Discussiones Mathematicae
Graph Theory 37 (2017) 221–238
doi:10.7151/dmgt.1930

THE DICHROMATIC NUMBER OF INFINITE FAMILIES OF CIRCULANT TOURNAMENTS

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Abstract

The *dichromatic number* $dc(D)$ of a digraph D is defined to be the minimum number of colors such that the vertices of D can be colored in such a way that every chromatic class induces an acyclic subdigraph in D. The *cyclic* circulant tournament is denoted by $T = \overrightarrow{C}_{2n+1}(1, 2, \ldots, n)$, where $V(T) = \mathbb{Z}_{2n+1}$ and for every jump $j \in \{1, 2, \ldots, n\}$ there exist the arcs $(a, a+j)$ for every $a \in \mathbb{Z}_{2n+1}$. Consider the circulant tournament $\overrightarrow{C}_{2n+1}\langle k\rangle$ obtained from the cyclic tournament by reversing one of its jumps, that is, $\overrightarrow{C}_{2n+1}\langle k\rangle$ has the same arc set as $\overrightarrow{C}_{2n+1}(1, 2, \ldots, n)$ except for $j = k$ in which case, the arcs are $(a, a-k)$ for every $a \in \mathbb{Z}_{2n+1}$. In this paper, we prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \in \{2, 3, 4\}$ for every $k \in \{1, 2, \ldots, n\}$. Moreover, we classify which circulant tournaments $\overrightarrow{C}_{2n+1}\langle k\rangle$ are vertex-critical r-dichromatic for every $k \in \{1, 2, \ldots, n\}$ and $r \in \{2, 3, 4\}$. Some previous results by Neumann-Lara are generalized.

Keywords: tournament, dichromatic number, vertex-critical r-dichromatic tournament.

2010 Mathematics Subject Classification: 05C20, 05C38.

1. Introduction

A *tournament* T is an orientation of the complete graph. If T contains no directed cycles, T is called *transitive* and denoted by TT_k, where $k \in \mathbb{N}$ is its order.

The definition of the dichromatic number of a digraph and the first important results were introduced by Neumann-Lara in 1982 [9]. Independently, Jacob and Meyniel defined the same notion in 1983, see [6]. In 1977, Erdős visited Mexico and began to work on the dichromatic number of a graph (see the definition below) with Neumann-Lara. The results of this collaboration were summarized in a survey by Erdős in 1979 (see [3] for details).

Other results concerning this topic can be found in a paper by Erdős, Gimbel and Kratsch [4]. According to this paper, the *dichromatic number of a digraph* D, denoted by $dc(D)$, is " the minimum number of parts the vertex set of D must be partitioned into, so that each part induces an acyclic digraph." Equivalently, the dichromatic number of D is the minimum number of colors such that the vertices of D can be colored in such a way that every chromatic class induces an acyclic subdigraph in D. The main result of paper [4] for tournaments is the following: every tournament with n vertices can be colored with $O(n/\log n)$ and there exists tournaments (for example, random tournaments) having dichromatic number $\Omega(n/\log n)$ (see Theorem 5 of the aforementioned paper). There are more interesting asymptotic results in [5] by Harutyunyan. In particular, Theorem 2.3.8 states that if T is a tournament of order n, then $dc(T) \leq \frac{n}{\log n}(1+o(1))$.

The *dichromatic number of a graph* G was defined by Erdős and Neumann-Lara in [3] as

$$dc(G) = \max\left\{ dc(\overrightarrow{G}) : \overrightarrow{G} \text{ is an orientation of } G \right\}.$$

Determining the dichromatic number of a general (di)graph is a very hard problem. Exact values of this parameter are only known for some special classes of digraphs, particularly in a few cases of circulant tournaments (see [1, 7, 9, 12, 10, 11] and [13]). In this paper, we prove that

(i) $$dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } 4 \leq n \leq 7, \\ 4 & \text{if } n \geq 8, \end{cases}$$

(Section 3, Corollary to Theorem 11),

(ii) $$dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8, \end{cases}$$

(Section 3, Corollary to Theorem 17),

(iii) if $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$ for $n \geq 7$ (Section 4, Theorem 22),

(iv) if $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$ (Section 5, Theorem 26), and

(v) $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$ for $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$, where $n \geq 3$ (Section 5, Theorem 29).

Our results generalize some theorems obtained by Neumann-Lara. At the end of Section 5, we characterize the vertex-critical r-dichromatic circulant tournaments $\overrightarrow{C}_{2n+1}\langle k\rangle$ for every $k \in \{1, 2, \ldots, n\}$ and $r \in \{2, 3, 4\}$, see Theorem 32, the main theorem of this paper.

2. Preliminaries

Let $\mathbb{Z}_m$ be the cyclic group of integers modulo m, where $m \in \mathbb{N}$ and J is a nonempty subset of $\mathbb{Z}_m \setminus \{0\}$ such that $w \in J$ if and only if $-w \notin J$ for every $w \in \mathbb{Z}_m$. The *circulant digraph* $\overrightarrow{C}_m(J)$ is defined by $V(\overrightarrow{C}_m(J)) = \mathbb{Z}_m$ and

$$A(\overrightarrow{C}_m(J)) = \{(i, j) : i, j \ \in \mathbb{Z}_m \ \text{ and } j - i \in J\}.$$

Notice that $\overrightarrow{C}_{2n+1}(J)$ is a circulant (or rotational) tournament if and only if $|J| = n$. We recall that circulant tournaments are regular and their automorphism group is vertex-transitive. We define

$$\begin{aligned} \overrightarrow{C}_{2n+1}(1, 2, \ldots, n) &:= \overrightarrow{C}_{2n+1}\langle\emptyset\rangle \quad \text{and} \\ \overrightarrow{C}_{2n+1}(1, \ldots, k-1, -k, k+1, \ldots, n) &= \\ \overrightarrow{C}_{2n+1}(1, \ldots, k-1, k+1, \ldots, n, 2n+1-k) &:= \overrightarrow{C}_{2n+1}\langle k\rangle. \end{aligned}$$

Observe that the circulant $\overrightarrow{C}_m(1) = \overrightarrow{C}_m$ is the directed cycle of length m. If $V(\overrightarrow{C}_m) = \{a_1, a_2, \ldots, a_m\}$, we denote $\overrightarrow{C}_m = (a_1, a_2, \ldots, a_m, a_1)$. The tournament $\overrightarrow{C}_{2n+1}\langle\emptyset\rangle$ is called the *cyclic tournament*. It is straightforward to check that there is only one cyclic tournament on n vertices up to isomorphism for every $n \in \mathbb{N}$. The isomorphism between digraphs G and H is denoted by $G \cong H$. A digraph D is called *r-dichromatic* if $dc(D) = r$. It is *vertex-critical* r-dichromatic if $dc(D) = r$ and $dc(D - v) < r$ for every $v \in V(D)$. For general terminology, see [2]. In what follows, we will need the following results of [11] and [13].

