RAINBOW CONNECTION NUMBER OF GRAPHS WITH DIAMETER 3

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Abstract

A path in an edge-colored graph $G$ is rainbow if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of $G$ is the smallest integer $k$ for which there exists a $k$-edge-coloring of $G$ such that every pair of distinct vertices of $G$ is connected by a rainbow path. Let $f(d)$ denote the minimum number such that $rc(G) \leq f(d)$ for each bridgeless graph $G$ with diameter $d$. In this paper, we shall show that $7 \leq f(3) \leq 9$.

Keywords: edge-coloring, rainbow path, rainbow connection number, diameter.

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1. Introduction

All graphs in this paper are undirected, finite, and simple. We refer to book [2] for notation and terminology not described here. A path $u_0u_1 \cdots u_k$ is called a $P_{uv}$ path, where $u = u_0$ and $u_k = v$. The distance between two vertices $x$ and $y$ in $G$, denoted by $d(x, y)$, is the number of edges of a shortest path between them. The eccentricity of a vertex $x$, denoted by $ecc(x)$, is $\max_{y \in V(G)} d(x, y)$. The radius and diameter of $G$, denoted by $rad(G)$ and $diam(G)$, are $\min_{x \in V(G)} ecc(x)$ and $\max_{x \in V(G)} ecc(x)$, respectively. A vertex $u$ is a center if $ecc(u) = rad(G)$. 
A path in an edge-colored graph $G$, where adjacent edges may have the same color, is *rainbow* if no two edges of the path are colored the same. An edge-coloring of a graph $G$ is a *rainbow-connected edge-coloring* if every pair of distinct vertices of $G$ is connected by a rainbow path. The *rainbow connection number* $rc(G)$ of $G$ is the minimum integer $k$ for which there exists a rainbow-connected $k$-edge-coloring of $G$. It is easy to see that $\text{diam}(G) \leq rc(G)$ for any connected graph $G$.

The rainbow connection number was introduced by Chartrand, Johns, McKeeon, and Zhang in [4]. It has application in transferring information of high security in multicomputer networks. We refer the readers to [3, 8] for details.

Chakraborty, Fischer, Matsliah, and Yuster [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph $G$, deciding if $rc(G) = 2$ is NP-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius and diameter, etc. [1, 5, 6, 7].

Let $f(d)$ denote the minimum number such that each bridgeless graph $G$ with diameter $d$ has a rainbow-connected $f(d)$-edge-coloring. It is easy to check that $f(1) = 1$. In [7], we showed that $f(2) = 5$. In this paper, we shall show that $7 \leq f(3) \leq 9$.

The following theorem will be used in this paper.

**Theorem 1** [5]. For every bridgeless graph $G$,

$$rc(G) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i+1, \eta(G)\} \leq \text{rad}(G) \eta(G),$$

where $\eta(G)$ is the smallest integer such that every edge of $G$ is contained in a cycle of length at most $\eta(G)$.

In this paper, we investigate the upper bound on the rainbow connection number of bridgeless graphs with diameter 3, and obtain the following result.

**Theorem 2.** For every bridgeless graph $G$ with diameter 3, $rc(G) \leq 9$.

If each edge of a bridgeless graph $G$ with diameter 3 belongs to a triangle, then $rc(G) \leq 9$ by Theorem 1. Thus, we suppose that there exists an edge $e$ such that $e$ does not belong to any triangle in $G$.

This paper is organized as follows. In Section 2, we partition $V(G)$, and present a partial edge-coloring of $G$ under this partition. In Section 3, we further partition $V(G)$ and give a complete edge-coloring of $G$ under this partition. In Section 4, we prove that the edge-coloring in Section 3 is a rainbow-connected 9-edge-coloring of $G$, and give a class of bridgeless graphs with diameter 3 and rainbow connection number at least 7.
2. A Partial Edge-Coloring

Let $G$ be a graph. For any integer $k \geq 1$, the $k$-step open neighborhood $N^k(X)$ is $\{ y \in V(G) : d(X, y) = k \}$. We simply write $N(X)$ for $N^1(X)$ and $N^k(x)$ for $N^k(\{x\})$. Similarly, the $k$-step closed neighborhood $N^k[X]$ is $\{ y \in V(G) : d(X, y) \leq k \}$. We simply write $N[X]$ for $N^1[X]$ and $N^k[x]$ for $N^k(\{x\})$.

Let $c$ be an edge-coloring of $G$, and let $P$ be a rainbow path in $G$. We use $c(P)$ to denote the set of colors used on $P$, that is, $c(P) = \{c(e) : \text{the edge } e \text{ belongs to } P\}$. If $c(P) \subseteq \{k_1, k_2, \ldots, k_r\}$, then $P$ is a $\{k_1, k_2, \ldots, k_r\}$-rainbow path. In particular, an edge $e$ is a $k$-color edge if $c(e) = k$. We use $x_0 \sim x_1 \sim \cdots \sim x_k$ to denote a rainbow path $x_0x_1\cdots x_k$ with $c(x_{i-1}x_i) = c_i$ for each $1 \leq i \leq k$. Let $X_1, X_2, \ldots, X_{k-1}$ be pairwise disjoint vertex subsets of $G$. The notation $x_0 \sim X_1 \sim \cdots \sim X_{k-1} \sim x_k$ means that there exists a rainbow path $x_0 \sim x_1 \sim \cdots \sim x_k$, where $x_i \in X_i$ for $1 \leq i \leq k - 1$.

