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ON THE *H*-FORCE NUMBER OF HAMILTONIAN GRAPHS AND CYCLE EXTENDABILITY

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Abstract

The H-force number $h(G)$ of a hamiltonian graph G is the smallest cardinality of a set $A \subseteq V(G)$ such that each cycle containing all vertices of A is hamiltonian. In this paper a lower and an upper bound of $h(G)$ is given. Such graphs, for which $h(G)$ assumes the lower bound are characterized by a cycle extendability property. The H-force number of hamiltonian graphs which are exactly 2-connected can be calculated by a decomposition formula.

Keywords: cycle, hamiltonian graph, H-force number, cycle extendability.

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1. Introduction

Throughout this paper, only finite graphs without loops or multiple edges are considered. The number of vertices of a graph G, i.e., its *order* will be denoted by n. We use the standard graph terminology according to [3].

Let G be a hamiltonian graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A nonempty vertex set $X \subseteq V(G)$ is called a *hamiltonian cycle enforcing set* (for short, *H-force set*) of G if every X-cycle of G (i.e., a cycle of G containing all vertices of X) is a hamiltonian one. Let $h(G)$ denote the smallest cardinality of an H-force set of G and call it the *H-force number* of G. The concepts of H-force set and H-force number were first given by Fabrici *et al.* (see [4]) and studied there for several special families of hamiltonian graphs. Timková (see [9]) determined the H-force number of generalized dodecahedral graphs. Note also, that the concepts of H-force set and H-force number were extended to hamiltonian digraphs and hypertournaments in [10] and [7], respectively.

The authors in [4] observed that the H-force number $h(G)$ of a hamiltonian graph G satisfies

- $h(G) = 1$ if and only if G is a cycle,
- $h(G) = n$ if and only if G is 1-*hamiltonian* (that is, if G is hamiltonian and $G - v$ is hamiltonian for every $v \in V$).

For a hamiltonian graph G, we define sets $S = S(G) = \{x \in V \mid G - x$ is hamiltonian$\}$ and $T = T(G) = \{x \in V \mid G - x$ is 2-connected$\}$. Then, we have $S \subseteq T$. Let $s(G) = |S(G)|$ and $t(G) = |T(G)|$.

Proposition 1. *Let G be a hamiltonian graph and P be a path of G containing no branch vertex of G, i.e., no vertex of degree at least 3 in G. Then, every smallest H-force set $F \subseteq V(G)$ contains at most one vertex of P.*

Let $\mathcal{H}$ be the family of hamiltonian graphs that do not contain adjacent vertices of degree 2. Also, let G' be the graph formed from a hamiltonian graph G by replacing each maximal path not containing a branch vertex by a single vertex. Then, G' is hamiltonian and has no adjacent vertices of degree 2, so $G' \in \mathcal{H}$. Because $h(G') = h(G)$, it is sufficient to restrict our study to the family $\mathcal{H}$.

The main results of this paper are Theorems 2, 7, 8 and 11. Theorem 2 shows that $s(G)$ and $t(G)$ form bounds for the H-force number $h(G)$. After this theorem, we discuss some consequences. Theorem 7 contains a decomposition formula for the H-force number of hamiltonian graphs which are exactly 2-connected. In Theorem 8 hamiltonian graphs G for which $S(G)$ is an H-force set are characterized by a cycle extendability property. Eventually, a sum formula for hamiltonian graphs G with $s(G) < h(G)$ is proved in Theorem 11.

2. Results and Proofs

Theorem 2. *Let $G \in \mathcal{H}$. Then*

$$s(G) \leq h(G) \leq t(G).$$

The proof of this theorem requires the following exchange property.

Lemma 3. *Let $G \in \mathcal{H}$ and let $F \subseteq V$ be a smallest H-force set of G. Then, for every vertex $v \in F \setminus T$ there exists a vertex $u \in T$ such that $(F \setminus \{v\}) \cup \{u\}$ is an H-force set of G.*

Proof. Suppose there exists a vertex $v \in V \setminus T$. Then G is exactly 2-connected. Let C be any fixed hamiltonian cycle of G and w be a cut-vertex of $G - v$. Then, C consists of two v-w-paths P_1 and P_2 both of which have at least one inner vertex but no inner vertex in common. Since G is not a cycle, C has a chord.

But, there is no chord connecting an inner vertex of P_1 with an inner vertex of P_2. Let $F \subseteq V$ be a smallest H-force set of G (i.e., $|F| = h(G)$) and suppose $v \in F$.