Theorem 1 ([13], Theorem 1)**.** *If T_{2n+1} is a regular tournament on $2n+1$ vertices, then $dc(T_{2n+1}) = 2$ if and only if $T_{2n+1} \cong \overrightarrow{C}_{2n+1}\langle\emptyset\rangle$.*

Theorem 2 ([13], Theorem 2)**.** *$\overrightarrow{C}_{2n+1}\langle n\rangle$ is a vertex-critical 3-dichromatic circulant tournament for $n \geq 3$.*

Theorem 3 ([11])**.** *$\overrightarrow{C}_{6m+1}\langle 2m\rangle$ is a vertex-critical 4-dichromatic circulant tournament for $m \geq 2$.*

Let T be a tournament and $k, l \in \mathbb{N}$. We recall that a transitive subtournament TT_k of T is *maximal* if there does not exist a transitive subtournament TT_l of T $(k < l)$ such that TT_k is a subtournament of TT_l.

Remark 4 ([8])**.** Up to isomorphism

(i) there exists a unique circulant tournament of order 5, that is, $\overrightarrow{C}_5(1,2) = \overrightarrow{C}_5\langle\emptyset\rangle$,

(ii) there exist two circulant tournaments of order 7 which are

$$\overrightarrow{C}_7(1,2,3) = \overrightarrow{C}_7\langle\emptyset\rangle \cong \overrightarrow{C}_7\langle 1\rangle \cong \overrightarrow{C}_7\langle 2\rangle \text{ and } \\ \overrightarrow{C}_7(1,2,4) = \overrightarrow{C}_7\langle 3\rangle,$$

(iii) there exist three circulant tournaments of order 9 which are

$$\overrightarrow{C}_9(1,2,3,4) = \overrightarrow{C}_9\langle\emptyset\rangle, \\ \overrightarrow{C}_9(1,2,3,5) = \overrightarrow{C}_9\langle 4\rangle \cong \overrightarrow{C}_9\langle 1\rangle \cong \overrightarrow{C}_9\langle 3\rangle \quad \text{and} \\ \overrightarrow{C}_9(1,3,4,7) = \overrightarrow{C}_9\langle 2\rangle,$$

(where $\overrightarrow{C}_9\langle 2\rangle \cong \overrightarrow{C}_3[\overrightarrow{C}_3]$ is the composition of $\overrightarrow{C}_3$ and $\overrightarrow{C}_3$).

In the following three sections we determine the exact value of $dc(\overrightarrow{C}_{2n+1}\langle k\rangle)$ for every $n, k \in \mathbb{N}$. We subdivide the calculations into five cases (the cases are illustrated in Figure 1):

(i) $k = 1$,
(ii) $k = 2$,
(iii) $3 \le k \le \left\lceil \frac{n}{2} \right\rceil$,
(iv) $\left\lceil \frac{n}{2} \right\rceil + 1 \le k \le \left\lfloor \frac{2}{3}n \right\rfloor$ and
(v) $\left\lfloor \frac{2}{3}n \right\rfloor + 1 \le k \le n$.

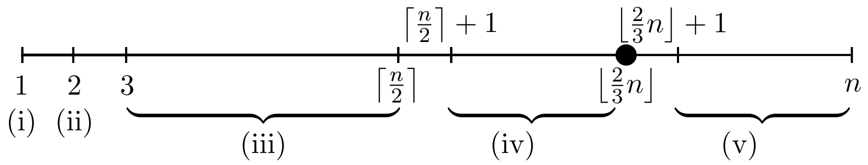

Figure 1

3. The Dichromatic Number of $\overrightarrow{C}_{2n+1}\langle 1\rangle$ and $\overrightarrow{C}_{2n+1}\langle 2\rangle$

To begin with, let us observe the following facts.

Remark 5.

(i) For every $j \in \mathbb{Z}_{2n+1}$, the set of vertices $\{j-2, j-1, j\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1\rangle$ and $\overrightarrow{C}_{2n+1}\langle 2\rangle$, respectively,

(ii) $\overrightarrow{C}_{11}\langle 1\rangle \cong \overrightarrow{C}_{11}\langle 4\rangle$,

(iii) $\overrightarrow{C}_{13}\langle 1\rangle \cong \overrightarrow{C}_{13}\langle 3\rangle$.

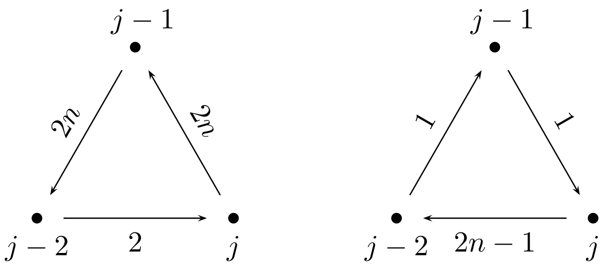

Figure 2. $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1\rangle$ and $\overrightarrow{C}_{2n+1}\langle 2\rangle$, respectively.

Proposition 6. $dc(\overrightarrow{C}_{11}\langle 1\rangle) = dc(\overrightarrow{C}_{13}\langle 1\rangle) = dc(\overrightarrow{C}_{15}\langle 1\rangle) = 3$.