Recall that $e$ is an edge not belonging to any triangle in $G$. Let $u$ and $v$ be the ends of $e$.

Since $e$ does not belong to any triangle, for the open neighborhood, $N(\{u, v\})$, of $\{u, v\}$ in $G$, we can divide it as follows:

$$A = N(u) \setminus \{v\},$$
$$B = N(v) \setminus \{u\}.$$ 

See Figure 1 for details.

For the 2-step open neighborhood, $N^2(\{u, v\})$, of $\{u, v\}$ in $G$, we can divide it as follows:

$$X = \{ x \in N(A) \setminus N(B) : x \notin A \cup B \cup \{u, v\} \},$$
$$Y = \{ x \in N(B) \setminus N(A) : x \notin A \cup B \cup \{u, v\} \},$$
$$Z = \{ x \in N(A) \cap N(B) : x \notin A \cup B \cup \{u, v\} \}.$$ 

See Figure 1 for details. It is easy to see that $x \in X$ if and only if $x \notin N[\{u, v\}]$, $d(x, u) = 2$ and $d(x, v) = 3$; $y \in Y$ if and only if $y \notin N[\{u, v\}]$, $d(y, u) = 3$ and $d(y, v) = 2$; $z \in Z$ if and only if $z \notin N[\{u, v\}]$, $d(x, u) = 2$ and $d(x, v) = 2$.

Note that for $x \in N^3(\{u, v\})$, we have $d(x, u) = d(x, v) = 3$, since $\text{diam}(G) = 3$, that is, $N(x) \cap N(A) \neq \emptyset$ and $N(x) \cap N(B) \neq \emptyset$.

For the 3-step open neighborhood, $N^3(\{u, v\})$, of $\{u, v\}$ in $G$, we can partition $N^3\{u, v\}$ based on the distribution of the neighbors of $x$ as follows:

$$W = \{ x \in N^3(\{u, v\}) : N(x) \cap X \neq \emptyset \text{ and } N(x) \cap Y \neq \emptyset \},$$
$$I = \{ x \in N^3(\{u, v\}) \setminus W : N(x) \cap X \neq \emptyset \text{ and } N(x) \cap Z \neq \emptyset \},$$

See Figure 1 for details.
\[ K = \{ x \in N^3(\{u, v\}) \setminus (W \cup I) : N(x) \cap Y \neq \emptyset \text{ and } N(x) \cap Z \neq \emptyset \}, \]
\[ J = \{ x \in N^3(\{u, v\}) \setminus (W \cup I \cup K) : N(x) \cap Z \neq \emptyset \}. \]

See Figure 1 for details. It is easy to see that \( N^3(\{u, v\}) = I \cup J \cup K \cup W \).

At this point, we further partition \( A \) and \( B \) as follows:
\[ A_1 = \{ x \in A : N(x) \cap (B \cup X \cup Z) \neq \emptyset \}, \]
\[ A_2 = \{ x \in A \setminus A_1 : N(x) \cap (A \setminus A_1) \neq \emptyset \}, \]
\[ A_3 = A \setminus (A_1 \cup A_2), \]
\[ B_1 = \{ x \in B : N(x) \cap (A \cup Y \cup Z) \neq \emptyset \}, \]
\[ B_2 = \{ x \in B \setminus B_1 : N(x) \cap (B \setminus B_1) \neq \emptyset \}, \]
\[ B_3 = B \setminus (B_1 \cup B_2). \]

That is, \( A_1 \) consists of vertices which have neighbors outside \( A \cup \{u\} \), \( A_2 \) consists of vertices which do not have neighbors outside \( A \) (apart from \( u \)) but have neighbors in \( A \setminus A_1 \), and \( A_3 \) consists of vertices which have neighbors only in \( A_1 \) (apart from \( u \)). It is clear that for each \( x \in A_2 \), there exists a vertex \( x' \in A_2 \) such that \( xx'u \) is a triangle. Similar results also hold for \( B_1, B_2 \) and \( B_3 \).

Note that there may exist edges between between \( A_1 \) and \( A_2 \), but it does not matter for our proof.

Meanwhile, we partition \( X \) and \( Y \) as follows:
\[ X_1 = \{ x \in X : N(x) \cap (Y \cup Z \cup I \cup W) \neq \emptyset \}, \]
\[ X_2 = \{ x \in X \setminus X_1 : N(x) \cap (X \setminus X_1) \neq \emptyset \}, \]
\[ X_3 = \{ x \in X \setminus (X_1 \cup X_2) : N(x) \subseteq A \}, \]
\[ X_4 = X \setminus (X_1 \cup X_2 \cup X_3), \]
\[ Y_1 = \{ y \in Y : N(y) \cap (X \cup Z \cup K \cup W) \neq \emptyset \}, \]
\[ Y_2 = \{ y \in Y \setminus Y_1 : N(y) \cap (Y \setminus Y_1) \neq \emptyset \}, \]
\[ Y_3 = \{ y \in Y \setminus (Y_1 \cup Y_2) : N(y) \subseteq B \}, \]
\[ Y_4 = Y \setminus (Y_1 \cup Y_2 \cup Y_3). \]

That is, \( X_1 \) consists of vertices which have neighbors outside \( X \) (apart from \( A_1 \)), \( X_2 \) consists of vertices which do not have neighbors outside \( X \) (apart from \( A_1 \)) but have neighbors in \( X \setminus X_1 \), \( X_3 \) consists of vertices which have neighbors only in \( A_1 \), and \( X_4 \) consists of vertices which have neighbors only in \( X_1 \) (apart from \( A_1 \)). Similar results also hold for \( Y_1, Y_2, Y_3 \) and \( Y_4 \).