Case 1. The cut-vertex w of $G - v$ can be chosen so that each P_i, for $i = 1, 2$, has a chord of C, say $x_i y_i$. Then, the subpath (x_i, y_i) of P_i contains an inner vertex z_i such that $z_i \in F$. Otherwise, the x_i-y_i-path on C which passes v forms together with $x_i y_i$ a non-hamiltonian F-cycle. By the choice of F, $F \setminus \{v\}$ is not an H-force set of G, i.e., G contains a non-hamiltonian $(F \setminus \{v\})$-cycle C' not passing v. Since z_1 and z_2 belong to different components of $G - \{v, w\}$ and since w is a cut-vertex of $G - v$, every z_1-z_2-path of $G - v$ is passing w which contradicts the fact that C' is a cycle.

Case 2. By any choice of the cut-vertex w of $G - v$ only one of P_1 and P_2 has a chord. Suppose for a fixed w that P_1 has no chord. Then P_1 has only one inner vertex u where $d_G(u) = 2$. Since every hamiltonian cycle of G passes the edge uv, $F' := (F \setminus \{v\}) \cup \{u\}$ is also an H-force set of G. Moreover, we have $u \in T$ because otherwise there exists a cut-vertex z of $G - u$ which is also a cut-vertex of $G - v$. Hence, C consists of two v-z-paths (with no common inner vertices) such that both of them have at least one chord, a contradiction. That proves the assertion. ■

***Proof of Theorem* 2.** Let $F \subseteq V$ be any smallest H-force set of G. Suppose that S contains a vertex x such that $x \notin F$. A hamiltonian cycle C of $G - x$ is, obviously, a non-hamiltonian F-cycle of G. That is a contradiction and proves $S \subseteq F$ and, consequently, $s(G) \leq h(G)$.

Let $F \subseteq V$ be a smallest H-force set of G. If $F \subseteq T$ then $h(G) \leq t(G)$ trivially holds. Otherwise, there exists an $x \in F \setminus T$. By Lemma 3 there is a $y \in T$ such that $(F \setminus \{x\}) \cup \{y\}$ is an H-force set of G, too. The repeated use of the above exchange property finally yields a smallest H-force set $F' \subseteq T$ and proves the upper bound. ■

From the proof of Theorem 2, we have $S \subseteq F$ and we can choose F such that $F \subseteq T$.

Corollary 4. *Let $G \in \mathcal{H}$. Then,*

(i) $s(G) = n$ *if and only if* $h(G) = n$.

(ii) *If* $s(G) = n - 1$, *then* $h(G) = n - 1$.

Proof. Statement (i) is an immediate consequence of the lower bound in Theorem 2.

If $s(G) = n - 1$, then the lower bound of Theorem 2 implies $h(G) \geq n - 1$, and by (i) we have $h(G) \neq n$ which proves (ii). ■

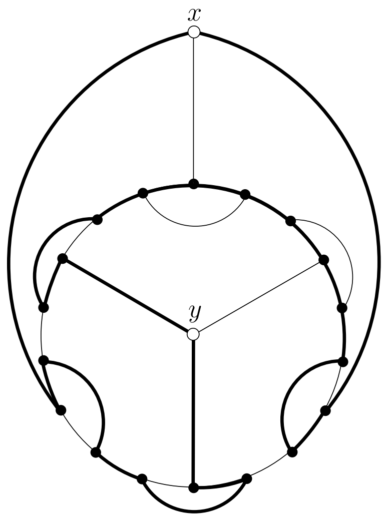

Figure 1

The graph G of order 20 shown in Figure 1 is hamiltonian (the bold painted edges form a hamiltonian cycle) with $S = V \setminus \{x, y\}$ and with $V \setminus \{x\}$ as a smallest H-force set confirms that the converse of statement (ii) does not hold.

Theorem 2 has the following two consequences. A planar graph is called *outerplanar* if it can be embedded in the plane in such a way that every vertex is incident with the unbounded face.

Theorem 5. *Let $G \in \mathcal{H}$ be outerplanar. Then $h(G)$ corresponds to the number of vertices of degree 2 whose two neighbours are adjacent.*

Proof. Let $G \in \mathcal{H}$ be outerplanar and let $x \in V$. If $d_G(x) \geq 3$ then $x \notin T$ and also $x \notin S$. Assume otherwise $d_G(x) = 2$ and let $y, z \in V$ denote the neighbours of x. If $yz \notin E$ then $x \notin T$ and also $x \notin S$. If $yz \in E$ then $G - x$ is hamiltonian which yields $x \in S$ and, consequently, $x \in T$. Hence, $S = T$ and the statement can be deduced from Theorem 2. ■

In [4], the H-force number of an outerplanar hamiltonian graph G different from a cycle was proved to be equal to the number of leafs of the weak dual of G. The *weak dual* of an outerplanar graph G is a tree and is obtained from the dual of G by removing the vertex corresponding to the unbounded face.