Proof. Consider $\overrightarrow{C}_{11}\langle 1\rangle$. Since $\overrightarrow{C}_{11}\langle 1\rangle \ncong \overrightarrow{C}_{11}\langle\emptyset\rangle$, by Theorem 1, $dc(\overrightarrow{C}_{11}\langle 1\rangle) \geq 3$. The partition of the vertices of $\overrightarrow{C}_{11}\langle 1\rangle$ given by $P_1 = \{0, 3, 2, 5\}$, $P_2 = \{7, 10, 9, 1\}$ and $P_3 = \{4, 6, 8\}$ implies that $dc(\overrightarrow{C}_{11}\langle 1\rangle) = 3$. Observe that P_1 and P_2 induce a TT_4, and P_3 induces a TT_3 in $\overrightarrow{C}_{11}\langle 1\rangle$, respectively. By Remark 5(ii), we have that $dc(\overrightarrow{C}_{11}\langle 4\rangle) = 3$. Notice that the transitive subtournaments induced by P_1, P_2 and P_3 are maximal. If $\langle P_1\rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is the vertex 10, but $(2, 5, 10, 2) \cong \overrightarrow{C}_3$. Then P_1 induces a maximal transitive subtournament. The arguments are similar for P_2 and P_3. Analogously we can prove the others cases $n = 6$ and $n = 7$. ■

The following lemmas will be useful tools in order to prove Theorem 11. Let a and b nonnegative integers such that $0 \leq a < b \leq n$. We define the interval $[a, b] = \{a, a + 1, \ldots, b\}$, $X_0 = [0, n]$, $X_3 = [0, 2n]$ and

$$Y_0 = \{j \in X_0 : j \equiv 1 \bmod 3\}.$$

Lemma 7. *The tournament $\overrightarrow{C}_{2n+1}\langle 1\rangle$ contains a maximal transitive subtournament of order $n + 1 - \left\lfloor \frac{n}{3} \right\rfloor$ if $n \equiv 0 \bmod 3$ and of order $n + 2 - \left\lceil \frac{n}{3} \right\rceil$ vertices if $n \equiv 1 \bmod 3$.*

Proof. Consider $\overrightarrow{C}_{2n+1}\langle 1\rangle = \overrightarrow{C}_{2n+1}(2, 3, 4, 5, 6, \ldots, n, 2n)$. Recall that circulant tournaments are vertex-transitive, so it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that

$$N^+(0) = \{2, 3, 4, 5, 6, \ldots, n, 2n\}.$$

We have two cases.

Case 1. $n \equiv 0 \bmod 3$. Let $n = 3k$ where $k \in \mathbb{N}$. Notice that the subset of vertices $j \equiv 0, 2 \bmod 3$ belonging to the set X_0 induces a transitive subtournament

$$H_0 = \langle X_0 \setminus Y_0\rangle,$$

by Remark 5(i). Observe that $|Y_0| = \lfloor \frac{n}{3} \rfloor$ and $|H_0| = n + 1 - \lfloor \frac{n}{3} \rfloor = 2k + 1$. It remains to prove that H_0 is maximal. If H_0 was not maximal, then the only vertex we can add is $2n$ by Remark 5(i). Observe that in this case the set $\{n, 2n, n-4\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1 \rangle$, which implies that $H_0 \cup \{2n\}$ cannot induce a maximal transitive subtournament.

Case 2. $n \equiv 1 \bmod 3$. This case is analogous to Case 1. The maximal transitive subtournament is

$$H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0 \rangle.$$

■

Lemma 8. *Let* $\overrightarrow{C}_{2n+1}\langle 1 \rangle$ *be such that* $n \equiv 0, 1 \bmod 3$. *Then the subtournaments induced by*

$$X_3 \setminus (X_0 \setminus Y_0) \quad \textit{if} \quad n \equiv 0 \bmod 3 \quad \textit{and}$$

$$X_3 \setminus ((X_0 \cup \{2n\}) \setminus Y_0) \quad \textit{if} \quad n \equiv 1 \bmod 3$$

contain a maximal transitive subtournament of $n - \lfloor \frac{n}{3} \rfloor$ *and* $n - \lfloor \frac{n}{3} \rfloor - 1$ *vertices, respectively.*

Proof. Suppose that $n \equiv 0 \bmod 3$. By Lemma 7, $\overrightarrow{C}_{2n+1}\langle 1 \rangle$ contains the transitive subtournament H_0. Consider $X_1 = [n + 1, 2n]$ and define

$$Y_1 = \{j \in X_1 : j \equiv 2 \bmod 3\} \text{ and } J_0 = \langle X_1 \setminus Y_1 \rangle.$$

By Remark 5(i), J_0 is transitive. Notice that J_0 has order $n - \lfloor \frac{n}{3} \rfloor$. We prove that J_0 is maximal in the same way as in the proof of Lemma 7. For a contradiction, if J_0 was not maximal, then the only vertex we can add is vertex 1 by Remark 5(i). Observe that in this case, the set $\{1, n+1, n+3\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1 \rangle$, which implies that $J_0 \cup \{1\}$ cannot induce a maximal transitive subtournament.

When $n \equiv 1 \bmod 3$, the arguments are similar. The maximal transitive subtournament J_1 is given by

$$J_1 = \langle X_1 \setminus (Y_2 \cup \{2n\}) \rangle,$$

where $Y_2 = \{j \in X_1 : j \equiv 0 \bmod 3\}$. ■

Lemma 9. *A maximal transitive subtournament contained in* $\overrightarrow{C}_{2n+1}\langle 1 \rangle$ *has* $n + 1 - \lceil \frac{n}{3} \rceil$ *vertices if* $n \equiv 2 \bmod 3$.

Proof. It is similar to the proof of Lemma 7. The maximal transitive subtournament is

$$H_2 = \langle X_0 \setminus Y_0 \rangle.$$

■

Lemma 10. *Let* $\overrightarrow{C}_{2n+1}\langle 1\rangle$ *be such that* $n \equiv 2 \bmod 3$. *Then the subtournament induced by*

$$X_3 \setminus (X_0 \setminus Y_0)$$

contains a maximal transitive subtournament of order $n - \left\lfloor \frac{n}{3} \right\rfloor$.

Proof. It is similar to the proof of Lemma 8. In this case, every vertex $j \equiv 0, 1 \bmod 3$ in X_1 induces a transitive subtournament

$$J_2 = \langle X_1 \setminus Y_3\rangle,$$

where $Y_3 = \{j \in X_1 : j \equiv 2 \bmod 3\}$. ∎

Theorem 11. *Let* $n \in \mathbb{N}$. *Then* $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 4$ *for every* $n \geq 8$.