By the definitions of sets \( A_1, A_2 \) and \( A_3 \), we know that \( N(X_3) \subseteq A_1 \) and \( N(Y_3) \subseteq B_1 \). Thus \( X_3 = \{ x \in X \setminus (X_1 \cup X_2) : N(x) \subseteq A_1 \} \) and \( Y_3 = \{ y \in Y \setminus (Y_1 \cup Y_2) : N(y) \subseteq B_1 \} \).
We denote the above set partition by $\mathcal{P}$. The following observation holds for $\mathcal{P}$ since $G$ is bridgeless.

**Lemma 3.**
1. For $x \in A_3$, $N(x) \cap A_1 \neq \emptyset$.
2. For $x \in B_3$, $N(x) \cap B_1 \neq \emptyset$.
3. For $x \in X_4$, $N(x) \cap X_1 \neq \emptyset$.
4. For $x \in Y_4$, $N(x) \cap Y_1 \neq \emptyset$.

We give a partial 9-edge-coloring of $G$ as follows:

$$c(e) = \begin{cases} 
1, & \text{if } e = uv; \\
2, & \text{if } e \in E[u, A_3] \cup E[v, B_1]; \\
3, & \text{if } e \in E[u, A_1] \cup E[v, B_3]; \\
4, & \text{if } e \in E[A_1, X_1 \cup Z] \cup E(G[A_1]); \\
5, & \text{if } e \in E[B_1, Y_1 \cup Z] \cup E(G[B_1]); \\
6, & \text{if } e \in E[A_1, B_1] \cup E[Z, K] \cup E[X_1, Z \cup I \cup W \cup Y_1]; \\
7, & \text{if } e \in E[Z, I] \cup E[Y_1, K \cup W \cup Z]; \\
8, & \text{if } e \in E[A_1, A_3] \cup E[B_1, B_3] \cup E[X_1, X_4] \\
& \cup E[Y_1, Y_4] \cup E[I, I \cup K \cup W]; \\
9, & \text{if } e \in E[A_1, X_4] \cup E[B_1, Y_4]. 
\end{cases}$$

See Figure 1 for details.
For each $x \in X_3$, $N(x) \subseteq A_1$ by the above set partition. Since $G$ is a bridgeless graph, $|N(x)| \geq 2$. Thus, we can color one edge incident to $x$ by 8, and color the others incident to $x$ by 9. Similarly, for each vertex $y \in Y_3$, we can color edges incident to $y$ by colors 8 and 9.

**Lemma 4.** (1) For $x \in X_1$, there exists an $x \overset{6}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{6}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{6}{\sim} I \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{6}{\sim} W \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring.

(2) For $y \in Y_1$, there exists a $y \overset{6}{\sim} I \overset{7}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{7}{\sim} W \overset{6}{\sim} Y_1 \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{7}{\sim} K \overset{6}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path under the above partial edge-coloring.

**Proof.** We only show (1) since the proofs are similar. For any $x \in X_1$, by the definition of set $X_1$, we know that $x$ has a neighbor, say $x'$, in $Y \cup Z \cup I \cup W$.

If $x' \in Y$, then $x' \in Y_1$ by the definition of set $Y_1$. Thus $xx'x''y$ is an $x \overset{6}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where $x''$ is a neighbor of $x'$ in $B_1$.

If $x' \in Z$, then $xx'x''y$ is an $x \overset{6}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where $x''$ is a neighbor of $x'$ in $B_1$.

If $x' \in I$, then $xx'x''y$ is an $x \overset{6}{\sim} I \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where $x''$ is a neighbor of $x'$ in $Z$ and $x''$ is a neighbor of $x'$ in $B_1$.

Otherwise, $x' \in W$, and then $xx'x''y$ is an $x \overset{6}{\sim} W \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where $x''$ is a neighbor of $x'$ in $Y_1$ and $x''$ is a neighbor of $x'$ in $B_1$.

**Lemma 5.** (1) For $x \in A_1$, there exists an $x \overset{6}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} I \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} W \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring.

(2) For $y \in B_1$, there exists a $y \overset{6}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{7}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{7}{\sim} K \overset{6}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path under the above partial edge-coloring.

**Proof.** We only show (1) since the proofs are similar. For any $x \in A_1$, by the definition of set $A_1$, we know that $x$ has a neighbor, say $x'$, in $B_1 \cup Z \cup X_1$.

If $x' \in B_1$, then $xx'y$ is an $x \overset{6}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring,
If \( x' \in Z \), then \( xx'x''v \) is an \( x \sim Z \sim B_1 \sim v \)-rainbow path, where \( x'' \) is a neighbor of \( x' \) in \( B_1 \).

