ALMOST SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

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Abstract

It is known that self-complementary 3-uniform hypergraphs on \( n \) vertices exist if and only if \( n \) is congruent to 0, 1 or 2 modulo 4. In this paper we define an almost self-complementary 3-uniform hypergraph on \( n \) vertices and prove that it exists if and only if \( n \) is congruent to 3 modulo 4. The structure of corresponding complementing permutation is also analyzed. Further, we prove that there does not exist a regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4, and it is proved that there exist a quasi regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4.

Keywords: uniform hypergraph, self-complementary hypergraph, almost complete 3-uniform hypergraph, almost self-complementary hypergraph, quasi regular hypergraph.

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1. Introduction

The study of self-complementary graphs was initiated by Sachs and Ringel, independently. Ringel [11] and Sachs [12] have both proved the results concerning order and cycle structure of a complementing permutation of self-complementary graphs. Das [3] introduced the concept of almost self-complementary graphs which is similar to graphs self-complementary in $K_n - e$ introduced by Clapham [1]. They proved similar results on order and cycle structure of complementing permutation of almost self-complementary graphs. Kocay [9] extended the results of self-complementary graphs to self-complementary 3-uniform hypergraphs. He has analysed the cycle structure of complementing permutation of self-complementary 3-uniform hypergraphs. Szymański and Wojda [13] have characterized $n$ and $k$ for which there exist $k$-uniform self-complementary hypergraphs and gave the structure of corresponding self-complementing permutations. Gosselin [5] has characterized all $n$ and $k$ for which there exists a regular $k$-uniform self-complementary hypergraph of order $n$.

Potočnik and Šajana [10] raised the following question strengthening Hartman’s conjecture [2, 6] about existence of large sets of (not necessarily isomorphic) designs.

**Question** [10]. Is it true that for every triple of integers $t < k < n$ such that $\binom{n-i}{k-i}$ is even for all $i = 0, \ldots, t$, there exists a self-complementary $t$-subset-regular $k$-uniform hypergraph of order $n$?

The answer to the above question is affirmative for $k = 2$ and $t = 1$ (see [12]). The answer was proved affirmative also for the case $k = 3$ and $t = 1$ (see [10]). And in [8] it is shown that the answer to the question above is affirmative for the remaining case of 3-uniform hypergraph, namely for the case $k = 3$, $t = 2$.

It is clear that if the number of triples in the complete design $(K^3_n)$ is odd, then there does not exist a self-complementary $t$-subset-regular 3-uniform hypergraph of order $n$. In this case one may modify the problem by “Does there exist a partition of $K^3_n - e$ into two isomorphic $t$-subset-regular 3-uniform hypergraphs of order $n”? Das and Rosa [4] proved that there exists a partition of Steiner triple system (STS) into two isomorphic 3-uniform hypergraphs of order $n$, if $n \equiv 3 \text{ or } 7 \pmod{12}$. In this paper we prove that there does not exist a partition of $K^3_n - e$ into two isomorphic 1-subset-regular 3-uniform hypergraphs of order $n$, if $n \equiv 3 \pmod{4}$ and some partial answers to the above question are given.

In Section 2, we define almost self-complementary 3-uniform hypergraph on $n$ vertices and further prove that such a hypergraph exists if and only if $n$ is congruent to 3 modulo 4.

In Section 3, the structure of a complementing permutation of such an almost self-complementary 3-uniform hypergraph is analyzed.
In Section 4, we prove that there does not exist a regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4. Further, we prove that there exists a quasi regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4.

2. Necessary and Sufficient Condition for Existence of Almost Self-Complementary 3-Uniform Hypergraph

Suppose \( H \) is a 3-uniform hypergraph with vertex set \( V \) and edge set \( E \). A partition of \( E = \bigcup_{i=1}^{s} E_i \) is called a factorization of \( H \) and the 3-uniform hypergraph \( H_i(V, E_i) \) is called a factor of \( H \) for \( i = 1, 2, \ldots, s \). A factorization in which all factors are isomorphic is called an isomorphic factorization.