Proof. By Theorem 1, we have that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) \geq 3$. In the first place, we prove that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) \geq 4$. For a contradiction, suppose that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 3$. Thus, $\overrightarrow{C}_{2n+1}\langle 1\rangle$ has a partition of its vertices inducing three transitive subtournaments. Suppose that $n \equiv 0 \bmod 3$ (it is similar when $n \equiv 1, 2 \bmod 3$). By Lemmas 7 and 8, two maximal disjoint transitive subtournaments in $\overrightarrow{C}_{2n+1}\langle 1\rangle$ are H_0 and J_0. Hence, the remaining vertex set $X_3 \setminus \{V(H_0) \cup V(J_0)\}$,

$$\{1, 4, 7, \dots, n+2, n+5, \dots\},$$

induces the third transitive subtournament. Observe that the vertex set $\{1, 7, n+2\}$ induces a $\overrightarrow{C}_3$, this is a contradiction. Therefore, $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) \geq 4$. We show that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 4$. By Lemmas 7 and 8, we have that the two maximal transitive subtournaments H_0 and J_0 have cardinality $n + 1 - \left\lfloor \frac{n}{3} \right\rfloor$ and $n - \left\lfloor \frac{n}{3} \right\rfloor$, respectively. Define a third subtournament

$$K_0 = \langle \{1\} \cup Y_1\rangle.$$

Notice that $|K_0| = \left\lfloor \frac{n}{3} \right\rfloor + 1$ and K_0 is transitive by the definition of Y_1. We will prove that K_0 is a maximal transitive subtournament in $\overrightarrow{C}_{2n+1}\langle 1\rangle \setminus \{H_0 \cup J_0\}$. If K_0 was not a maximal transitive subtournament, then we can add at least one vertex of $Y_0 \setminus \{1\}$. Notice that if $i \in Y_0 \setminus \{1\}$, we have that $(i, i+n-2, i+n+1, i) \cong \overrightarrow{C}_3$. Therefore, K_0 is a maximal transitive subtournament. Finally, the subtournament $L_0 = \langle Y_0 \setminus \{1\}\rangle$ is transitive by the definition of Y_0 and maximal. Thus, $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 4$. The proof is completely analogous for the cases when $n \equiv 1, 2 \bmod 3$. The partition into transitive subtournaments is

$$H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0\rangle,\ J_1 = \langle X_1 \setminus Y_2 \cup \{2n\}\rangle,\ K_1 = \langle Y_2 \cup \{1\}\rangle,\ L_1 = \langle Y_0 \setminus \{1\}\rangle,$$

for $n \equiv 1 \bmod 3$. For $n \equiv 2 \bmod 3$, we have that

$$H_2 = \langle X_0 \setminus Y_0\rangle,\ J_2 = \langle X_1 \setminus Y_3\rangle,\ K_2 = \langle (Y_3 \cup \{1\})\rangle,\ L_2 = \langle Y_0 \setminus \{1\}\rangle.$$

∎

From Proposition 6, Theorems 1, 2 and 11 and Remark 4(iii), we obtain the following consequence.

Corollary 12.

$$dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = \begin{cases} 2 & \textit{if } n = 3, \\ 3 & \textit{if } 4 \leq n \leq 7, \\ 4 & \textit{if } n \geq 8. \end{cases}$$

Theorem 13. *Let* $r \in \{2,3,4\}$. *Then* $\overrightarrow{C}_{2n+1}\langle 1\rangle$ *is a vertex-critical* r*-dichromatic circulant tournament if and only if* $n \in \{1,4\}$.

Proof. If $r = 2$, clearly $\overrightarrow{C}_3\langle 1\rangle$ is a vertex-critical 2-dichromatic.

If $r = 3$, we need to check for which values of $4 \leq n \leq 7$, the circulant tournament $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical 3-dichromatic. Notice that $\overrightarrow{C}_9\langle 1\rangle \cong \overrightarrow{C}_9\langle 4\rangle$ by Remark 4(iii). It is a vertex-critical 3-dichromatic by Theorem 2. For $n = 5$, the circulant tournament $\overrightarrow{C}_{11}\langle 1\rangle$ is not a vertex-critical 3-dichromatic by Proposition 6. Using analogous arguments, $\overrightarrow{C}_{13}\langle 1\rangle$ and $\overrightarrow{C}_{15}\langle 1\rangle$ are not vertex-critical.

If $r = 4$, the circulant tournament $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is 4-dichromatic for every $n \geq 8$ by Theorem 11. It was partitioned into four maximal transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0,1,2$. Notice that $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical 4-dichromatic if the cardinality of L_i is equal to one for $i = 0,1,2$. Since $|L_i| = |Y_0| - 1 = \lfloor \frac{n}{3} \rfloor - 1$, we have that $|L_i| = 1$ if and only if $\lfloor \frac{n}{3} \rfloor = 2$. It occurs when $n = 6, 7$ or 8. By Theorem 11, it is only possible for $n \geq 8$. Observe that $|L_2| \geq 2$ for $T = \overrightarrow{C}_{2n+1}\langle 1\rangle$ if $n \geq 8$. Since this partition is maximal, T is not vertex-critical.

Therefore, $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical r-dichromatic circulant tournament if and only if n is 1 or 4. ■

Let us recall that

Remark 14. $\overrightarrow{C}_9\langle 2\rangle = \overrightarrow{C}_3[\overrightarrow{C}_3]$ is 3-dichromatic, a particular case of Theorem 8 from [9]. Notice that it is not vertex-critical.

Remark 15 ([10], Theorem 2.6)**.** $\overrightarrow{C}_{11}\langle 2\rangle$ is vertex-critical 4-dichromatic.

Proposition 16. *If* $n = 6$ *and* 7, *then* $dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = 3$.