Otherwise, \( x' \in X_1 \). By Lemma 4, there exists a desired rainbow path. ■

**Lemma 6.** (1) For \( x \in Z \), there exists an \( x \sim B_1 \sim v \sim u \sim A_1 \sim x \)-rainbow cycle under the above partial edge-coloring.

(2) For \( x \in I \), there exists an \( x \sim Z \sim B_1 \sim v \sim u \sim A_1 \sim X_1 \sim x \)-rainbow cycle under the above partial edge-coloring.

(3) For \( x \in K \), there exists an \( x \sim Y_1 \sim B_1 \sim v \sim u \sim A_1 \sim Z \sim x \)-rainbow cycle under the above partial edge-coloring.

(4) For \( x \in W \), there exists an \( x \sim Y_1 \sim B_1 \sim v \sim u \sim A_1 \sim X_1 \sim x \)-rainbow cycle under the above partial edge-coloring.

**Proof.** We only show (4) since (1), (2) and (3) can be proved similarly. For any \( x \in W \), by the definition of set \( W \), the vertex \( x \) has a neighbor \( v_1 \in X_1 \) and a neighbor \( v_2 \in Y_1 \). Moreover, by the definitions of sets \( X_1 \) and \( Y_1 \), the vertex \( v_1 \) has a neighbor \( v_3 \in A_1 \), and the vertex \( v_2 \) has a neighbor \( v_4 \in B_1 \). Thus \( x \sim v_2 \sim v_4 \sim v_3 \sim v_1 \sim x \) is a rainbow cycle, that is, there exists an \( x \sim Y_1 \sim B_1 \sim v \sim u \sim A_1 \sim X_1 \sim x \)-rainbow cycle under the above partial edge-coloring. ■

**Lemma 7.** For any two vertices \( x, y \in V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J) \), there exists a rainbow path joining \( x \) and \( y \) under the above partial edge-coloring.

**Proof.** Let \( x \) and \( y \) be any two vertices in \( V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J) \). It is easy to see that there exists a rainbow path between \( u \) (respectively \( v \)) and another vertex \( w \in V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J) \) in the partial edge-color graph \( G \). Thus suppose that \( \{u, v\} \cap \{x, y\} = \emptyset \).

Case 1. \( x, y \in A_1 \cup B_1 \cup X_1 \cup Y_1 \cup Z \cup I \cup K \cup W \). By Lemmas 4, 5 and 6, we can pick a special rainbow path \( P_1 \) between \( x \) and \( v \) and a special rainbow path \( P_2 \) between \( y \) and \( v \) such that \( c(P_1) \cap c(P_2) = \emptyset \). Thus we can obtain a rainbow path joining \( x \) and \( y \) by combining the paths \( P_1 \) and \( P_2 \).

Case 2. Exactly one of \( x \) and \( y \) belongs to \( A_1 \cup B_1 \cup X_1 \cup Y_1 \cup Z \cup I \cup K \cup W \). Without loss of generality, say \( x \in A_1 \cup B_1 \cup X_1 \cup Y_1 \cup Z \cup I \cup K \cup W \) and \( y \in A_3 \cup B_3 \cup X_3 \cup X_4 \cup Y_3 \cup Y_4 \). We only check the case \( y \in A_3 \cup X_3 \cup X_4 \) since the case \( y \in B_3 \cup Y_3 \cup Y_4 \) can be checked similarly.

For \( y \in A_3 \cup X_3 \cup X_4 \), there exists a \( y \sim B_4 \sim u \sim v \)-rainbow path \( P_1 \) joining \( y \) and \( v \). Moreover, there exists a \( \{2, 4, 5, 6, 7\} \)-rainbow path \( P_2 \) joining \( x \) and \( v \). Thus a rainbow path joining \( x \) and \( y \) can be obtained from \( P_1 \) and \( P_2 \).

Case 3. \( x \in A_3 \cup X_3 \cup X_4 \) and \( y \in B_3 \cup Y_3 \cup Y_4 \).
Subcase 3.1. $x \in A_3$. There exist an $x \sim u$-rainbow path $P_1$ and an $x \sim A_1 3 \sim u$-rainbow path $P_2$ by Figure 1 and Lemma 3.

If $y \in Y_3 \cup Y_4$, then there exists a $y \sim B_1 2 \sim v 1 \sim u$-rainbow path $P_3$. Thus a rainbow path joining $x$ and $y$ can be obtained from $P_2$ and $P_3$.

If $y \in B_3$, then there exists a $y 2 \sim v 1 \sim u$-rainbow path $P_4$. Thus a rainbow path joining $x$ and $y$ can be obtained from $P_1$ and $P_4$.

Subcase 3.2. $x \in X_3 \cup X_4$. There exists an $x \sim A_1 3 \sim u$-rainbow path $P_1$ by Figure 1. Moreover, there exists a $y \sim B_1 2 \sim v 1 \sim u$-rainbow path $P_2$ if $y \in B_3 \cup Y_3$, or there exists a $y \sim B_1 2 \sim v 1 \sim u$-rainbow path $P_2$ if $y \in Y_4$. Thus a rainbow path joining $x$ and $y$ can be obtained from $P_1$ and $P_2$.

