we obtain the following consequence.

**Corollary 12.**

$$dc(\overrightarrow{C}_{2n+1}(1)) = \begin{cases} 
2 & \text{if } n = 3, \\
3 & \text{if } 4 \leq n \leq 7, \\
4 & \text{if } n \geq 8.
\end{cases}$$

**Theorem 13.** Let \( r \in \{2, 3, 4\} \). Then \( \overrightarrow{C}_{2n+1}(1) \) is a vertex-critical \( r \)-dichromatic circulant tournament if and only if \( n \in \{1, 4\} \).

**Proof.** If \( r = 2 \), clearly \( \overrightarrow{C}_3(1) \) is a vertex-critical 2-dichromatic.

If \( r = 3 \), we need to check for which values of \( 4 \leq n \leq 7 \), the circulant tournament \( \overrightarrow{C}_{2n+1}(1) \) is a vertex-critical 3-dichromatic. Notice that \( \overrightarrow{C}_9(1) \cong \overrightarrow{C}_9(4) \) by Remark 4(iii). It is a vertex-critical 3-dichromatic by Theorem 2. For \( n = 5 \), the circulant tournament \( \overrightarrow{C}_{11}(1) \) is not a vertex-critical 3-dichromatic by Proposition 6. Using analogous arguments, \( \overrightarrow{C}_{13}(1) \) and \( \overrightarrow{C}_{15}(1) \) are not vertex-critical.

If \( r = 4 \), the circulant tournament \( \overrightarrow{C}_{2n+1}(1) \) is 4-dichromatic for every \( n \geq 8 \) by Theorem 11. It was partitioned into four maximal transitive subtournaments, where \( |L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\} \) for \( i = 0, 1, 2 \). Notice that \( \overrightarrow{C}_{2n+1}(1) \) is a vertex-critical 4-dichromatic if the cardinality of \( L_i \) is equal to one for \( i = 0, 1, 2 \). Since \( |L_i| = |Y_0| - 1 = \left\lfloor \frac{n}{3} \right\rfloor - 1 \), we have that \( |L_i| = 1 \) if and only if \( \left\lfloor \frac{n}{4} \right\rfloor = 2 \). It occurs when \( n = 6, 7 \) or 8. By Theorem 11, it is only possible for \( n \geq 8 \). Observe that \( |L_2| \geq 2 \) for \( T = \overrightarrow{C}_{2n+1}(1) \) if \( n \geq 8 \). Since this partition is maximal, \( T \) is not vertex-critical.

Therefore, \( \overrightarrow{C}_{2n+1}(1) \) is a vertex-critical \( r \)-dichromatic circulant tournament if and only if \( n \) is 1 or 4.

Let us recall that

**Remark 14.** \( \overrightarrow{C}_9(2) = \overrightarrow{C}_3[\overrightarrow{C}_3] \) is 3-dichromatic, a particular case of Theorem 8 from [9]. Notice that it is not vertex-critical.

**Remark 15** ([10], Theorem 2.6). \( \overrightarrow{C}_{11}(2) \) is vertex-critical 4-dichromatic.

**Proposition 16.** If \( n = 6 \) and 7, then \( dc(\overrightarrow{C}_{2n+1}(2)) = 3 \).

**Proof.** Observe that \( \overrightarrow{C}_{15}(2) \not\cong \overrightarrow{C}_{15}(\emptyset) \). Then by Theorem 1, \( dc(\overrightarrow{C}_{15}(2)) \geq 3 \). Consider the following partition of \( V(\overrightarrow{C}_{15}(2)) \):

\[ P_1 = \{0, 1, 3, 4, 6, 7\}, P_2 = \{5, 8, 9, 11, 12\} \text{ and } P_3 = \{2, 10, 13, 14\}. \]
We have that $\langle P_1 \rangle \cong TT_6$, $\langle P_2 \rangle \cong TT_5$ and $\langle P_3 \rangle \cong TT_4$. Therefore, $dc(\overrightarrow{C}_{15}(2)) = 3$. Note that the transitive subtournaments induced by $P_1$, $P_2$ and $P_3$ are maximal. If $\langle P_1 \rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is vertex 13. We cannot add the vertex 5 by Remark 5(i). But $(4, 7, 13, 4) \cong \overrightarrow{C}_3$. Then $P_1$ induces a maximal transitive subtournament. The same conclusion is valid for $P_2$ and $P_3$. Observe that $\overrightarrow{C}_{15}(2)$ is not vertex-critical. The proof is analogous for $n = 6$.