A factor in a factorization of complete 3-uniform hypergraph \( K_3^3 \) with only two isomorphic factors is nothing but a self complementary 3-uniform hypergraph. A partitioning of the edge set of \( K_3^3 \) into two isomorphic factors is not possible when \( K_3^3 \) has an odd number of edges, i.e., when \( n \) is congruent to 3 modulo 4. However, after deleting some odd number of edges from \( K_3^3 \) the remaining 3-uniform hypergraph may be partitioned into two isomorphic factors.

In this paper we delete one edge from \( K_3^3 \) and define an almost self-complementary 3-uniform hypergraph. We always denote by \( e \) the edge deleted from \( K_3^3 \), call it the missing edge and the corresponding vertices of \( e \) the special vertices.

**Definition.** The hypergraph \( \tilde{K}_3^3 = K_3^3 - e \) is called an almost complete 3-uniform hypergraph.

**Definition.** A 3-uniform hypergraph \( H \) with \( n \) vertices is almost self-complementary if it is isomorphic with its complement \( \tilde{H} \) with respect to \( \tilde{K}_3^3 \).

This means that a 3-uniform hypergraph \( H \) with \( n \) vertices is almost self-complementary if \( K_3^3 \) can be decomposed into two isomorphic factors with \( H \) as one factor.

Since \( K_3^3 \) has \( \binom{n}{3} - 1 \) edges, such a factorization is possible only if this number is divisible by 2. Thus it is necessary that \( n \equiv 3 \pmod{4} \). We can compare this with the fact that isomorphic factorizations of \( K_3^3 \), into 2 factors, i.e., self-complementary 3-uniform hypergraphs, exist only if \( n \equiv 0, 1 \) or 2 (mod 4). Almost self-complementary 3-uniform hypergraphs in a sense fill the gap where self-complementary 3-uniform hypergraphs do not exist.

Following theorem gives a necessary and sufficient condition on the order of an almost self-complementary 3-uniform hypergraph.

**Theorem 1.** There exists an almost self-complementary 3-uniform hypergraph on \( n \) vertices if and only if \( n \equiv 3 \pmod{4} \).
Figure 1. The types of triples making up the edge set of an almost self-complementary 3-uniform hypergraph on \( n = 4m + 3 \) vertices.

**Proof.** Necessity is obvious from the above discussions. To prove sufficiency, we construct a 3-uniform hypergraph which is self-complementary in \( \tilde{K}_3^n \) on \( n \) vertices with \( n \equiv 3 \) (mod 4). Denote the missing edge by \( e = \{x, y, z\} \).

Let \( m \) be a positive integer such that \( n = 4m + 3 \) and \( V = V_0 \cup V_1 \cup V_2 \cup V_3 \cup \{x, y, z\} \), where \( V_i = \{v^i_j : j \in \mathbb{Z}_m\} \) for all \( i \in \mathbb{Z}_4 \).

For pairwise distinct \( i, i', i'' \in \mathbb{Z}_4 \) we consider the following partition of edges of \( K^3_n \):

- \( E_i = V_i^{(3)} = \) all 3-subsets of \( V_i \),
- \( E_{i,i'} = \{\{v^i_{j1}, v^{i'}_{j2}, v^{i''}_{j''}\} : j_1, j_2, j'' \in \mathbb{Z}_m, j_1 \neq j_2\} \),
- \( E_{i,j,j'} = \{\{v^i_j, v^{i'}_{j''}\} : j, j' \in \mathbb{Z}_m\} \),
- \( E_x = \{\{x, v^i_{j1}, v^{i'}_{j2}\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\} \),
- \( E_y = \{\{y, v^i_{j1}, v^{i'}_{j2}\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\} \),
- \( E_z = \{\{z, v^i_{j1}, v^{i'}_{j2}\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\} \),
- \( E_x^{i,i'} = \{\{x, v^i_j, v^{i'}_{j''}\} : j, j'' \in \mathbb{Z}_m\} \),
- \( E_y^{i,i'} = \{\{y, v^i_j, v^{i'}_{j''}\} : j, j'' \in \mathbb{Z}_m\} \),
- \( E_z^{i,i'} = \{\{z, v^i_j, v^{i'}_{j''}\} : j, j'' \in \mathbb{Z}_m\} \).
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$$E_i^{(x,y)} = \{\{x, y, v^i_j\} : j \in \mathbb{Z}_m\},$$
$$E_i^{(x,z)} = \{\{x, z, v^i_j\} : j \in \mathbb{Z}_m\},$$
$$E_i^{(y,z)} = \{\{y, z, v^i_j\} : j \in \mathbb{Z}_m\}.$$

