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DISTINGUISHING CARTESIAN PRODUCTS OF COUNTABLE GRAPHS

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Abstract

The distinguishing number $\mathrm{D}(G)$ of a graph G is the minimum number of colors needed to color the vertices of G such that the coloring is preserved only by the trivial automorphism. In this paper we improve results about the distinguishing number of Cartesian products of finite and infinite graphs by removing restrictions to prime or relatively prime factors.

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1. Introduction

This paper is concerned with automorphisms breaking of Cartesian products of graphs by vertex colorings. Our focus is on breaking the automorphisms of a graph G with a minimum number of colors. This number is called the *distinguishing number* $\mathrm{D}(G)$. It is defined as the least natural number d such that G has a vertex coloring with d colors that is only preserved by the trivial automorphism.

The distinguishing number was introduced by Albertson and Collins in [2] and has spawned a wealth of interesting papers. There also exists a sizable literature on this problem for the Cartesian product. In particular, in [7] it was shown that the distinguishing number of the product $G \,\square\, H$ of two finite connected graphs that are relatively prime with respect to the Cartesian product is 2 if $|G| \leq |H| \leq 2^{|G|} - |G| + 1$. Here we prove that G and H need not be relatively prime if $G \,\square\, H$ is different from three exceptional graphs with at most nine vertices.

Then we consider countably infinite graphs and extend a result from [14], where it was shown that $\mathrm{D}(G \,\square\, H) = 2$ if G and H are connected graphs of infinite diameter. Here we prove that the result still holds even when the diameters are finite. For the proof we rely on the weak Cartesian product and some of its basic properties.

We use standard graph theoretic notation, but will denote the order (the number of vertices) of a graph G by $|G|$. Also, we restrict attention to undirected graphs without multiple edges and loops.

The *Cartesian product* $G \,\square\, H$ has as its vertex set the Cartesian product $V(G) \times V(H)$. Its edge set $E(G \,\square\, H)$ is the set

$$\big\{(x,u)(y,v) \mid (xy \in E(G) \wedge u = v) \vee (x = y \wedge uv \in E(H))\big\}.$$

The Cartesian product is commutative, associative, and has K_1 as a unit. $G \,\square\, H$ is connected if and only if both G and H are connected.

The graphs G and H are called *factors* of $G \,\square\, H$. We write G^2 for the second power $G \,\square\, G$ of G, and recursively define the r-th Cartesian power of G as $G^r = G \,\square\, G^{r-1}$. A non-trivial graph G is called *prime* if $G = G_1 \,\square\, G_2$ implies that either G_1 or G_2 is K_1. It was proven independently by Sabidussi [13] and Vizing [15] that every connected graph has a prime factor decomposition with respect to the Cartesian product that it is unique up to the order and isomorphisms of the factors. Two graphs G and H are called *relatively prime* if K_1 is the only common factor of G and H.

2. Finite Cartesian Products

The distinguishing number of the Cartesian powers of finite graphs has been thoroughly investigated. It was first proved by Albertson [1] that if G is a connected prime graph, then $\mathrm{D}(G^k) = 2$ whenever $k \geq 4$, and, if $|V(G)| \geq 5$, then $\mathrm{D}(G^k) = 2$ for $k \geq 3$. Next, Klavžar and Zhu showed in [9] that for any connected graph G with a prime factor of order at least 3 the distinguishing number $\mathrm{D}(G^k) = 2$ for $k \geq 3$. Both results were obtained using the Motion Lemma [12]. Finally, Imrich and Klavžar [7] provided the complete solution for the problem of the distinguishing number of the Cartesian powers of connected graphs.

Theorem 1 [7]**.** *Let G be a connected graph and $k \geq 2$. Then $\mathrm{D}(G^k) = 2$ except for the graphs K_2^2, K_2^3, K_3^2, whose distinguishing number is three.*

Their proof is based on the algebraic properties of the automorphism group of the Cartesian product of graphs. In the same paper Imrich and Klavžar considered the Cartesian product of distinct factors and obtained a sufficient condition when the distinguishing number of the Cartesian product of two relatively prime graphs equals 2.

Theorem 2 [7]**.** *Let G and H be connected, relatively prime graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $\mathrm{D}(G \Box H) \leq 2$.

They also proved several lemmas that will be useful in this paper.

Lemma 3 [7]**.** *Let G and H be two connected, relatively prime graphs such that* $2 \leq \mathrm{D}(G) \leq 3$ *and* $\mathrm{D}(H) = 2$. *Then* $\mathrm{D}(G \Box H) = 2$.