**Theorem 17.** Let $n \in \mathbb{N}$. Then $dc(\overrightarrow{C}_{2n+1}(2)) = 4$ for every $n \geq 8$.

**Proof.** It is analogous to the proof of Theorem 11. Therefore, Remark 5(i) is applied for $\overrightarrow{C}_{2n+1}(2)$. The corresponding partitions are following.

(i) $n \equiv 0 \pmod{3}$, we define $Y_4 = \{j \in X_0 : j \equiv 2 \pmod{3}\}$,

$H_0 = \langle X_0 \setminus Y_4 \rangle$, $J_0 = \langle X_1 \setminus Y_2 \rangle$, $K_0 = \langle Y_2 \cup \{2\} \rangle$, $L_0 = \langle Y_4 \setminus \{2\} \rangle$.

(ii) $n \equiv 1 \pmod{3}$, we define $Y_5 = \{j \in X_1 : j \equiv 1 \pmod{3}\}$,

$H_1 = \langle X_0 \setminus Y_4 \rangle$, $J_1 = \langle X_1 \setminus Y_3 \rangle$, $K_1 = \langle Y_3 \cup \{2\} \rangle$, $L_1 = \langle Y_4 \setminus \{2\} \rangle$.

(iii) $n \equiv 2 \pmod{3}$, we define $Y_6 = \{j \in X_1 : j \equiv 1 \pmod{3}\}$,

$H_2 = \langle X_0 \setminus Y_4 \rangle$, $J_2 = \langle (X_1 \cup \{n\}) \setminus Y_6 \rangle$, $K_2 = \langle Y_6 \rangle$, $L_2 = \langle Y_4 \setminus \{n\} \rangle$.

The next corollary is an immediate consequence of Remarks 4(ii)–(iii), 14, 15, Proposition 16 and Theorems 1, 2 and 17.

**Corollary 18.**

$$dc(\overrightarrow{C}_{2n+1}(2)) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8. \end{cases}$$

**Theorem 19.** Let $r \in \{2, 3, 4\}$. Then $\overrightarrow{C}_{2n+1}(2)$ is a vertex-critical $r$-dichromatic circulant tournament if and only if $n = 5$.

**Proof.** If $r = 2$, by Theorem 1, $\overrightarrow{C}_7(2) \cong \overrightarrow{C}_7(\emptyset)$ is 2-dichromatic, but it is not vertex-critical.

Let $r = 3$. For $n = 4$, we have that $\overrightarrow{C}_9(2)$ is not vertex-critical by Remark 14. For $n = 6$ and 7 by Proposition 16, $\overrightarrow{C}_{13}(2)$ and $\overrightarrow{C}_{15}(2)$ are not vertex-critical.

If $r = 4$, then by Remark 15, $\overrightarrow{C}_{11}(2)$ is vertex-critical. By Theorem 17, $\overrightarrow{C}_{2n+1}(2)$ is 4-dichromatic for every $n \geq 8$. It was partitioned into four maximal
transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0, 1, 2$. Notice that $\overrightarrow{C}_{2n+1}(2)$ is vertex-critical if and only if $n = 6, 7$ or $8$. By Theorem 17, it is only possible for $n \geq 8$. Observe that $|L_2| \geq 2$ for $T = \overrightarrow{C}_{2n+1}(2)$ if $n \geq 8$.