Let

$$E = \bigcup_{k=0}^{1} \left( E_k \cup E_{2,k} \cup E_{3,k} \cup E_{k,1,2,3} \cup E_{k}^x \cup E_k^y \cup E_k^z \cup E_{k}^{(x,y)} \cup E_{k}^{(x,z)} \cup E_{k}^{(y,z)} \right)$$
$$\cup E_{0,1} \cup E_{1,0} \cup E_{0,1}^x \cup E_{0,1}^y \cup E_{0,3} \cup \bigcup_{k=1,2,3} \left( E_{k}^y \cup E_{k}^z \right).$$

Let $H$ be the 3-uniform hypergraph with vertex set $V$ and edge set $E$. Figure 1 gives a diagrammatic construction of $H$.

To prove that $H$ is almost self-complementary, we define a permutation $\phi : V \to V$ by $\phi(x) = x, \phi(y) = y, \phi(z) = z, \phi(v^0_j) = v^3_j, \phi(v^1_j) = v^1_j, \phi(v^2_j) = v^2_j$, and $\phi(v^3_j) = v^0_j$, for all $j \in \mathbb{Z}_m$. It is checked easily that $\phi$ is a complementing permutation of $H$ and therefore $H$ is almost self-complementary.

3. The Complementing Permutation

It is known (see [9]) that for a 3-uniform self-complementary (s.c.) hypergraph, if $\tau$ is a complementing permutation of the vertices that maps $H$ onto its complement $\bar{H}$, then

(i) every cycle of $\tau$ has even length, or
(ii) $\tau$ has 1 or 2 fixed points, and the length of all other cycles is a multiple of 4.

We prove similar results for the complementing permutation of an almost self-complementary 3-uniform hypergraph.

Given an almost self-complementary 3-uniform hypergraph $H$, let the edges of $H$ be coloured red and the remaining edges of $K^3_n$ be coloured green. Since the 2 factors are isomorphic, there is a permutation $\tau$ of the vertices of $K^3_n$ that induces a mapping of the red edges onto the green edges. We consider $\tau$ as a permutation of the vertices of $K^3_n$, and denote by $\tau'$ the corresponding mapping induced on the set of edges of $K^3_n$. Thus $\tau'$ maps each red edge onto a green edge. However, the mapping $\tau'$ need not necessarily map each green edge onto a red edge. This would be so if $\tau'$ mapped $e$ onto itself, but it may be that $\tau'$ maps $e$ onto a red edge and some green edge onto $e$. Such a $\tau$ (which, for definiteness we shall always assume induces a mapping from red to green) will (as for s.c. 3-uniform hypergraphs) be called a complementing permutation. It will be useful to consider the cycles of the induced mapping $\tau'$. 
The following remarks regarding the cycles of induced mapping $\tau'$ will be used to prove a number of results about the structure of complementing permutation $\tau$.

**Remark 2.** A cycle of $\tau'$ that does not include $e$ must be of even length, consisting of edges alternately red and green.

**Remark 3.** The cycle of $\tau'$ that includes $e$ has odd length, consisting of $e$ followed by red and green edges alternately. Further, this length equals 1 when $\tau'$ maps $e$ onto itself.

**Lemma 4.** If the special vertices $x, y, z$ occur in different cycles of $\tau$, then they must be three fixed points $(x)(y)(z)$.

**Proof.** Suppose that $x$ occurs in a cycle of length $L_1$, $y$ occurs in a cycle of length $L_2$ and $z$ occurs in a cycle of length $L_3$.

Consider the cycle of $\tau'$ including $e$. The number of edges in this cycle is the least common multiple of $L_1, L_2$ and $L_3$. By Remark 3 this number must be odd. If $L_1 \geq 3$ then any triple $\{i, j, k\}$ of this cycle gives rise to a sequence of triples $\{i, j, k\}, \tau\{i, j, k\}, \tau^2\{i, j, k\}, \ldots$ etc. These must be alternately edges of $H$ and $\overline{H}$. This is possible only if $L_1$ is even. Hence $L_1 = 1$. Similarly $L_2 = L_3 = 1$.