Proof. Observe that $\overrightarrow{C}_{15}\langle 2\rangle \ncong \overrightarrow{C}_{15}\langle \emptyset\rangle$. Then by Theorem 1, $dc(\overrightarrow{C}_{15}\langle 2\rangle) \geq 3$. Consider the following partition of $V(\overrightarrow{C}_{15}\langle 2\rangle)$:

$$P_1 = \{0,1,3,4,6,7\}, P_2 = \{5,8,9,11,12\} \text{ and } P_3 = \{2,10,13,14\}.$$

We have that $\langle P_1\rangle \cong TT_6$, $\langle P_2\rangle \cong TT_5$ and $\langle P_3\rangle \cong TT_4$. Therefore, $dc(\overrightarrow{C}_{15}\langle 2\rangle) = 3$. Note that the transitive subtournaments induced by P_1, P_2 and P_3 are maximal. If $\langle P_1\rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is vertex 13. We cannot add the vertex 5 by Remark 5(i). But $(4,7,13,4) \cong \overrightarrow{C}_3$. Then P_1 induces a maximal transitive subtournament. The same conclusion is valid for P_2 and P_3. Observe that $\overrightarrow{C}_{15}\langle 2\rangle$ is not vertex-critical. The proof is analogous for $n = 6$. ■

Theorem 17. *Let $n \in \mathbb{N}$. Then $dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = 4$ for every $n \geq 8$.*

Proof. It is analogous to the proof of Theorem 11. Therefore, Remark 5(i) is applied for $\overrightarrow{C}_{2n+1}\langle 2\rangle$. The corresponding partitions are following.

(i) $n \equiv 0 \bmod 3$, we define $Y_4 = \{j \in X_0 : j \equiv 2 \bmod 3\}$,

$$H_0 = \langle X_0 \setminus Y_4\rangle,\ J_0 = \langle X_1 \setminus Y_2\rangle,\ K_0 = \langle Y_2 \cup \{2\}\rangle,\ L_0 = \langle Y_4 \setminus \{2\}\rangle.$$

(ii) $n \equiv 1 \bmod 3$, we define $Y_5 = \{j \in X_1 : j \equiv 1 \bmod 3\}$,

$$H_1 = \langle X_0 \setminus Y_4\rangle,\ J_1 = \langle X_1 \setminus Y_5\rangle,\ K_1 = \langle Y_5 \cup \{2\}\rangle,\ L_1 = \langle Y_4 \setminus \{2\}\rangle.$$

(iii) $n \equiv 2 \bmod 3$, we define $Y_6 = \{j \in X_1 : j \equiv 1 \bmod 3\}$,

$$H_2 = \langle X_0 \setminus Y_4\rangle,\ J_2 = \langle (X_1 \cup \{n\}) \setminus Y_6\rangle,\ K_2 = \langle Y_6\rangle,\ L_2 = \langle Y_4 \setminus \{n\}\rangle.$$ ■

The next corollary is an immediate consequence of Remarks 4(ii)–(iii), 14, 15, Proposition 16 and Theorems 1, 2 and 17.

Corollary 18.

$$dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8. \end{cases}$$

Theorem 19. *Let $r \in \{2,3,4\}$. Then $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is a vertex-critical r-dichromatic circulant tournament if and only if $n = 5$.*

Proof. If $r = 2$, by Theorem 1, $\overrightarrow{C}_7\langle 2\rangle \cong \overrightarrow{C}_7\langle \emptyset\rangle$ is 2-dichromatic, but it is not vertex-critical.

Let $r = 3$. For $n = 4$, we have that $\overrightarrow{C}_9\langle 2\rangle$ is not vertex-critical by Remark 14. For $n = 6$ and 7 by Proposition 16, $\overrightarrow{C}_{13}\langle 2\rangle$ and $\overrightarrow{C}_{15}\langle 2\rangle$ are not vertex-critical.

If $r = 4$, then by Remark 15, $\overrightarrow{C}_{11}\langle 2\rangle$ is vertex-critical. By Theorem 17, $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is 4-dichromatic for every $n \geq 8$. It was partitioned into four maximal transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0, 1, 2$. Notice that $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is vertex-critical 4-dichromatic if the cardinality of L_i is equal to one for $i = 0, 1, 2$. Since $|L_i| = |Y_4| - 1 = \lfloor \frac{n}{3} \rfloor - 1$, we have that $|L_i| = 1$ if and only if $\lfloor \frac{n}{3} \rfloor = 2$. It occurs when $n = 6, 7$ or 8. By Theorem 17 it is only possible for $n \geq 8$. Observe that $|L_2| \geq 2$ for $T = \overrightarrow{C}_{2n+1}\langle 2\rangle$ if $n \geq 8$. Since this partition is maximal, T is not vertex-critical. Therefore, $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is vertex-critical if and only if $n = 5$. ■

4. The Dichromatic Number of $\overrightarrow{C}_{2n+1}\langle k\rangle$ for $3 \leq k \leq \lceil \frac{n}{2} \rceil$

We prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$, for $3 \leq k \leq \lceil \frac{n}{2} \rceil$ and $n \geq 7$.

Lemma 20. *If $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains a maximal transitive subtournament H.*

Proof. Let n and k be nonnegative integers and consider the interval $[0, n]$. Applying the Euclidean division algorithm to $n+1$ and $2k-1$, there exist unique α, $r \in \mathbb{N}$ such that

$$n + 1 = \alpha(2k - 1) + r \quad \text{where} \quad 0 \leq r < 2k - 1.$$

Consider the partition of the interval $[0, 2k-2] = [0, k-1] \cup [k, 2k-2]$ and define

$$n + 1 = \begin{cases} \alpha(2k - 1) + s_1 & \text{if } s_1 \in [0, k - 1], \\ \alpha(2k - 1) + s_2 & \text{if } s_2 \in [k, 2k - 2]. \end{cases}$$

Observe that since $3 \leq k \leq \lceil \frac{n}{2} \rceil$, we have that $s_1 \in [1, k-1]$.

Let

$$W = \bigcup_{i=0}^{\alpha-1} [i(2k - 1), i(2k - 1) + (k - 1)].$$

We define the subtournament H of $\overrightarrow{C}_{2n+1}\langle k\rangle$ in the following way.

(i) If $s_1 \in [1, k-1]$, then $H = \langle W \cup [\alpha(2k-1), n]\rangle$. Moreover, if $k = \frac{n+1}{2}$ and n is odd, then $H = \langle W \cup \{n, 2n+1-k\}\rangle$.