Since this partition is maximal, $T$ is not vertex-critical. Therefore, $\overrightarrow{C}_{2n+1}(2)$ is vertex-critical if and only if $n = 5$.

4. The Dichromatic Number of $\overrightarrow{C}_{2n+1}(k)$ for $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$

We prove that $dc(\overrightarrow{C}_{2n+1}(k)) = 4$, for $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$ and $n \geq 7$.

Lemma 20. If $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$, then $\overrightarrow{C}_{2n+1}(k)$ contains a maximal transitive subtournament $H$.

Proof. Let $n$ and $k$ be nonnegative integers and consider the interval $[0, n]$. Applying the Euclidean division algorithm to $n+1$ and $2k-1$, there exist unique $\alpha, r \in \mathbb{N}$ such that

$$n + 1 = \alpha(2k-1) + r \quad \text{where} \quad 0 \leq r < 2k-1.$$ 

Consider the partition of the interval $[0, 2k-2] = [0, k-1] \cup [k, 2k-2]$ and define

$$n + 1 = \begin{cases} 
\alpha(2k-1) + s_1 & \text{if} \ s_1 \in [0, k-1], \\
\alpha(2k-1) + s_2 & \text{if} \ s_2 \in [k, 2k-2]. 
\end{cases}$$

Observe that since $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$, we have that $s_1 \in [1, k-1]$.

Let

$$W = \bigcup_{i=0}^{\alpha-1} [i(2k-1), i(2k-1) + (k-1)].$$

We define the subtournament $H$ of $\overrightarrow{C}_{2n+1}(k)$ in the following way.

(i) If $s_1 \in [1, k-1]$, then $H = (W \cup [\alpha(2k-1), n])$. Moreover, if $k = \frac{n+1}{2}$ and $n$ is odd, then $H = (W \cup \{n, 2n+1-k\})$.

(ii) $H = (W \cup [\alpha(2k-1), \alpha(2k-1) + k-1])$ for every $s_2 \in [k, 2k-2]$.

Note that $H$ is a transitive subtournament by construction, since its vertex set does not contain induced $\overrightarrow{C}_3$'s. We prove that $H$ is maximal by contradiction. Since $\overrightarrow{C}_{2n+1}(k)$ is vertex-transitive, without loss of generality, choose the vertex 0. Observe that $N^+(0) = \{1, 2, \ldots, k-1, k+1, \ldots, n, 2n+1-k\}$ and

$$N^+(0) \setminus V(H) = (X_0 \cup \{2n+1-k\}) \setminus (V(H) \cup \{k\}).$$
For every vertex \( x \in N^+(0) \setminus V(H) \) there exist \( h_1, h_2 \in V(H) \) such that the vertex set \( \{h_1, h_2, x\} \) induces a \( \overrightarrow{C}_3 \) (for instance, \( x = k + 1, h_1 = 1 \) and \( h_2 = k - 1 \)), a contradiction. Therefore, \( H \) is maximal.

\[ \text{Theorem 22. If } 3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil, \text{ then } dc(\overrightarrow{C}_{2n+1}(k)) = 4 \text{ for } n \geq 7. \]

\[ \text{Proof. The construction of } J \text{ is similarly obtained as in the proof of Lemma 20 for } H, \text{ but we have two ways of defining } J. \]

\[ \begin{align*}
\text{Case 1. } & \alpha = 1. \\
(\text{i}) & \text{ If } s_1 \in [1, k - 1], \text{ then } J = [k, 2k - 2] \cup [3k - 2, 3k + s_1 - 2]. \text{ Notice that if } k = \frac{n + 1}{2} \text{ with } n \text{ odd if and only if } s_1 = 1. \text{ Then } J = [k, 2k - 2] \cup \{3k - 2\} \text{ by the construction of } H. \\
(\text{ii}) & \text{ If } s_2 \in [k, 2k - 2], \text{ then } J = [k, 2k - 2] \cup [3k - 1, 4k - 2].
\end{align*} \]