Theorem 6. *For $G \in \mathcal{H}$, $h(G) = 2$ if and only if $t(G) = 2$.*

Proof. Suppose first $h(G) = 2$. Then by Lemma 3 there exists a smallest H-force set $F = \{x, y\}$ of G such that $F \subseteq T$. Assume that there exists a vertex

$v \in T \setminus F$ which means that $G-v$ is 2-connected. Then, $G-v$ and, consequently, G has two different x-y-paths with no common inner vertices. Hence, G has an F-cycle not passing v, a contradiction. That proves $F = T$ and $t(G) = 2$.

Suppose now $t(G) = 2$. Since G is not a cycle we have $h(G) \geq 2$. And, by Theorem 2 we have $h(G) \leq 2$ which completes the proof. ∎

In [4], hamiltonian graphs with H-force number 2 have been characterized already by a condition on crossed chords of a hamiltonian cycle. In [4] they also noted that every hamiltonian graph with $h(G) = 2$ is planar.

Now, we give a decomposition formula with respect to the H-force number of a hamiltonian graph which is exactly 2-connected. To that end, let $G \in \mathcal{H}$ be a graph with vertices $u, v \in V$ such that $G - \{u, v\}$ is disconnected, i.e., $u, v \notin T$. Any given hamiltonian cycle C of G can be divided into two u-v-paths P_1 and P_2 which have no inner vertices in common. For $i = 1, 2$, let G_i denote the graph which results from $G[V(P_i)]$ (the subgraph of G induced by $V(P_i)$) by introducing an additional vertex w_i ($w_1 \neq w_2$) and edges uv, uw_i, vw_i. Obviously, G_i is also a member of $\mathcal{H}$.

Theorem 7. *Let $G \in \mathcal{H}$ with $u, v \in V(G)$ such that $G - \{u, v\}$ is disconnected, and let G_1, G_2 be graphs derived from G as described above. Then,*

$$h(G) = h(G_1) + h(G_2) - 2.$$

Proof. On the one hand, from $u, v \notin T(G_i)$ and Lemma 3 it follows that G_i has a smallest H-force set $F_i \subseteq V(G_i)$ such that $u, v \notin F_i$. F_i contains w_i because $G_i - w_i$ is hamiltonian. Let $F := (F_1 \setminus \{w_1\}) \cup (F_2 \setminus \{w_2\})$ and let C_F denote an F-cycle of G. $F_i \setminus \{w_i\}$ is not empty for $i = 1, 2$ which implies that neither G_1 nor G_2 contains C_F as a cycle. Suppose that C_F is not a hamiltonian cycle of G. Then, without loss of generality, there exists a vertex $x \in V(G) \setminus V(G_2)$ which is not contained in F. Let $P_{F,1}$ denote the u-v-path of C_F which is completely contained in G_1. Then, the cycle obtained by connecting $P_{F,1}$ with the u-v-path (u, w_1, v) is an F_1-cycle of G_1 which is not hamiltonian, a contradiction. Consequently, F is an H-force set of G and

$$\begin{aligned} h(G) \leq |F| &= |F_1 \setminus \{w_1\}| + |F_2 \setminus \{w_2\}| = (|F_1| - 1) + (|F_2| - 1) \\ &= h(G_1) + h(G_2) - 2. \end{aligned}$$

On the other hand, Lemma 3 implies that G has an H-force set $F \subseteq V(G)$ where $|F| = h(G)$ and $u, v \notin F$. Clearly, $F_i := (F \cap V(G_i)) \cup \{w_i\}$ is a subset of $V(G_i)$. If C_i denotes an F_i-cycle of G_i, then C_i contains w_i and also the vertices u and v. Hence, $C_i - w_i$ is a u-v-path of G_i and also of G. By connecting the u-v-paths $C_1 - w_1$ and $C_2 - w_2$ we obtain an F-cycle $\tilde{C}$ in G. If C_i for $i = 1$ or 2 would not be hamiltonian in G_i, then $\tilde{C}$ could not be hamiltonian in G.

This contradicts the fact that F is an H-force set of G and implies that F_i is an H-force set of G_i. Hence,

$$h(G) = |F| = (|F_1|-1)+(|F_2|-1) \geq (h(G_1)-1)+(h(G_2)-1) = h(G_1)+h(G_2)-2$$

which proves the statement of Theorem 7 ■

If, for example, G_t denotes the hamiltonian graph which consists of a "chain" of $t \geq 1$ cube graphs (see Figure 2) then by induction and using Theorem 7 we obtain for the H-force-number $h(G_t) = 2t + 2$.