Case 4. $x, y \in A_3 \cup X_3 \cup X_4$ or $x, y \in B_3 \cup Y_3 \cup Y_4$. We only check the case $x, y \in A_3 \cup X_3 \cup X_4$ since the case $x, y \in B_3 \cup Y_3 \cup Y_4$ can be checked similarly.

Subcase 4.1. $x \in A_3$ or $y \in A_3$. Without loss of generality, say $x \in A_3$. Then there exists a $x 2 \sim u 3 \sim A_1 8(9) \sim y$-rainbow path connecting $x$ and $y$.

Subcase 4.2. At least one of $x$ and $y$ belongs to $X_3$. Without loss of generality, assume that $x \in X_3$. Let $x'$ and $y'$ be neighbors of $x$ and $y$ in $A_1$ such that $c(xx') = 8$ and $c(yy') = 9$. By Lemma 5, there exists a $\{2, 4, 5, 6, 7\}$-rainbow path $P$ joining $y'$ and $v$. Thus $yy'Pvux'x$ is a rainbow path connecting $x$ and $y$.

Subcase 4.3. Both $x$ and $y$ belong to $X_4$. Let $x'$ be a neighbor of $x$ in $A_1$, and let $y'$ be a neighbor of $y$ in $X_1$. By Lemma 4, there exists a $\{2, 5, 6, 7\}$-rainbow path $P$ joining $y'$ and $v$. Thus $yy'Pvux'x$ is a rainbow path connecting $x$ and $y$.

3. A Complete Edge-Coloring

To complete our edge-coloring, we further partition $J$ as follows:

- $J_0 = \{ x \in J : x$ is not an isolated vertex in $G[J]$ $\}$,
- $J_1 = \{ x \in J \setminus J_0 : x$ has at least a neighbor in $K$ $\}$,
- $J_2 = \{ x \in J \setminus (J_0 \cup J_1) : x$ has at least a neighbor in $W$ $\}$,
- $J_3 = \{ x \in J \setminus (J_0 \cup J_1 \cup J_2) : x$ has at least a neighbor in $I$ $\}$,
- $J_4 = J \setminus (J_0 \cup J_1 \cup J_2 \cup J_3)$.

Now we further color the edges of $G$ as follows: color the edges in $E[Z, J_1 \cup J_2 \cup J_3]$ by color 7; for any $x \in J_4$, color one in $E[x, Z]$ by 8, color the others in $E[x, Z]$ by 9 (there exists at least one such edge since $G$ is bridgeless).

To color the remaining edges, we need the following lemma.
Figure 2. A complete edge-coloring of $G$ (we omit the line between $Z$ and $J_1$, the line between $Z$ and $J_2$, and the line between $Z$ and $J_3$).

**Lemma 8.** Let $S$ and $T$ be two disjoint vertex sets of a graph $G$ such that $S \subseteq N(T)$. If the induced subgraph $G[S]$ has no trivial components, then there is an $\{\alpha, \beta, \gamma\}$-edge-coloring of $G[S] \cup E[S,T]$ such that there exist two rainbow paths $P_1$ and $P_2$ joining $s$ and $T$ for every $s \in S$. Furthermore, if $P_1$ has color $\{\alpha\}$, then $P_2$ has colors $\{\beta, \gamma\}$; if $P_1$ has color $\{\beta\}$, then $P_2$ has colors $\{\alpha, \gamma\}$.

**Proof.** Let $F$ be a maximal spanning forest of $G[S]$, and let $(X,Y)$ be any of the bipartitions defined by this forest $F$. We give a 3-edge-coloring $c : E(G[S]) \cup E[S,T] \rightarrow \{\alpha, \beta, \gamma\}$ of $G$ by defining

$$c(e) = \begin{cases} 
\alpha, & \text{if } e \in E[T,X]; \\
\beta, & \text{if } e \in E[T,Y]; \\
\gamma, & \text{otherwise}. 
\end{cases}$$

Clearly, for the edge-coloring above, there exist two rainbow paths $P_1$ and $P_2$ joining $s$ and $T$ for every $s \in S$. Furthermore, if $P_1$ has color $\{\alpha\}$, then $P_2$ has colors $\{\beta, \gamma\}$; if $P_2$ has color $\{\beta\}$, then $P_2$ has colors $\{\alpha, \gamma\}$. \qed
Remark. The edge-coloring in Lemma 8 is called an \(\langle \alpha, \beta, \gamma \rangle\)-edge-coloring for \(T\) and \(X \cup Y\). Let \(T_{A_2}, T_{B_2}, T_{X_2}, T_{Y_2}\) and \(T_{J_0}\) be maximal spanning forests of \(G[A_2], G[B_2], G[X_2], G[Y_2]\) and \(G[J_0]\), respectively. Clearly, the forests have no isolated vertex. Let \(A_{02}^1\) and \(A_{12}^1\), \(B_{02}^1\) and \(B_{21}^1\), \(X_{02}^1\) and \(X_{21}^1\), \(Y_{02}^1\) and \(Y_{21}^1\), and \(J_{01}^1\) and \(J_{10}^1\) be bipartitions of \(T_{A_2}, T_{B_2}, T_{X_2}, T_{Y_2}\) and \(T_{J_0}\). Now we give a \((2, 3, 8)\)-edge-coloring for \(u\) and \(A_{02}^1 \cup A_{12}^1\), a \((2, 3, 8)\)-edge-coloring for \(v\) and \(B_{02}^1 \cup B_{21}^1\), an \((8, 9, 7)\)-edge-coloring for \(A_1\) and \(X_{02}^1 \cup X_{21}^1\), an \((8, 9, 7)\)-edge-coloring for \(B_1\) and \(Y_{02}^1 \cup Y_{21}^1\), a \((7, 9, 8)\)-edge-coloring for \(Z\) and \(J_{01}^1 \cup J_{10}^1\) as shown in Figure 2.