\[ \begin{align*}
\text{Case 2. } & \alpha > 1. \text{ Let } \\
U & = \bigcup_{i=0}^{\alpha-1} [(n + 1) + i(2k - 1), (n + 1) + i(2k - 1) + (k - 1)]. \\
(\text{i}) & \text{ If } s_1 \in [1, k - 1], \text{ then } J = (U \cup [(n + 1) + \alpha(2k - 1), 2n]). \\
(\text{ii}) & J = (U \cup [(n+1)+\alpha(2k-1), (n+1)+\alpha(2k-1)+k-1]) \text{ for every } s_2 \in [k, 2k-2].
\end{align*} \]

Notice that \( H \) is a maximal transitive subtournament in \( \overrightarrow{C}_{2n+1}(k) \) by Lemma 20. We claim that \( J \) is maximal in \( V(\overrightarrow{C}_{2n+1}(k)) \setminus V(H) \). If \( J \) was not maximal, we could add at least one vertex of \( V(\overrightarrow{C}_{2n+1}(k)) \setminus (V(H) \cup V(J)). \)

For Case 1, consider
\[ \{n + k + 1, 3k - 3\} \subseteq V(\overrightarrow{C}_{2n+1}(k)) \setminus (V(H) \cup V(J)). \]

We have that \( (k, n+k, n+k+1, k) \cong \overrightarrow{C}_3 \) or \( (2k-2, 3k-3, 3k-2, 2k-2) \cong \overrightarrow{C}_3. \) Therefore, \( J \) is maximal.

For Case 2, consider
\[ k \in V(\overrightarrow{C}_{2n+1}(k)) \setminus (V(H) \cup V(J)). \]

We have that \( (k, n+k, n+2k+1, k) \cong \overrightarrow{C}_3. \) Hence, \( J \) is maximal.

\[ \text{Theorem 22. If } 3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil, \text{ then } dc(\overrightarrow{C}_{2n+1}(k)) = 4 \text{ for } n \geq 7. \]
Proof. By Theorem 1, $dc(\overrightarrow{C}_{2n+1}(k)) \geq 3$. We prove that $dc(\overrightarrow{C}_{2n+1}(k)) \geq 4$. For a contradiction, suppose that $dc(\overrightarrow{C}_{2n+1}(k)) = 3$. Thus, $\overrightarrow{C}_{2n+1}(k)$ has a partition of its vertices consisting of three transitive subtournaments. By Lemmas 20 and 21, two maximal transitive disjoint subtournaments in $\overrightarrow{C}_{2n+1}(k)$ are $H$ and $J$. Hence, the remaining vertex set $X_3 \setminus (V(H) \cup V(J))$ induces the third transitive subtournament.

We consider three cases.

Case 1. $J = \langle [k, 2k-2] \cup [3k-2, 3k+s_1-2] \rangle$ obtained by Case 1(i) of Lemma 21. Therefore, $K = \langle [3k+s_1-1, 2n] \rangle$. Moreover, $|J| = k+s_1$ and $|H| = n-k+2$. Since $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, we have that $|K| = 2n+1 - (|H| + |J|) = 2k-3 > k$. In this case, $K$ is induced by at least $k+1$ consecutive vertices. Therefore, $K$ cannot be a transitive subtournament by the definition of $\overrightarrow{C}_{2n+1}(k)$. Hence, $dc(\overrightarrow{C}_{2n+1}(k)) \geq 4$. The following cases are necessary because the structure of $K$ and $L$ changes with different values of $s_1$.