**Lemma 5.** If all special vertices $x, y, z$ occur in the same cycle $C$ of $\tau$, then $C$ has length 3.

**Proof.** Suppose all the special vertices $x, y, z$ occur in a cycle $C$ of length $L$. Consideration of the cycle of $\tau'$ including $e$ shows that $L$ must be odd. If $L > 3$, one finds that there is another cycle of $\tau'$ not including $e$, of odd length, which is a contradiction to Remark 2. Thus $L = 3$.

**Lemma 6.** If any two of the special vertices, say $x, y$, occur in the same cycle $C_1$ of $\tau$, then $C_1$ has length $4h + 2, h \geq 0$, with $\tau^{2h+1}(x) = y$ and special vertex $z$ fixed.

**Proof.** Suppose special vertices $x, y$ occur in the same cycle $C_1$ of length $L_1$. Let the remaining special vertex $z$ occur in a cycle $C_2$ of length $L_2$. Clearly $L_1 \geq 2$ and $L_2 \geq 1$. Since $L_1 \geq 2$, it must be even as argued in Lemma 4. If $L_2 > 1$ we get a contradiction to Remark 3. Hence $L_2 = 1$.

Let $\tau^m(x) = y$, therefore $\tau^{L_1-m}(y) = x$. Consider the sequence of triples $\{x, y, z\}, \tau\{x, y, z\}, \ldots$. This cycle has length either $L_1$ or $m$ if $m = \frac{L_1}{2}$ and it must be odd. Since $L_1$ is even the only possibility is that the length is $m$ and $m$ is odd. Hence $L_1 = 2m = 4h + 2, h \geq 0$.

**Lemma 7.** The cycles of $\tau$ that do not include the special vertices are of length multiple of 4.
**Proof.** If a cycle \((u_1 u_2 \cdots u_L)\) does not involve the special vertices, then by Remark 2 \(L\) is even. We get the following cases depending on occurrence of special vertices in \(\tau\).

*Case (i)* Special vertices \(x, y, z\) are fixed points.

*Case (ii)* Special vertices \(x, y, z\) occur in the cycle of length 3, say \(C' = (x, y, z)\).

*Case (iii)* Two special vertices say \(x, y\) occur in the same cycle and \(z\) is fixed.

In all these cases consider the cycle of \(\tau'\) including the edge \(\{u_1, u_{(L+1)/2}, z\}\) which is of length \(L/2\). From Remark 2 we find that \(L/2\) must be even. Thus \(L\) must be a multiple of 4.

Complete description of complementing permutation of almost self-complementary 3-uniform hypergraph is given below.

**Theorem 8.** Let \(\tau\) be a complementing permutation of an almost self-complementary 3-uniform hypergraph on \(n \geq 3\) vertices and \(e = \{x, y, z\}\) be the deleted edge. Then \(n \equiv 3 \pmod{4}\). Further

(a) \(\tau\) consists of 3 fixed special vertices and all other cycles of length multiple of 4, or

(b) \(\tau\) consists of a cycle of length \(4h + 2\), \(h \geq 0\), including two special vertices \(x\) and \(y\) with \(\tau^{2h+1}(x) = y\), one fixed special vertex \(z\) and all other cycles of length multiple of 4, or

(c) \(\tau\) consists of the cycle \((x, y, z)\) and all other cycles are of length multiple of 4.

4. **Regular and Quasi Regular Almost Self-Complementary 3-Uniform Hypergraph**

It is known that (see [10]) a regular self-complementary 3-uniform hypergraph on \(n\) vertices exists if and only if \(n \geq 5\) and \(n\) is congruent to 1 or 2 modulo 4. In the next theorem we prove that there does not exist a regular almost self-complementary 3-uniform hypergraph on \(n\) vertices where \(n\) is congruent to 3 modulo 4.

**Theorem 9.** There does not exist a regular almost self-complementary 3-uniform hypergraph on \(n\) vertices where \(n\) is congruent to 3 modulo 4.