Furthermore, we color the edges in subgraphs \(G[A_1], G[X_{02}^1]\) and \(G[X_{21}^1]\) by 4, the edges in subgraphs \(G[B_1], G[Y_{02}^1]\) and \(G[Y_{21}^1]\) by 5, the edges in \(E[X_1, X_2]\) and \(E[Y_1, Y_2]\) by 8, and the edges in \(E[X_1, X_3]\) and \(E[Y_1, Y_3]\) by 9.

For the remaining edges, we can color them arbitrarily. Up to now, we give the graph \(G\) a complete edge-coloring. Let \(\mathcal{P}\) be our final vertex set partition and let \(c\) be our final edge-coloring.

**Lemma 9.** For any two vertices \(x \in A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J\) and \(y \in V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J)\), there exists a rainbow path under the above partial edge-coloring.

**Proof.** We consider the following three cases.

**Case 1.** \(x \in A_2 \cup B_2\). We only consider the case \(x \in A_2\) since the case \(x \in B_2\) can be checked similarly.

**Subcase 1.1.** \(x \in A_{02}^1\). By observing Figure 2, there exist an \(x \sim \sim u\)-rainbow path \(P_1\) joining \(x\) and \(u\), or an \(x \sim \sim A_{12}^1 \sim \sim u\)-rainbow path \(P_2\) joining \(x\) and \(u\).

If \(y \in A_3\), then \(P_2 y\) is a rainbow path joining \(x\) and \(y\).

If \(y \in B_3\), then \(P_1 v y\) is a rainbow path joining \(x\) and \(y\).

If \(y \in B_1 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z \cup I \cup K \cup W\), then there exists a \(\{1, 2, 5, 6, 7, 9\}\)-rainbow path \(Q_1\) joining \(u\) and \(y\). Thus a rainbow path joining \(x\) and \(y\) can be obtained by combining \(P_2\) and \(Q_1\).

If \(y \in A_1 \cup X_1 \cup X_3 \cup X_4\), then there exists a \(\{3, 4, 9\}\)-rainbow path \(Q_2\) joining \(u\) and \(y\). Thus a rainbow path joining \(x\) and \(y\) can be obtained by combining \(P_1\) and \(Q_2\).

**Subcase 1.2.** \(x \in A_{12}^1\). By observing Figure 2, there exist an \(x \sim \sim \sim \sim u\)-rainbow path \(P_1\) joining \(x\) and \(u\), or an \(x \sim \sim \sim \sim A_{02}^1 \sim \sim \sim \sim u\)-rainbow path \(P_2\) joining \(x\) and \(u\).

If \(y \in A_3\), then \(P_1 y\) is a rainbow path joining \(x\) and \(y\).

If \(y \in B_3\), then \(P_2 v y\) is a rainbow path joining \(x\) and \(y\).

If \(y \in B_1 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z \cup I \cup K \cup W\), then there exists a \(\{1, 2, 5, 6, 7, 9\}\)-rainbow path \(Q_1\) joining \(u\) and \(y\). Thus a rainbow path joining \(x\) and \(y\) can be obtained by combining \(P_1\) and \(Q_1\).

If \(y \in A_1 \cup X_1 \cup X_3 \cup X_4\), then there exists a \(\{3, 4, 9\}\)-rainbow path \(Q_2\) joining \(u\) and \(y\). Thus a rainbow path joining \(x\) and \(y\) can be obtained by combining \(P_2\) and \(Q_2\).
Case 2. \( x \in X_2 \cup Y_2 \). We only consider the case \( x \in X_2 \) since the case \( x \in Y_2 \) can be checked similarly.

Subcase 2.1. \( x \in X_2^0 \). By observing Figure 2, there exists an \( x \sim A_1 \sim u \)-rainbow path \( P_1 \) joining \( x \) and \( u \).

If \( y \in A_3 \), then \( P_1 y \) is a rainbow path joining \( x \) and \( y \).

If \( y \in B_3 \), then \( x \sim X_2^1 \sim A_1 \sim u \sim v \sim B_1 \sim y \) is a rainbow path joining \( x \) and \( y \).

If \( y \in B_1 \cup Y_1 \cup Y_2 \cup Z \cup I \cup K \cup W \), then there exists a \( \{1, 2, 5, 6, 7, 9\} \)-rainbow path \( Q_1 \) joining \( v \) and \( y \). Thus a rainbow path joining \( x \) and \( y \) can be obtained by combining \( P_1 \) and \( Q_1 \).