(i) If $s_1 = 1, 2$ or 3, then

$$K = \langle [n+1, 3k-3] \cup [4k-3, 2n] \rangle$$

and

$$L = \langle [3k+s_1-1, 4k-4] \rangle.$$

(ii) If $s_1 \in [4, k-5]$, then

$$K = \langle [n+1, 3k-3] \cup [4k-3, 5k-4] \rangle$$

and

$$L = \langle [3k+s_1-1, 4k-4] \cup [5k-3, 2n] \rangle.$$

(iii) If $s_1 = k-4$, then

$$K = \langle [n+1 = 3k-4, 3k-3] \cup [4k-3, 5k-4] \rangle$$

and

$$L = \langle [4k-5, 4k-4] \cup [5k-3, 2n] \rangle.$$

(iv) If $s_1 = k-3$, then

$$K = \langle [n+1 = 3k-5, 3k-3] \cup [4k-3, 5k-4] \rangle$$

and

$$L = \langle [4k-4] \cup [5k-3, 2n] \rangle.$$

(v) If $s_1 = k-1$ or $k-2$, then

$$K = \langle [n+k+1, n+2k] \rangle$$

and

$$L = \langle [n+2k+1, 2n] \rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}(k)$, the subtournaments $K$ and $L$ are transitive. Observe that if $s_1 \in [1, k-3]$, $4k-4 \notin V(K)$ and $(4k-4, 4k-3, 3k-3, 4k-4) \cong \overrightarrow{C}_3$. Therefore, $K$ is maximal in $\overrightarrow{C}_{2n+1}(k) \setminus (H \cup J)$. If $s_1 = k-1$ or $k-2$, then $n+2k+1 \notin V(K)$ and $(n+k+1, n+2k, n+2k+1, n+k+1) \cong \overrightarrow{C}_3$. Thus, $K$ is maximal in $\overrightarrow{C}_{2n+1}(k) \setminus (H \cup J)$. Hence, $dc(\overrightarrow{C}_{2n+1}(k)) = 4.$
Case 2. $J = [k, 2k - 2] \cup [3k - 1, 4k - 2]$ obtained by Case 1(ii) of Lemma 21. We have that $X_3 \setminus (V(H) \cup V(J)) = [4k - 1, 2n]$, but $2n - 4k + 2 > k$ implies that the subtournament induced by $X_3 \setminus (V(H) \cup V(J))$ has at least $k + 1$ consecutive vertices and a $C_3$ is induced by $X_3 \setminus (V(H) \cup V(J))$. Therefore, $dc(\overrightarrow{C}_{2n+1}(k)) \geq 4$. The following cases show the partition of $\overrightarrow{C}_{2n+1}(k)$ into transitive subtournaments.

(i) If $s_2 = k$, then

$$K = \langle [4k - 1, 5k - 2] \rangle$$

and $L = \langle [5k - 1, 2n] \rangle$.

(ii) If $s_2 = k + 1$, then

$$K = \langle [4k - 1, 5k - 2] \cup \{6k - 2 = 2n\} \rangle$$

and $L = \langle [5k - 1, 6k - 3] \rangle$.

(iii) If $s_2 = k + 2$, then

$$K = \langle [4k - 1, 5k - 2] \cup [6k - 2, 6k] \rangle$$

and $L = \langle [5k - 1, 6k - 3] \rangle$.

(iv) If $s_2 \in [k + 3, 2k - 2]$, then

(a) if $2n \leq 7k - 3$, we have that

$$K = \langle [4k - 1, 5k - 2] \cup [6k - 2, 2n] \rangle$$

and $L = \langle [5k - 1, 6k - 3] \rangle$,

(b) if $2n > 7k - 3$, then

$$K = \langle [4k - 1, 5k - 2] \cup [6k - 2, 7k - 3] \rangle$$

and $L = \langle [5k - 1, 6k - 3] \cup [7k - 2, 2n] \rangle$.

By construction and the definition of $\overrightarrow{C}_{2n+1}(k)$, the subtournaments $K$ and $L$ are transitive. Observe that $5k - 1 \notin V(K)$ and $(4k - 1, 5k - 2, 5k - 1, 4k - 1) \cong \overrightarrow{C}_3$. Therefore, $K$ is maximal in $\overrightarrow{C}_{2n+1}(k) \setminus (H \cup J)$. Hence, $dc(\overrightarrow{C}_{2n+1}(k)) = 4$.