(ii) $H = \langle W \cup [\alpha(2k-1), \alpha(2k-1) + k - 1]\rangle$ for every $s_2 \in [k, 2k-2]$.

Note that H is a transitive subtournament by construction, since its vertex set does not contain induced $\overrightarrow{C}_3$'s. We prove that H is maximal by contradiction. Since $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex-transitive, without loss of generality, choose the vertex 0. Observe that $N^+(0) = \{1, 2, \ldots, k-1, k+1, \ldots, n, 2n+1-k\}$ and

$$N^+(0) \setminus V(H) = (X_0 \cup \{2n + 1 - k\}) \setminus (V(H) \cup \{k\}).$$

For every vertex $x \in N^{+}(0)\backslash V(H)$ there exist $h_1, h_2 \in V(H)$ such that the vertex set $\{h_1, h_2, x\}$ induces a $\overrightarrow{C}_3$ (for instance, $x = k+1$, $h_1 = 1$ and $h_2 = k-1$), a contradiction. Therefore, H is maximal. ■

Lemma 21. *If* $3 \leq k \leq \lceil \frac{n}{2} \rceil$, *then* $X_3 \setminus V(H)$ *contains a maximal transitive subtournament* J, *where* H *is the subtournament defined in Lemma* 20.

Proof. The construction of J is similarly obtained as in the proof of Lemma 20 for H, but we have two ways of defining J.

Case 1. $\alpha = 1$.

(i) If $s_1 \in [1, k-1]$, then $J = [k, 2k-2] \cup [3k-2, 3k+s_1-2]$. Notice that if $k = \frac{n+1}{2}$ with n is odd if and only if $s_1 = 1$. Then $J = [k, 2k-2] \cup \{3k-2\}$ by the construction of H.

(ii) If $s_2 \in [k, 2k-2]$, then $J = [k, 2k-2] \cup [3k-1, 4k-2]$.

Case 2. $\alpha > 1$. Let

$$U = \bigcup_{i=0}^{\alpha-1} [(n+1) + i(2k-1), (n+1) + i(2k-1) + (k-1)].$$

(i) If $s_1 \in [1, k-1]$, then $J = \langle U \cup [(n+1) + \alpha(2k-1), 2n] \rangle$.

(ii) $J = \langle U \cup [(n+1)+\alpha(2k-1), (n+1)+\alpha(2k-1)+k-1] \rangle$ for every $s_2 \in [k, 2k-2]$.

Notice that H is a maximal transitive subtournament in $\overrightarrow{C}_{2n+1}\langle k \rangle$ by Lemma 20. We claim that J is maximal in $V(\overrightarrow{C}_{2n+1}\langle k \rangle \setminus V(H)$. If J was not maximal, we could add at least one vertex of $V(\overrightarrow{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J))$.

For Case 1, consider

$$\{n+k+1, 3k-3\} \subseteq V(\overrightarrow{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J))).$$

We have that $(k, n+k, n+k+1, k) \cong \overrightarrow{C}_3$ or $(2k-2, 3k-3, 3k-2, 2k-2) \cong \overrightarrow{C}_3$. Therefore, J is maximal.

For Case 2, consider

$$k \in V(\overrightarrow{C}_{2n+1}\langle k \rangle \setminus (V(H) \cup V(J)).$$

We have that $(k, n+k, n+2k+1, k) \cong \overrightarrow{C}_3$. Hence, J is maximal. ■

Theorem 22. *If* $3 \leq k \leq \lceil \frac{n}{2} \rceil$, *then* $dc(\overrightarrow{C}_{2n+1}\langle k \rangle) = 4$ *for* $n \geq 7$.

Proof. By Theorem 1, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 3$. We prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. For a contradiction, suppose that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$. Thus, $\overrightarrow{C}_{2n+1}\langle k\rangle$ has a partition of its vertices consisting of three transitive subtournaments. By Lemmas 20 and 21, two maximal transitive disjoint subtournaments in $\overrightarrow{C}_{2n+1}\langle k\rangle$ are H and J. Hence, the remaining vertex set $X_3 \setminus (V(H) \cup V(J))$ induces the third transitive subtournament.

We consider three cases.

Case 1. $J = \langle [k, 2k-2] \cup [3k-2, 3k+s_1-2]\rangle$ obtained by Case 1(i) of Lemma 21. Therefore, $K = \langle [3k+s_1-1, 2n]\rangle$. Moreover, $|J| = k+s_1$ and $|H| = n-k+2$. Since $k \leq \lceil \frac{n}{2} \rceil$, we have that $|K| = 2n+1-(|H|+|J|) = 2k-3 > k$. In this case, K is induced by at least $k+1$ consecutive vertices. Therefore, K cannot be a transitive subtournament by the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. The following cases are necessary because the structure of K and L changes with different values of s_1.

(i) If $s_1 = 1, 2$ or 3, then

$$K = \langle [n+1, 3k-3] \cup [4k-3, 2n]\rangle \text{ and } L = \langle [3k+s_1-1, 4k-4]\rangle.$$

(ii) If $s_1 \in [4, k-5]$, then

$$K = \langle [n+1, 3k-3] \cup [4k-3, 5k-4]\rangle \text{ and}$$
$$L = \langle [3k+s_1-1, 4k-4] \cup [5k-3, 2n]\rangle.$$

(iii) If $s_1 = k-4$, then

$$K = \langle [n+1 = 3k-4, 3k-3] \cup [4k-3, 5k-4]\rangle \text{ and}$$
$$L = \langle [4k-5, 4k-4] \cup [5k-3, 2n]\rangle.$$

(iv) If $s_1 = k-3$, then

$$K = \langle [n+1 = 3k-5, 3k-3] \cup [4k-3, 5k-4]\rangle \text{ and}$$
$$L = \langle \{4k-4\} \cup [5k-3, 2n]\rangle.$$

(v) If $s_1 = k-1$ or $k-2$, then

$$K = \langle [n+k+1, n+2k]\rangle \text{ and } L = \langle [n+2k+1, 2n]\rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the subtournaments K and L are transitive. Observe that if $s_1 \in [1, k-3]$, $4k-4 \notin V(K)$ and $(4k-4, 4k-3, 3k-3, 4k-4) \cong \overrightarrow{C}_3$. Therefore, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$. If $s_1 = k-1$ or $k-2$, then $n+2k+1 \notin V(K)$ and $(n+k+1, n+2k, n+2k+1, n+k+1) \cong \overrightarrow{C}_3$. Thus, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle = 4$.