Lemma 4 [7]**.** *Let G and H be two connected graphs such that G is prime,* $2 \leq |G| \leq |H| + 1$ *and* $\mathrm{D}(H) = 2$. *Then* $\mathrm{D}(G \Box H) = 2$.

We shall use the following strengthened version of this lemma.

Lemma 5. *Let G and H be two connected graphs such that $|G| \leq |H| + 1$ and* $\mathrm{D}(H) = 2$. *Then* $\mathrm{D}(G \Box H) = 2$.

Proof. If $|G| = 1$, then the conclusion follows trivially. Suppose then that $G = G_1 \Box G_2$, where G_2 is prime. Then $G \Box H = G_1 \Box (G_2 \Box H)$ and $D(G_2 \Box H) = 2$ by Lemma 4 since G_2 is prime and $|G_2| \leq |H| + 1$ as $|G| \leq |H| + 1$. We can continue this way pulling off prime factors from G and putting them with H. ■

We now prove a generalization of Theorem 2 for graphs that are not necessarily relatively prime.

Theorem 6. *Let G and H be connected graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1. \tag{1}$$

Then $\mathrm{D}(G\Box H) \leq 2$ *unless* $G\Box H \in \{K_2^2, K_2^3, K_3^2\}$.

Proof. The case when G and H are relatively prime was settled in Theorem 2. Let then G and H have at least one common factor. Let $G = G_1^{k_1}\Box \cdots \Box G_t^{k_t}$ and $H = H_1^{l_1}\Box \cdots \Box H_s^{l_s}$ be the prime factor decompositions of G and H. Assume that the first r prime factors are common, i.e., $G_i = H_i, i = 1, \ldots, r$. Define

$$G_c = G_1^{k_1}\Box \cdots \Box G_r^{k_r}, \qquad H_c = H_1^{l_1}\Box \cdots \Box H_r^{l_r}.$$

Hence, $G = G_c\Box G_d$ and $H = H_c\Box H_d$. We begin with finding the distinguishing number of the Cartesian product

$$G_c\Box H_c = G_1^{l_1+k_1}\Box \cdots \Box G_r^{l_r+k_r}.$$

Due to Theorem 1, for each $i = 1, \ldots, r$, either $\mathrm{D}(G_i^{l_i+k_i}) = 2$ or $\mathrm{D}(G_i^{l_i+k_i}) = 3$ if $G_i^{l_i+k_i} \in \{K_2^2, K_2^3, K_3^2\}$. The distinguishing number of the Cartesian product of two graphs from the set $\{K_2^2, K_2^3, K_3^2\}$ equals 2 by Theorem 2. It follows from Lemma 3 that $\mathrm{D}(G_c\Box H_c) = 2$ unless $G_c\Box H_c \in \{K_2^2, K_2^3, K_3^2\}$. In this case $\mathrm{D}(G_c\Box H_c) = 3$.

Now assume that $G\Box H \notin \{K_2^2, K_2^3, K_3^2\}$ and consider the graphs $G' = G_c\Box H_c\Box G_d$ and $H' = H_d$. They are relatively prime by definition and

$$|H'| \leq H \leq 2^{|G|} - |G| + 1 \leq 2^{|G'|} - |G'| + 1,$$

since the function $f(x) = 2^x - x + 1$ is increasing for $x > 0$. If $|G'| \leq |H'|$ the result follows from Theorem 2. Therefore, we assume $|G'| > |H'|$ throughout the rest of the proof. Note that this means that our result follows from Lemma 5, if we can show $D(G') = 2$.

We consider two cases. For the first case, suppose that $|G_c\Box H_c| \geq |G_d|$. We assume that $G_c\Box H_c \in \{K_2^2, K_2^3, K_3^2\}$, as otherwise $D(G') = 2$ by Lemma 5. But if $|G_d| > 3$, then $2^{|G_d|} - |G_d| + 1 > 9 > |G_c\Box H_c|$ and $D(G') = 2$ by Theorem 2. If $|G_d| = 2$, then $D(G_d) = 2$, so $D(G') = 2$ by Lemma 3 since $D(G_c\Box H_c) = 3$. If $|G_d| = 3$, then $G' = K_2^2\Box K_3$ or $G' = K_2^3\Box K_3$. For the former, Theorem 2 gives $D(G') = 2$ since $4 < 2^3 - 3 + 1$. For the latter, observe that $D(K_2\Box K_3) = 2$ by Lemma 3, hence $D(G') = 2$ by Lemma 5 since $G' = K_2^2\Box(K_2\Box K_3)$. Finally, suppose that $|G_d| = 1$, so $G' = G_c\Box H_c$. Since we are assuming $|G'| = |G_c\Box H_c| > |H_d|$, we can proceed with H_d replacing G_d in the previous arguments to show that $D(G\Box H) = D(G_c\Box H_c\Box H_d) = 2$.