Case 3. $J$ is obtained by Case 2(i) and (ii) of Lemma 21. If $dc(\overrightarrow{C}_{2n+1}(k)) = 3$, then $X_3 \setminus (V(H) \cup V(J))$ induces a transitive subtournament, but the vertex set $\{k, 3k-1, n+k+1\} \subseteq X_3 \setminus (V(H) \cup V(J))$ induces a $\overrightarrow{C}_3$. Hence, $dc(\overrightarrow{C}_{2n+1}(k)) \geq 4$. The partition of $\overrightarrow{C}_{2n+1}(k)$ into transitive subtournaments is $H, J, K = \langle X_1 \cup \{k\} \setminus V(J) \rangle$ and

$$L = \langle X_3 \setminus (V(H) \cup V(J) \cup V(K)) \rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}(k)$, the subtournaments $K$ and $L$ are maximal transitive. Observe that $k + 1 \notin V(K)$. Then the vertex set $\{k, k + 1, n + k + 1\}$ induces a $\overrightarrow{C}_3$. Therefore, $K$ is maximal in $\overrightarrow{C}_{2n+1}(k) \setminus (H \cup J)$.

This proves that $dc(\overrightarrow{C}_{2n+1}(k)) = 4$. \qed
The following example illustrates Theorem 22, Case 2(ii). The tournament $\overrightarrow{C}_{29}(5)$ has the following partition into four transitive subtournaments $H = \langle [0, 4] \cup [9, 13] \rangle$, $K = \langle [5, 8] \cup [14, 18] \rangle$, $J = \langle [19, 23] \cup \{28\} \rangle$ and $L = \langle [24, 27] \rangle$. Observe that $H \cong TT_{10}$, $J \cong TT_9$, $K \cong TT_6$ and $L \cong TT_4$.

**Theorem 23.** If $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$, then $\overrightarrow{C}_{2n+1}(k)$ is a vertex-critical 4-dichromatic circulant tournament if and only if

1. $n = 7$ and $k \in \{3, 4\}$,
2. $n = 9$ and $k = 4$,
3. $n = 10$ and $k = 5$,
4. $n = 13$ and $k = 6$.

**Proof.** By Theorem 22, $\overrightarrow{C}_{2n+1}(k)$ is 4-dichromatic, where $H$, $J$, $K$ and $L$ are maximal transitive subtournaments. Note that by the partition of the vertices of $\overrightarrow{C}_{2n+1}(k)$, the cases that need to be considered are when $\alpha = 1$, because it is when the order of $L$ can be one. In this case, $\overrightarrow{C}_{2n+1}(k)$ is a vertex-critical 4-dichromatic. We have two cases when $\alpha = 1$.

**Case 1.** $s_1 \in [1, k-1]$.

1. If $s_1 \in \{1, 2, 3\}$, then by Theorem 22 Case 1(i), we have that $|L| = k - 3, k - 4, k - 5$, respectively. The tournament is vertex-critical if and only if $|L| = 1$ if and only if $k = 4$ and $n = 7$, $k = 5$ and $n = 10$, $k = 6$ and $n = 13$, respectively.
2. If $s_1 = k - 3$, then by the proof of Theorem 22 Case 1(iv), it is vertex-critical if and only if $|L| = 1$ if and only if $2n = 5k - 4$ and $n = 3k - 5$ if and only if $k = 6$ and $n = 13$.
3. If $s_1 = k - 2$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = k - 2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
4. If $s_1 = k - 1$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = n - 2k$. It is vertex-critical if and only if $|L| = 1$ if and only if $n = 2k + 1$ and $n = 3k - 3$ if and only if $k = 4$ and $n = 9$.