Case 2. $J = [k, 2k-2] \cup [3k-1, 4k-2]$ obtained by Case 1(ii) of Lemma 21. We have that $X_3 \setminus (V(H) \cup V(J)) = [4k-1, 2n]$, but $2n - 4k + 2 > k$ implies that the subtournament induced by $X_3 \setminus (V(H) \cup V(J))$ has at least $k+1$ consecutive vertices and a $\overrightarrow{C}_3$ is induced by $X_3 \setminus (V(H) \cup V(J))$. Therefore, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle \geq 4$. The following cases show the partition of $\overrightarrow{C}_{2n+1}\langle k\rangle$ into transitive subtournaments.

(i) If $s_2 = k$, then

$$K = \langle [4k-1, 5k-2] \rangle \text{ and } L = \langle [5k-1, 2n] \rangle.$$

(ii) If $s_2 = k+1$, then

$$K = \langle [4k-1, 5k-2] \cup \{6k-2 = 2n\} \rangle \text{ and } L = \langle [5k-1, 6k-3] \rangle.$$

(iii) If $s_2 = k+2$, then

$$K = \langle [4k-1, 5k-2] \cup [6k-2, 6k] \rangle \text{ and } L = \langle [5k-1, 6k-3] \rangle.$$

(iv) If $s_2 \in [k+3, 2k-2]$, then

(a) if $2n \leq 7k-3$, we have that

$$K = \langle [4k-1, 5k-2] \cup [6k-2, 2n] \rangle \text{ and}$$
$$L = \langle [5k-1, 6k-3] \rangle,$$

(b) if $2n > 7k-3$, then

$$K = \langle [4k-1, 5k-2] \cup [6k-2, 7k-3] \rangle \text{ and}$$
$$L = \langle [5k-1, 6k-3] \cup [7k-2, 2n] \rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the subtournaments K and L are transitive. Observe that $5k-1 \notin V(K)$ and $(4k-1, 5k-2, 5k-1, 4k-1) \cong \overrightarrow{C}_3$. Therefore, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$.

Case 3. J is obtained by Case 2(i) and (ii) of Lemma 21. If $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$, then $X_3 \setminus (V(H) \cup V(J))$ induces a transitive subtournament, but the vertex set $\{k, 3k-1, n+k+1\} \subseteq X_3 \setminus (V(H) \cup V(J))$ induces a $\overrightarrow{C}_3$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. The partition of $\overrightarrow{C}_{2n+1}\langle k\rangle$ into transitive subtournaments is H, J, $K = \langle X_1 \cup \{k\} \setminus V(J) \rangle$ and

$$L = \langle X_3 \setminus (V(H) \cup V(J) \cup V(K)) \rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the subtournaments K and L are maximal transitive. Observe that $k+1 \notin V(K)$. Then the vertex set $\{k, k+1, n+k+1\}$ induces a $\overrightarrow{C}_3$. Therefore, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$.

This proves that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$. ■

The following example illustrates Theorem 22, Case 2(ii). The tournament $\overrightarrow{C}_{29}\langle 5\rangle$ has the following partition into four transitive subtournaments

$H = \langle [0,4] \cup [9,13]\rangle$, $K = \langle [5,8] \cup [14,18]\rangle$, $J = \langle [19,23] \cup \{28\}\rangle$ and $L = \langle [24,27]\rangle$.

Observe that $H \cong TT_{10}$, $J \cong TT_9$, $K \cong TT_6$ and $L \cong TT_4$.

Theorem 23. *If* $3 \leq k \leq \lceil \frac{n}{2} \rceil$, *then* $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is a vertex-critical* 4*-dichromatic circulant tournament if and only if*

(i) $n = 7$ *and* $k \in \{3,4\}$,
(ii) $n = 9$ *and* $k = 4$,
(iii) $n = 10$ *and* $k = 5$,
(iv) $n = 13$ *and* $k = 6$.

Proof. By Theorem 22, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is 4-dichromatic, where H, J, K and L are maximal transitive subtournaments. Note that by the partition of the vertices of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the cases that need to be considered are when $\alpha = 1$, because it is when the order of L can be one. In this case, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is a vertex-critical 4-dichromatic. We have two cases when $\alpha = 1$.

Case 1. $s_1 \in [1, k-1]$.

(i) If $s_1 \in \{1,2,3\}$, then by Theorem 22 Case 1(i), we have that $|L| = k-3, k-4, k-5$, respectively. The tournament is vertex-critical if and only if $|L| = 1$ if and only if $k = 4$ and $n = 7$, $k = 5$ and $n = 10$, $k = 6$ and $n = 13$, respectively
(ii) If $s_1 = k-3$, then by the proof of Theorem 22 Case 1(iv), it is vertex-critical if and only if $|L| = 1$ if and only if $2n = 5k-4$ and $n = 3k-5$ if and only if $k = 6$ and $n = 13$.
(iii) If $s_1 = k-2$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = k-2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
(iv) If $s_1 = k-1$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = n-2k$. It is vertex-critical if and only if $|L| = 1$ if and only if $n = 2k+1$ and $n = 3k-3$ if and only if $k = 4$ and $n = 9$.

Case 2. $s_2 \in [k, 2k-2]$.

(i) If $s_2 = k$, then by the proof of Theorem 22 Case 2(i), we have that $|L| = k-2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
(ii) If $s_2 \in [k+1, 2k-2]$, then by the proof of Theorem 22 Case 2(ii)–(iv)(a), we have that $|L| = k-2$, but it is not necessarily vertex-critical if $|L| = 1$, because the last vertices remain in K. When L is obtained by the proof of Theorem 22 Case 2(iv)(b), $|L|$ never is one. In any case, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is not a vertex-critical 4-dichromatic circulant tournament. ■

5. The Dichromatic Number of $\overrightarrow{C}_{2n+1}\langle k\rangle$ for $\lceil \frac{n}{2} \rceil + 1 \le k \le n$.