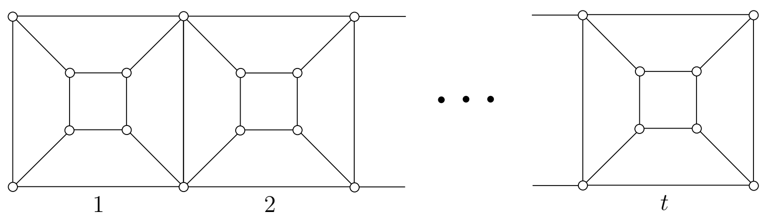

Figure 2

Next, we will give a characterization of hamiltonian graphs G such that $S(G)$ is an H-force set of G and, consequently, $h(G) = s(G)$. To this end, let us consider the concept of cycle extendable graphs (which was first investigated by Hendry in [5]) and weaken it in a suitable sense.

A cycle C of a graph G is called *extendable* if G contains a $V(C)$-cycle C' which has exactly one vertex more than C. A graph G is called *cycle extendable* if G contains a cycle and if every non-hamiltonian cycle is extendable. Cycle extendable graphs are obviously hamiltonian ones.

In [5], Hendry raised the problem whether every hamiltonian chordal graph is cycle extendable or not. Jiang proved in [6] that every planar hamiltonian chordal graph is also cycle extendable. Moreover, a hamiltonian graph which is an interval graph or a split graph has been proved to be cycle extendable, see [1] and also [2].

Now, we call a non-hamiltonian cycle C of a graph G *weakly extendable* if G contains a $V(C)$-cycle of length $n-1$. And, a graph G is called *weakly cycle extendable* if G is hamiltonian and if every non-hamiltonian cycle is weakly extendable. Trivially, every cycle extendable graph is weakly cycle extendable. Every outerplanar graph which belongs to $\mathcal{H}$ is also weakly cycle extendable.

Theorem 8. *Let $G \in \mathcal{H}$. Then, the following conditions are equivalent.*

(i) *$S(G)$ is an H-force set, i.e., $h(G) = s(G)$.*

(ii) *G is weakly cycle extendable.*

Proof. Suppose that $S = S(G)$ is an H-force set and that G contains a cycle C which is not weakly extendable. Then, $G - x$ is not hamiltonian for each $x \in V(G) \setminus V(C)$ which implies $x \notin S$. Hence, C is an S-cycle which contradicts our claim that S is an H-force set. Thus, G is weakly cycle extendable.

Now, let G be weakly cycle extendable and suppose that S is not an H-force set. If S is empty then $G - x$ is not hamiltonian for each $x \in V(G)$. Since G is not a cycle, there exists a cycle C in G of length at most $n - 2$, and C is not weakly extendable, a contradiction. So, suppose that S is not empty and let C be a non-hamiltonian S-cycle of G. Then, C is weakly extendable, i.e., G has a $V(C)$-cycle C' of length $n - 1$. Suppose C' does not contain a vertex $x \in V(G)$. Then $G - x$ is hamiltonian and, consequently, $x \in S$. That together with

$$x \in V(G) \setminus V(C') \subseteq V(G) \setminus V(C) \subseteq V(G) \setminus S$$

yields a contradiction which proves that S is an H-force set. ■

Hence, every weakly cycle extendable graph $G \in \mathcal{H}$ has a uniquely determined smallest H-force set. In Figure 3, a not weakly cycle extendable graph with a unique smallest H-force set (the two black vertices) is presented.

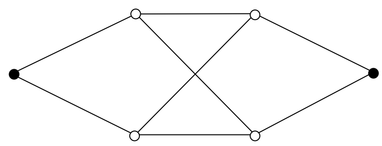

Figure 3

Theorem 9. *Let $G \in \mathcal{H}$.*

(i) *If $s(G) \geq n - 1$, then G is weakly cycle extendable.*

(ii) *If $s(G) \leq 1$, then G is not weakly cycle extendable.*

Proof. (i) If $s(G) = n$ then G is 1-hamiltonian which implies that every non-hamiltonian cycle of G is weakly extendable. If $s(G) = n - 1$ then every S-cycle is hamiltonian. For every other non-hamiltonian cycle C of G, there is an $x \in S$ which is not contained in C. Since $G - x$ is hamiltonian, C is a cycle of $G - x$ and, consequently, weakly extendable in G.