**Case 2.** $s_2 \in [k, 2k - 2]$.

1. If $s_2 = k$, then by the proof of Theorem 22 Case 2(i), we have that $|L| = k - 2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
2. If $s_2 \in [k + 1, 2k - 2]$, then by the proof of Theorem 22 Case 2(ii)–(iv)(a), we have that $|L| = k - 2$, but it is not necessarily vertex-critical if $|L| = 1$, because the last vertices remain in $K$. When $L$ is obtained by the proof of Theorem 22 Case 2(iv)(b), $|L|$ never is one. In any case, $\overrightarrow{C}_{2n+1}(k)$ is not a vertex-critical 4-dichromatic circulant tournament. □
5. The Dichromatic Number of $\overrightarrow{C}_{2n+1}(k)$ for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n$.

In this part we prove that the tournaments $\overrightarrow{C}_{2n+1}(k)$ are 4-dichromatic if $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lceil \frac{3n}{2} \right\rceil$ for $n \geq 8$.

**Lemma 24.** If $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lceil \frac{3n}{2} \right\rceil$, then $\overrightarrow{C}_{2n+1}(k)$ contains a maximal transitive subtournament of order $k$.

**Proof.** Since $\overrightarrow{C}_{2n+1}(k)$ is vertex-transitive, it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that $N^+(0) = \{1, 2, \ldots, k-1, k+1, \ldots, n, 2n+1-k\}$. We define $H = \langle [0, k-1] \rangle$. It is transitive by the definition of $\overrightarrow{C}_{2n+1}(k)$. If $H$ was not maximal, then we could add one vertex of $N^+(0) \setminus [0, k-1]$. Let $j \in [k+1, n]$. Without loss of generality, choose $j = k+1$. Thus, the set of vertices $\{1, t, k+1\}$ with $t \in [2, k-1]$ induces a $\overrightarrow{C}_3$. The same occurs for the vertex $2n+1-k$. Observe that $(3, k-1, 2n+1-k, 3) \cong \overrightarrow{C}_3$, a contradiction. Therefore, $H$ is maximal. ■

**Lemma 25.** If $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lceil \frac{3n}{2} \right\rceil$, then $\overrightarrow{C}_{2n+1}(k)$ contains three maximal transitive subtournaments of $k$ vertices.

**Proof.** By Lemma 24, $\overrightarrow{C}_{2n+1}(k)$ contains a maximal transitive subtournament $H$. Notice that $|N^+(0)| - |H| < k$. Consider the following subtournaments

$$J = \langle [k, 2k-1] \rangle \quad \text{and} \quad K = \langle [2k, 3k-1] \rangle.$$

Observe that $J$ and $K$ are isomorphic to $H$. Let $\varphi_1 : H \rightarrow J$ such that $\varphi_1(j) = j + k$ with $0 \leq j \leq k-1$, ($\varphi_1$ is bijective and it is clear that $H$ is isomorphic to $J$). Analogously, $\varphi_2 : H \rightarrow K$ is an isomorphism between $H$ and $K$. As in Lemma 24, we can prove that $J$ and $K$ are maximal transitive subtournaments. Then $\overrightarrow{C}_{2n+1}(k)$ contains three maximal transitive subtournaments on $k$ vertices. ■

**Theorem 26.** If $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lceil \frac{3n}{2} \right\rceil$, then $dc(\overrightarrow{C}_{2n+1}(k)) = 4$.

**Proof.** First we prove that $dc(\overrightarrow{C}_{2n+1}(k)) \geq 4$. By Lemma 25, we have that $\overrightarrow{C}_{2n+1}(k)$ contains three maximal transitive subtournaments of $k$ vertices. Then $|\overrightarrow{C}_{2n+1}(k)| - 3k > 0$. Thus, $V(\overrightarrow{C}_{2n+1}(k))$ cannot be partitioned into three transitive subtournaments. Then $dc(\overrightarrow{C}_{2n+1}(k)) \geq 4$. We verify that $dc(\overrightarrow{C}_{2n+1}(k)) = 4$. By Lemma 25, we have that $H$, $J$ and $K$ are maximal transitive subtournaments of order $k$. The fourth transitive subtournament is $L = \langle [3k, 2n] \rangle$. Therefore, $\overrightarrow{C}_{2n+1}(k)$ is 4-dichromatic. ■
Theorem 27. If \(\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor\), then \(\overrightarrow{C}_{2n+1}(k)\) is a vertex-critical 4-dichromatic circulant tournament if and only if \(n \equiv 0 \mod 3\).