In this part we prove that the tournaments $\overrightarrow{C}_{2n+1}\langle k\rangle$ are 4-dichromatic if $\lceil \frac{n}{2} \rceil + 1 \le k \le \lfloor \frac{2}{3}n \rfloor$ for $n \ge 8$.

Lemma 24. *If $\lceil \frac{n}{2} \rceil + 1 \le k \le \lfloor \frac{2}{3}n \rfloor$, then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains a maximal transitive subtournament of order k.*

Proof. Since $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex-transitive, it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that $N^+(0) = \{1, 2, \dots, k-1, k+1, \dots, n, 2n+1-k\}$. We define $H = \langle [0, k-1] \rangle$. It is transitive by the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$. If H was not maximal, then we could add one vertex of $N^+(0) \setminus [0, k-1]$. Let $j \in [k+1, n]$. Without loss of generality, choose $j = k+1$. Thus, the set of vertices $\{1, t, k+1\}$ with $t \in [2, k-1]$ induces a $\overrightarrow{C}_3$. The same occurs for the vertex $2n+1-k$. Observe that $(3, k-1, 2n+1-k, 3) \cong \overrightarrow{C}_3$, a contradiction. Therefore, H is maximal. ∎

Lemma 25. *If $\lceil \frac{n}{2} \rceil + 1 \le k \le \lfloor \frac{2}{3}n \rfloor$, then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains three maximal transitive subtournaments of k vertices.*

Proof. By Lemma 24, $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains a maximal transitive subtournament H. Notice that $|N^+(0)| - |H| < k$. Consider the following subtournaments

$$J = \langle [k, 2k-1] \rangle \quad \text{and } K = \langle [2k, 3k-1] \rangle.$$

Observe that J and K are isomorphic to H. Let $\varphi_1 : H \to J$ such that $\varphi_1(j) = j + k$ with $0 \le j \le k-1$, (φ_1 is bijective and it is clear that H is isomorphic to J). Analogously, $\varphi_2 : H \to K$ is an isomorphism between H and K. As in Lemma 24, we can prove that J and K are maximal transitive subtournaments. Then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains three maximal transitive subtournaments on k vertices. ∎

Theorem 26. *If $\lceil \frac{n}{2} \rceil + 1 \le k \le \lfloor \frac{2}{3}n \rfloor$, then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$.*

Proof. First we prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \ge 4$. By Lemma 25, we have that $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains three maximal transitive subtournaments of k vertices. Then $|\overrightarrow{C}_{2n+1}\langle k\rangle| - 3k > 0$. Thus, $V(\overrightarrow{C}_{2n+1}\langle k\rangle)$ cannot be partitioned into three transitive subtournaments. Then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \ge 4$. We verify that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$. By Lemma 25, we have that H, J and K are maximal transitive subtournaments of order k. The fourth transitive subtournament is $L = \langle [3k, 2n] \rangle$. Therefore, $\overrightarrow{C}_{2n+1}\langle k\rangle)$ is 4-dichromatic. ∎

Theorem 27. *If* $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, *then* $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is a vertex-critical* 4-*dichromatic circulant tournament if and only if* $n \equiv 0 \bmod 3$.

Proof. By Theorem 26, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is 4-dichromatic. Observe that the order of H, J and K is k and $|L| = 2n - 3k + 1$. Notice that $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex critical 4-dichromatic if the cardinality of L is equal to one, and it occurs if and only if $k = \frac{2}{3}n$ when $n \equiv 0 \bmod 3$. By Theorem 3, $\overrightarrow{C}_{2n+1}\langle \frac{2}{3}n\rangle$ with $n \equiv 0 \bmod 3$ is a vertex-critical circulant tournament 4-dichromatic. ■

Corollary 28 ([11])**.** $\overrightarrow{C}_{6m+1}\langle 2m\rangle$ *is a vertex-critical* 4-*dichromatic circulant tournament for* $m \geq 2$.

Theorem 29. *Let* $n \geq 3$. *Then* $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$ *for* $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$.

Proof. Let $n \geq 3$. By Theorem 1, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 3$. Take the following partition of the vertices of $\overrightarrow{C}_{2n+1}\langle k\rangle$:

$$H = [0, k-1], J = [k, 2k-1] \text{ and } K = [2k, 2n].$$

Observe that H induces a TT_k because $N^+(i) = \{i+1, i+2, \ldots, k+1\}$ for $k \leq i \leq 2k-1$, also J and K induce a TT_k and a $TT_{2n-2k+1}$, respectively. Then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$. ■

Theorem 30. *If* $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$, $n \geq 3$. *Then* $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is a vertex-critical* 3-*dichromatic circulant tournament if and only if* $n = k$.

Proof. By Theorem 29, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is 3-dichromatic and its partition into three maximal transitive subtournaments was

$$|H| = |J| = k \text{ and } |K| = 2n - 2k + 1.$$

Since $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$, we have that $k \geq 2n - 2k + 1$. Hence, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex-critical if and only if $2n - 2k + 1 = 1$, if and only if $n = k$. ■

Corollary 31 ([13], Theorem 2)**.** $\overrightarrow{C}_{2n+1}\langle n\rangle$ *is a vertex-critical* 3-*dichromatic circulant tournament for* $n \geq 3$.

By Theorems 13, 19, 23, 27 and 30, we have the following.

Theorem 32. *Let* $r \in \{2, 3, 4\}$, $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is vertex-critical* r-*dichromatic if and only if*

(i) $r = 2$, $n = 1$ *and* $k = 1$;

(ii) $r = 3$,

  (a) $n = 4$ *and* $k = 1$,

(b) $n \geq 3$ *and* $k = n$;

(iii) $r = 4$,

(a) $n = 5$ *and* $k = 2$,

(b) $n = 7$ *and* $k \in \{3, 4\}$,

(c) $n = 9$ *and* $k = 4$,

(d) $n = 10$ *and* $k = 5$,

(e) $n = 13$ *and* $k = 6$,

(f) $n = 3m$ *and* $k = 2m$ $(m \geq 2)$.

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Received 30 June 2016
Revised 7 April 2016
Accepted 7 April 2016