For the second case, suppose $|G_c \square H_c| < |G_d|$. Note that this implies $|G_d| > 4$. Since $|G'| > |H'|$ throughout, we have

$$|H_d| < |G'| = |G_c \square H_c| \cdot |G_d| \leq |G_d|^2 \leq 2^{|G_d|} - |G_d| + 1,$$

where the latter inequality follows from the fact that $x^2 \leq 2^x - x + 1$ for $x > 4$. If $|H_d| \leq |G_d|$, then $D(G_d \square H_d) = 2$ by Theorem 2, and the result follows from Lemma 3 since $2 \leq D(G_c \square H_c) \leq 3$. Assume instead that $|H_d| > |G_d|$. Let $G'' = G_d$ and $H'' = G_c \square H_c \square H_d$. Then

$$|H''| = |G_c| \cdot |H_c| \cdot |H_d| < |H_d|^2 < |G_d|^2 < 2^{|G_d|} - |G_d| + 1 = 2^{|G''|} - |G''| + 1.$$

Thus by Theorem 2, we have $D(G'' \square H'') = D(G \square H) = 2$. ■

3. Infinite Cartesian Products

It was shown in [8] that the distinguishing number of the Cartesian product $G \square H$ of two graphs of the same, but arbitrary, cardinality is 2 if G and H are either relatively prime or prime and isomorphic. In this section we remove the condition that G and H are relatively prime or isomorphic if G and H are both countable (Theorem 9 below). For the proof, we need some more results concerning the automorphism group of the Cartesian product of graphs, and we recall them first.

Let $G = G_1 \square G_2 \square \cdots \square G_r$, where $G_i, i = 1, \ldots, r$, is a finite or infinite graph. For a given $v = (g_1, \ldots, g_r) \in V(G)$, the subgraph G_i^v of G induced by the vertex set

$$\{(g_1, g_2, \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_r) \mid x \in V(G_i)\}$$

is called the G_i*-layer* containing v. Clearly, every G_i-layer is isomorphic to G_i.

Notice, in the proof of Theorem 10 we consider the layers of a product $G \square H$, where $V(G) = V(H) = \mathbb{N}$. In this case the vertices of $G \square H$ are pairs (i, j) of integers, and the notation of $G^{(i,j)}$ for the G-layer through (i, j) and $H^{(i,j)}$ for the H-layer through (i, j), which we use there, is consistent with the above definition.

The automorphism group of the Cartesian product of connected graphs is described by the following theorem of Imrich and Miller [4, 10]. We use the description from [3, Theorem 6.10].

Theorem 7. *Suppose φ is an automorphism of a connected graph G with prime factor decomposition $G = G_1 \square G_2 \square \cdots \square G_r$. Then there is a permutation π of $\{1, 2, \ldots, r\}$ and an isomorphism $\varphi_i \colon G_{\pi(i)} \mapsto G_i$ for every i such that*

$$\varphi(x_1, x_2, \ldots, x_r) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_r(x_{\pi(r)})). \tag{2}$$

There are two important special cases. In the first, π is the identity permutation and only one φ_i is nontrivial. Then the mapping φ_i^* defined by

$$\varphi_i^*(x_1, \ldots, x_r) = (x_1, \ldots x_{i-1}, \varphi_i(x_i), x_{i+1}, \ldots, x_r)$$

is an automorphisms and we say that φ_i^* is *induced by* $\varphi_i \in \mathrm{Aut}(G_i)$. Clearly φ_i^* preserves every G_i-layer and preserves every set of G_j-layers for fixed j.

The second case is the transposition $\varphi_{i,j}$ of isomorphic factors $G_i \cong G_j$. If we assume that $G_i = G_j$, where $i < j$, then $\varphi_{i,j}$ can be defined by

$$\varphi_{i,j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k).$$

It is called a *transposition of isomorphic factors* and interchanges the set of G_i-layers with the set of G_j-layers.