Proof. By Theorem 26, \(\overrightarrow{C}_{2n+1}(k)\) is 4-dichromatic. Observe that the order of \(H, J\) and \(K\) is \(k\) and \(|L| = 2n - 3k + 1\). Notice that \(\overrightarrow{C}_{2n+1}(k)\) is vertex critical 4-dichromatic if the cardinality of \(L\) is equal to one, and it occurs if and only if \(k = \frac{2}{3}n\) when \(n \equiv 0 \mod 3\). By Theorem 3, \(\overrightarrow{C}_{2n+1}(\frac{2}{3}n)\) with \(n \equiv 0 \mod 3\) is a vertex-critical circulant tournament 4-dichromatic.

Corollary 28 ([11]). \(\overrightarrow{C}_{6n+1}(2m)\) is a vertex-critical 4-dichromatic circulant tournament for \(m \geq 2\).

Theorem 29. Let \(n \geq 3\). Then \(\text{dc}(\overrightarrow{C}_{2n+1}(k)) = 3\) for \(k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n\).

Proof. Let \(n \geq 3\). By Theorem 1, \(\text{dc}(\overrightarrow{C}_{2n+1}(k)) \geq 3\). Take the following partition of the vertices of \(\overrightarrow{C}_{2n+1}(k)\):

\[H = [0, k - 1], J = [k, 2k - 1] \text{ and } K = [2k, 2n].\]

Observe that \(H\) induces a \(TT_k\) because \(N^+(i) = \{i + 1, i + 2, \ldots, k + 1\}\) for \(k \leq i \leq 2k - 1\), also \(J\) and \(K\) induce a \(TT_k\) and a \(TT_{2n-2k+1}\), respectively. Then \(\text{dc}(\overrightarrow{C}_{2n+1}(k)) = 3\).

Theorem 30. If \(k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n, n \geq 3\). Then \(\overrightarrow{C}_{2n+1}(k)\) is a vertex-critical 3-dichromatic circulant tournament if and only if \(n = k\).

Proof. By Theorem 29, \(\overrightarrow{C}_{2n+1}(k)\) is 3-dichromatic and its partition into three maximal transitive subtournaments was

\[|H| = |J| = k \text{ and } |K| = 2n - 2k + 1.\]

Since \(k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n\), we have that \(k \geq 2n - 2k + 1\). Hence, \(\overrightarrow{C}_{2n+1}(k)\) is vertex-critical if and only if \(2n - 2k + 1 = 1\), if and only if \(n = k\).

Corollary 31 ([13], Theorem 2). \(\overrightarrow{C}_{2n+1}(n)\) is a vertex-critical 3-dichromatic circulant tournament for \(n \geq 3\).

By Theorems 13, 19, 23, 27 and 30, we have the following.

Theorem 32. Let \(r \in \{2, 3, 4\}\), \(\overrightarrow{C}_{2n+1}(k)\) is vertex-critical \(r\)-dichromatic if and only if

- (i) \(r = 2, n = 1\) and \(k = 1\);
- (ii) \(r = 3, n \geq 3\)
  - (a) \(n = 4\) and \(k = 1\),
(b) $n \geq 3$ and $k = n$;

(iii) $r = 4$,

(a) $n = 5$ and $k = 2$,
(b) $n = 7$ and $k \in \{3, 4\}$,
(c) $n = 9$ and $k = 4$,
(d) $n = 10$ and $k = 5$,
(e) $n = 13$ and $k = 6$,
(f) $n = 3m$ and $k = 2m$ ($m \geq 2$).

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