If \( y \in B_1 \cup X_1 \), then there exists a \( \{2, 4, 5, 6, 7\} \)-rainbow path \( Q_1 \) joining \( v \) and \( y \) by Lemmas 4 and 5. Thus a \( \{1, 2, 3, 4, 5, 6, 7, 8\} \)-rainbow path joining \( x \) and \( y \) can be obtained by combining \( P_1 \), \( Q_1 \) and edge \( uv \).

If \( y \in X_3 \cup X_4 \), then \( y \) has a neighbor \( y' \) in \( A_1 \) such that \( c(yy') = 9 \). Note that there exists a \( \{1, 2, 3, 4, 5, 6, 7, 8\} \)-rainbow path \( P \) joining \( x \) and \( y' \) by the arguments of the above paragraph. Thus \( Py \) is a rainbow path joining \( x \) and \( y \).

Subcase 2.2. \( x \in X_2^1 \). By observing Figure 2, there exist an \( x \sim A_1 \sim u \)-rainbow path \( P_1 \) joining \( x \) and \( u \).

If \( y \in A_3 \), then \( P_1 y \) is a rainbow path joining \( x \) and \( y \).

If \( y \in B_3 \), then \( P_1 vy' y \) is a rainbow path joining \( x \) and \( y \), where \( y' \) is a neighbor of \( y \) in \( B_1 \).

If \( y \in B_1 \cup Y_1 \cup Y_2 \cup Z \cup I \cup K \cup W \), then there exists a \( \{1, 2, 5, 6, 7, 8\} \)-rainbow path \( Q_1 \) joining \( u \) and \( y \). Thus a rainbow path joining \( x \) and \( y \) can be obtained by combining \( P_1 \) and \( Q_1 \).

If \( y \in A_1 \cup X_1 \), then there exists a \( \{2, 4, 5, 6, 7\} \)-rainbow path \( Q_1 \) joining \( v \) and \( y \) by Lemmas 4 and 5. Thus a \( \{1, 2, 3, 4, 5, 6, 7, 9\} \)-rainbow path joining \( x \) and \( y \) can be obtained by combining \( P_1 \), \( Q_1 \) and edge \( uv \).

If \( y \in X_3 \cup X_4 \), then \( y \) has a neighbor \( y' \) in \( A_1 \) or \( X_1 \) such that \( c(yy') = 8 \). Note that there exists a \( \{1, 2, 3, 4, 5, 6, 7, 9\} \)-rainbow path \( P \) joining \( x \) and \( y' \) by the arguments of the above paragraph. Thus \( Py \) is a rainbow path joining \( x \) and \( y \).

Case 3. \( x \in J \). By observing Figure 2, there exists a \( \{7, 9\} \)-rainbow path \( P \) joining \( x \) and some vertex \( z \in Z \). Furthermore, there exist a \( z \sim A_1 \sim u \)-rainbow path \( Q_1 \) joining \( z \) and \( v \), and a \( z \sim B_1 \sim v \)-rainbow path \( Q_2 \) joining \( z \) and \( u \). Thus a \( \{1, 3, 4, 7, 9\} \)-rainbow path \( Q'_1 \) joining \( x \) and \( v \) can be obtained from \( P \) and \( Q_1 \), and a \( \{1, 2, 5, 7, 9\} \)-rainbow path \( Q'_2 \) joining \( x \) and \( u \) can be obtained from \( P \) and \( Q_2 \).

If \( y \in B_1 \cup B_3 \cup Y_1 \cup Y_2 \cup Y_4 \), then there exists a \( \{2, 5, 8\} \)-rainbow path \( R_1 \) between \( v \) and \( y \). Thus a rainbow path joining \( x \) and \( y \) can be obtained from \( Q'_1 \) and \( R_1 \).
If $y \in A_1 \cup A_3 \cup X_1 \cup X_3 \cup X_4 \cup Z \cup I \cup K \cup W$, then there exists a $\{3, 4, 6, 8\}$-rainbow path $R_2$ between $u$ and $y$. Thus a rainbow path joining $x$ and $y$ can be obtained from $Q'_2$ and $R_2$.

4. **9-Rainbow-Connected Edge-Coloring**

In this section, we check that the above 9-edge-coloring is rainbow-connected 9-edge-coloring. It suffices to check that for any two vertices $x, y \in A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J$, there exists a rainbow path under the above partial edge-coloring.

**Lemma 10.** There exists a rainbow path joining any two vertices of $X_2$ under the edge-coloring $c$.

**Proof.** Let $x$ and $y$ be any two vertices in $X_2$. We consider the following two cases.

Case 1. $x \in X_2^0$ and $y \in X_2^1$, or $x \in X_2^0$ and $y \in X_2^1$. Without loss of generality, assume that $x \in X_2^0$ and $y \in X_2^1$. Let $x'$ and $y'$ be neighbors of $x$ and $y$ in $A_1$, respectively. By Figure 2, we know that $c(xx') = 8$ and $c(yy') = 9$. By Lemma 5, there exists a $\{2, 4, 5, 6, 7\}$-rainbow path $P'_{y',u}$. Thus, a $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$-rainbow path joining $x$ and $y$ is obtained from the edge $yy'$, rainbow paths $P'_{y',v}$ and $vux'x$.

Case 2. $x, y \in X_2^0$ or $x, y \in X_2^1$. We only check the case $x, y \in X_2^0$ since the case $x, y \in X_2^1$ can be checked similarly.