The automorphisms induced by the automorphisms of the factors, together with the transposition of isomorphic factors, generate $\mathrm{Aut}(G)$. Thus every automorphism $\varphi \in \mathrm{Aut}(G)$ permutes the sets of G_i-layers in the sense that φ maps the set of G_i-layers into the set of $G_{\pi(i)}$-layers, where π is the permutation from equation 2.

We now extend the definition of the Cartesian product to arbitrarily many factors. Given an index set I and graphs G_ι, $\iota \in I$, we let the Cartesian product

$$G = \mathop{\square}_{\iota \in I} G_\iota$$

be defined on the vertex set consisting of all functions $x : \iota \to x_\iota$ with $x_\iota \in V(G_\iota)$, where two vertices x and y are adjacent if there exists a $\kappa \in I$ such that $x_\kappa y_\kappa \in E(G_\kappa)$ and $x_\iota = y_\iota$ for $\iota \in I \setminus \{\kappa\}$.

For finite I we obtain the usual Cartesian product, which is connected if and only if all factors are connected. However, the Cartesian product $G = \square_{\iota \in I} G_\iota$ of infinitely many non-trivial connected graphs is disconnected. The connected components are called *weak Cartesian products*, and we denote the connected component containing a vertex $a \in V(G)$ by

$$\mathop{\square}_{\iota \in I}^{a} G_\iota.$$

Clearly, $\square^a_{\iota \in I} G_\iota = \square^b_{\iota \in I} G_\iota$ if and only if a and b differ in only finitely many coordinates.

Note that the distance $d(x, y)$ between two vertices x and y that differ in k coordinates is at least k. Hence, there are vertices of arbitrarily large distance in any weak Cartesian product G of infinitely many non-trivial factors. We say the G has infinite *diameter*.

For the weak Cartesian product we have the following theorem of Imrich and Miller [5, 11].

Theorem 8. *Every connected graph is uniquely representable as a weak Cartesian product of connected prime graphs.*

Again every $\varphi \in \mathrm{Aut}(G)$ can be represented in the form

$$\varphi(x)_i = \varphi_i(x_{\pi(i)}), \tag{3}$$

where $i \in I$, $\varphi_i \in \mathrm{Aut}(G_i)$, and π is a permutation of I. As in the finite case all automorphisms of a weak Cartesian product are generated by automorphisms induced by automorphisms of factors and transpositions of isomorphic factors.

We now state the main result of this section.

Theorem 9. *Let G and H be countably infinite, connected graphs. Then* $\mathrm{D}(G\Box H) \leq 2$.

It generalizes the result from [8] for countably infinite graphs, which we now state and prove for the sake of completeness.

Theorem 10. *Let G and H be two countably infinite, connected graphs that are relatively prime, or prime and isomorphic. Then* $\mathrm{D}(G\Box H) \leq 2$.

Proof. Suppose G and H satisfy the assumptions of the theorem. Let $V(G) = V(H) = \mathbb{N}$. We color the vertices $(i, j) \in V(G \Box H)$ black if $1 \leq j \leq i$, and white otherwise. Then all vertices of $G^{(1,1)}$ are black. But, because every H-layer $H^{(i,1)}$ has i black vertices, each H-layer has only finitely many black vertices. Hence, the set of G-layers cannot be interchanged with the set of H-layers. Furthermore, notice that every G-layer $G^{(1,i)}$ has $i-1$ white vertices. Thus any two G layers have a different number of white vertices and any two H-layers different numbers of black vertices. Thus every color-preserving automorphism must fix all H-layers and all G-layers. The only automorphism with this property is the identity automorphism. ∎

If both G and H are complete, then we obtain $\mathrm{D}(K_{\aleph_0} \Box K_{\aleph_0}) = 2$ as a special case. This was shown in [6] with essentially the same coloring.

We first note that Theorem 9 is true if at least one of the graphs G and H has infinitely many factors because of the following theorem of Smith, Tucker and Watkins.

Theorem 11 [14]**.** *If G and H are countably infinite, connected graphs of infinite diameter, then* $\mathrm{D}(G\Box H) = 2$.

Corollary 12. *Let G and H be connected graphs. If H has infinitely many non-trivial factors, then* $\mathrm{D}(G \Box H) = 2$.