**Proof.** Suppose there exists a regular almost self-complementary 3-uniform hypergraph, say \(H\) of regular degree \(r\). Then the total number of edges in \(H\) is

\[
\frac{1}{2} \left( \binom{n}{3} - 1 \right) = \frac{n(n-1)(n-2)-6}{12}.\]

Since \(H\) is regular we get \(rn = 3 \times \text{number of edges}\)
in \( H \), i.e., \( rn = \frac{3(n(n-1)(n-2)-6)}{12} \). Hence \( r = \frac{n(n-1)(n-2)-6}{4n} \) which is not an integer for any \( n \) congruent to 3 modulo 4, a contradiction. Hence, there does not exist a regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4.

A hypergraph \( H \) is said to be quasi regular if the degree of every vertex is either \( r \) or \( r - 1 \) for some positive integer \( r \). In [7], it is proved that there exists a quasi regular self-complementary 3-uniform hypergraph on \( n \) vertices if and only if \( n \) is congruent to 0 modulo 4. In the following theorem we prove that there exists a quasi regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4.

**Theorem 10.** There exists a quasi regular almost self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \) is congruent to 3 modulo 4.

**Proof.** \( H \) constructed in Theorem 1 is already shown to be almost self-complementary 3-uniform hypergraph. We show that \( H \) is quasi regular. Considering the same notation as in proof of Theorem 1, take any vertex \( v_j \).

**Case (i)** If \( i \in \{0,1\} \) then, for fixed \( i', i'' \in \mathbb{Z}_4 \) distinct from \( i \), the vertex \( v_j \) lies in \( \binom{n-1}{2} \) triples of \( E_i \), \( 3 \binom{m}{2} \) triples of \( E_{i,i'} \), \( (m-1)m \) triples of \( E_{i,i,i''} \), \( m^2 \) triples of \( E_{i,i',i''} \), \( (m-1) \) triples of each \( E_{i,i,i'}^x \), \( E_{i,i}^y \), \( E_{i,i}^z \), \( 4m \) triples of \( E_{i,i,i'}^{x,y} \), \( E_{i,i,i'}^{y,z} \) and \( 1 \) triple of each \( E_{i,i,i'}^{x,y} \), \( E_{i,i,i'}^{x,z} \), \( E_{i,i,i'}^{y,z} \). Hence, for every vertex \( v_j \) in \( H \) with \( i \in \{0,1\} \), we have

\[
\deg(v_j) = \left( \frac{m-1}{2} \right) + 3 \left( \frac{m}{2} \right) + m(m-1) + m^2 + 3(m-1) + 4m + 3 = 4m^2 + 3m + 1.
\]

**Case (ii)** If \( i \in \{2,3\} \) then the vertex \( v_j \) lies in \( 2(m-1)m \) triples of \( E_{i,i'} \), \( 2m^2 \) triples of \( E_{i,i',i''} \), \( 5m \) triples of \( E_{i,i,i'}^x \), \( E_{i,i,i'}^y \), \( E_{i,i,i'}^z \). Hence, for every vertex \( v_j \) in \( H \) with \( i \in \{2,3\} \), we obtain

\[
\deg(v_j) = 2(m-1)m + 2m^2 + 5m = 4m^2 + 3m.
\]

**Case (iii)** \( x \) lies in \( \binom{m}{2} \) triples of \( E_i^x \), \( m \) triples of each \( E_i^{(x,y)} \), \( E_i^{(x,z)} \) and \( 3m^2 \) triples of \( E_{i,i'}^{x,y} \). Hence

\[
\deg(x) = 2 \left( \frac{m}{2} \right) + 4m + 3m^2 = 4m^2 + 3m.
\]

\( y \) lies in \( \binom{m}{2} \) triples of \( E_i^y \), \( 4m \) triples of \( E_i^{(x,y)} \), \( E_i^{(y,z)} \) and \( 3m^2 \) triples of \( E_{i,i'}^{y,z} \). Hence

\[
\deg(y) = 2 \left( \frac{m}{2} \right) + 4m + 3m^2 = 4m^2 + 3m.
\]
Similarly, 
\[
\deg(z) = 2 \binom{m}{2} + 4m + 3m^2 = 4m^2 + 3m.
\]

Hence, \( H \) is quasi regular with degrees \( r = 4m^2+3m+1 \) and \( r-1 = 4m^2+3m \).

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**References**


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