Subcase 2.1. $d(x, B_1) = d(y, B_1) = 2$. Without loss of generality, assume $d(x, B_1) = 2$. Let $x' \in N(x) \cap N(B_1)$. By the definition of the above set partition, we know $x' \in A_1$. So, $xx'x''vu$ is a $\{1, 2, 6, 8\}$-rainbow path, where $x''$ is a neighbor of $x'$ in $B_1$. By Figure 2 and Lemma 8, $u$ and $y$ are connected by a $\{3, 7, 9\}$-rainbow path $P$. Thus a $\{1, 2, 3, 6, 7, 8, 9\}$-rainbow path joining $x$ and $y$ can be obtained from rainbow paths $xx'x''vu$ and $P$.

Subcase 2.2. $d(x, B_1) = d(y, B_1) = 3$. Let $xx_1x_2x_3$ be a path joining $x$ and some vertex $x_3 \in B_1$. By the set partition above, $x_1 \in A_1 \cup X_1 \cup X_2$.

Subsubcase 2.2.1. $x_1 \in A_1$. By the definition of $\mathcal{P}$, $x_2 \in A_1 \cup B_1 \cup Z$. So $xx_1x_2x_3$ is a $\{4, 5, 6\}$-rainbow path. Furthermore, $xx_1x_2x_3vu$ is $\{1, 2, 4, 5, 6, 8\}$-rainbow. By Figure 2, there exists a $\{3, 7, 9\}$-rainbow path $P$ joining $u$ and $y$. Hence a rainbow path joining $x$ and $y$ can be obtained from rainbow paths $xx_1x_2x_3vu$ and $P$.

Subsubcase 2.2.2. $x_1 \in X_1$. By the definition of the above set partition, $x_2 \in A_1 \cup Z$. Thus $xx_1x_2x_3$ is a $\{4, 5, 6, 9\}$-rainbow path. Thus $xx_1x_2x_3vu'y$ is a $\{1, 2, 3, 4, 5, 6, 8, 9\}$-rainbow path joining $x$ and $y$, where $y_1$ is a neighbor of $y$ in $A_1$. 

Subsubcase 2.2.3. $x_1 \in X_2$. If $x_1 \in X_0^3$, then $c(xx_1) = 4$. Furthermore, $x_2 \in A_1$. Thus $xx_1x_2x_3vu$ is a $\{1, 2, 4, 6, 8\}$-rainbow path. By Figure 2, there exists a $\{3, 7, 9\}$-rainbow path $P$ joining $u$ and $y$. Hence a rainbow path joining $x$ and $y$ can be obtained from $xx_1x_2x_3vu$ and $P$.

If $x_1 \in X_1^3$, then $c(xx_1) = 7$. Furthermore, $x_2 \in A_1$. Thus $xx_1x_2x_3vu$ is a $\{1, 2, 4, 6, 7, 9\}$-rainbow path. By Figure 2, there exists a $\{3, 8\}$-rainbow path $P$ joining $u$ and $y$. Hence a rainbow path joining $x$ and $y$ can be obtained from $xx_1x_2x_3vu$ and $P$.

Similarly to Lemma 8, the following lemma holds.

**Lemma 11.** There exists a rainbow path joining any two vertices of $Y_2$ under the edge-coloring above.

**Lemma 12.** For any two vertices $x, y \in A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J$, there exists a rainbow path under the above partial edge-coloring.

**Proof.** For $x, y \in X_2$ or $x, y \in Y_2$, there exists a rainbow path joining $x$ and $y$ by Lemmas 10 or 11. For the others, we can easily check them by Lemmas 4, 5, 6 and 8 in a similar way.

Combining Lemmas 7, 9 and 12, we have the following result.

**Theorem 13.** Let $G$ be a bridgeless graph with diameter 3. If there exists an edge $e$ such that $e$ does not belong to any triangle in $G$, then $rc(G) \leq 9$.

For a bridgeless graph $G$ with diameter 3, if each edge belongs to a triangle in $G$, then $rc(G) \leq 9$ by Theorem 1. Combining this result with Theorem 13, we know that Theorem 2 holds.

We can give the following example of graphs with diameter 3 for which the rainbow connection number reaches 7.

**Example 2.** Let $K_n$ be a complete graph with vertex set $\{v_1, \ldots, v_n\}$, where $n \geq 217$. For every $v_i$, we add a pendant path $\langle v_i, v_{i1}, v_{i2}, v_{i3}\rangle$, denoted by $P_i$, and then we identify the vertex $v_{i3}$ with a vertex $v$. The resulting graph is denoted by $G$. Clearly, $diam(G) = 3$. Let $c$ be any 6-edge-coloring of $G$ with colors $\{1, \ldots, 6\}$. Since $6^3 = 216$, at least two of them are colored the same. Without loss generality, say $P_1$ and $P_2$, that is, $c(v_1v_{11}) = c(v_2v_{21}), c(v_{11}v_{12}) = c(v_{21}v_{22})$ and $c(v_{12}v) = c(v_{22}v)$. By the structure of $G$, it is easy to see that there exists no rainbow path joining $v_{11}$ and $v_{21}$ in $G$ under $c$. Thus $rc(G) \geq 7$.

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