Proof. If H has infinitely many non-trivial factors, then this is also true for $G \,\Box\, H$. Hence, we can represent $G \,\Box\, H$ as a product $G' \Box H'$, where both G' and H' are weak Cartesian products with infinitely many factors. Since both graphs must have infinite diameter we get $\mathrm{D}(G \,\Box\, H) = 2$. ∎

Proof [3] ***of Theorem*** **9.** By Corollary 12 we only have to consider the case where the prime factorizations of both G and H consist of only finitely many factors. Thus both G and H contain at least one infinite prime factor. Let G' and H' be infinite prime divisors of G and H, respectively. Their product is 2-distinguishable by Theorem 10, and hence $G \,\Box\, H$ is the product of the countably infinite, 2-distinguishable graph $G' \,\Box\, H'$ with finitely many prime graphs, say $A_1, \ldots, A_k$, which can be finite or infinite.

Lemma 13, see below, shows that the product of a connected prime graph with a countably infinite 2-distinguishable graph is also 2-distinguishable. The theorem follows by repeated application of Lemma 13. ∎

Lemma 13. *Let G and H be connected graphs, where G is finite or infinite, and H countably infinite. If G is prime and* $\mathrm{D}(H) = 2$, *then* $\mathrm{D}(G \Box H) = 2$.

Proof. We argue similarly as in the proof of Lemma 3.2 in [7]. We color one H-layer with a distinguishing 2-coloring c. We can assume without loss of generality that c colors infinitely many vertices of H white. Clearly the set of G-layers cannot be permuted as all automorphisms of this H-layer are broken. If G and H are relatively prime, we color all remaining H-layers with distinct 2-colorings. This is possible since $|G| < 2^{|H|}$. Thus all permutations of the H-layers are also broken.

If G and H are not relatively prime and $G \neq K_2$, we color all vertices of another H-layer black and the remaining H-layers such that each layer contains only one black vertex, each of them with a different projection into a white vertex of H. Then every G-layer is colored with both black and white vertices. If an automorphism maps a G-layer into an H-layer, then all G-layers are mapped into H-layers, but one H-layer contains only black vertices, hence this is not possible.

Suppose now that $G = K_2$ and that H contains K_2 as a factor. Recall that the k-th power of K_2^k is the hypercube Q_k. Then $G \,\Box\, H = K_2 \,\Box\, (Q_k \,\Box\, H') = Q_{k+1} \,\Box\, H'$. Because K_2 and H' are relatively prime, the H'-layers of G are preserved by every automorphism of G.

We now color $G \,\Box\, H$. Recall that $G \,\Box\, H = K_2 \,\Box\, H$ has two H-layers, say H^0 and H^1, both isomorphic to H. We color H^0 with a distinguishing 2-coloring c.

[3] The proof of this theorem was roughly outlined in W. Imrich, On the Weak Cartesian Product of Graphs, Topics In Graph Theory. A tribute to A.A. and T.E. Zykova on the occasion of A.A. Zykov's 90th birthday, University of Illinois at Urbana-Champaign, The personal web page of Professor Alexandr V. Kostochka, http://www.math.uiuc.edu/ kostochk/ (2013), 5663, viewed on August 4, 2015.

This coloring induces 2-colorings of the H'-layers that are in H^0. These colorings need not be distinguishing colorings of the H'- layers, nor need they be different (in the sense that they are equivalent with respect to an automorphism of $H^0 \cong Q_k \,\square\, H'$ that is induced by an automorphism of Q_k). Both H^0 and H^1 contain finitely many H'-layers, namely 2^k. Because H' is infinite, it is possible to color the H'-layers of $G \,\square\, H$ that are in H^1 such that they are pairwise different and different from the 2-colorings of the H'-layers in H^0 that are induced by c.

This means that no automorphism of $G \,\square\, H$ can map an H'-layer of H^1 into one of H^0. Hence, H^1, and thus also H^0, is preserved. Since c is distinguishing on H^0, we infer that we have constructed a distinguishing 2-coloring.

If $k = \aleph_0$, then $G \,\square\, H = K_2 \,\square\, (Q_{\aleph_0} \,\square\, H') \cong Q_{\aleph_0} \,\square\, H'$, and $\mathrm{D}(G \,\square\, H) = 2$ by Theorem 10. ■

Corollary 14. *If G is a countably infinite, connected graph and $2 \leq k \leq \aleph_0$, then $\mathrm{D}(G^k) = 2$.*

Proof. Let k be finite. Then the theorem follows by repeated application of Theorem 9. If $k = \aleph_0$, then the theorem follows from Corollary 12. ■

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