(ii) If $s(G) = 0$ then G has no cycle of length $n-1$, i.e., every non-hamiltonian cycle is not weakly extendable. If $s(G) = 1$ then, obviously, G has at least five vertices. Let be $S = \{x\}$ and let C be a hamiltonian cycle of $G - x$. Moreover, let y and z be two neighbors of x. Then, C passes y and z and consists of two y-z-paths P_1 and P_2 with no common inner vertex. At least one of these paths has more than one inner vertex. Otherwise, because of $n \geq 5$, each of P_1 and

P_2 would have exactly one inner vertex which implies $s(G) > 1$, a contradiction. Suppose, now, that P_1 has at least two inner vertices. Then, $V(P_2) \cup \{x\}$ is the vertex set of a cycle C' of length at most $n-2$. C' cannot be weakly extendable in G because otherwise there would exist a $V(C')$-cycle of length $n-1$ in G which is different from C. That contradicts the claim $S(G) = \{x\}$. ∎

For every integer $n \geq 9$ and all k with $2 \leq k \leq n-2$ we were able to construct a weakly cycle extendable graph of order n with H-force number k.

Now, let $\mathcal{F} = \mathcal{F}(G)$ for a given graph $G \in \mathcal{H}$ denote the family of all H-force sets of G. As is easily seen, $\bar{\mathcal{F}} = \{X \subseteq V \mid X \notin \mathcal{F}\}$ is an independence system on V which means that $\bar{\mathcal{F}}$ satisfies the following two properties.

(M1) $\emptyset \in \bar{\mathcal{F}}$.

(M2) $X \in \bar{\mathcal{F}}, Y \subseteq X$ implies $Y \in \bar{\mathcal{F}}$.

In general, the independence system $(V, \bar{\mathcal{F}})$ is not also a matroid which means that the property

(M3) If $X, Y \in \bar{\mathcal{F}}$ and $|X| = |Y| + 1$, then there exists an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \bar{\mathcal{F}}$.

is not satisfied for every graph $G \in \mathcal{H}$ (see, also [8]). Consider the hamiltonian graph G with vertex set $V = \{1, 2, \ldots, 7\}$ which consists of the cycle $(1, 2, \ldots, 7)$ and the chords 14 and 36. For G we have $\{1, 2, 3, 4\} \in \bar{\mathcal{F}}$ and $\{1, 2, 3, 6, 7\} \in \bar{\mathcal{F}}$ but, property (M3) is not satisfied for these two sets.

Theorem 10. *If G is a weakly cycle extendable graph, then $(V, \bar{\mathcal{F}})$ is a matroid.*

Proof. Let $X, Y \in \bar{\mathcal{F}}$ be two sets where $|X| = |Y| + 1$. As G is weakly cycle extendable, G contains a Y-cycle C of length $n-1$. Let $v \in V$ be the only vertex which does not belong to C. Hence, $X \setminus \{v\}$ is a subset of $V(C)$. If there is a vertex $x \in X \setminus \{v\}$ with $x \notin Y$, then we have $Y \cup \{x\} \in \bar{\mathcal{F}}$ and, consequently, $Y \setminus \{x\} \in \bar{\mathcal{F}}$. Otherwise, we have $Y = X \setminus \{v\}$. That yields $Y \cup \{v\} = X \in \bar{\mathcal{F}}$ and proves the property (M3). ∎

The maximal independent sets of the matroid $(V, \bar{\mathcal{F}})$, which are the members of $\bar{\mathcal{F}}$ of maximal cardinality, are just the vertex sets of the cycles of length $n-1$ of G.

If $\mathcal{C} = \mathcal{C}(G)$ denotes the set of all cycles in G which are not weakly extendable, then let $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ denote a partition of $\mathcal{C}$, i.e., $\mathcal{C}$ is the union of $m \geq 1$ nonempty and disjoint subsets $\mathcal{C}_i$ of $\mathcal{C}(G)$. We call a partition $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ *vertex-unsaturated* (for short, *unsaturated*) if $V(\mathcal{C}_i)$ where

$$V(\mathcal{C}_i) := \bigcup_{C \in \mathcal{C}_i} V(C)$$

is different from $V(G)$ for $i = 1, 2, \ldots, m$. Now, let $p(G)$ denote the smallest integer m for which there exists an unsaturated partition $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of $\mathcal{C}(G)$.

Theorem 11. *Let $G \in \mathcal{H}$ be a graph that is not weakly cycle extendable. Then,*

$$h(G) = s(G) + p(G).$$

Proof. First, let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ be an unsaturated partition of $\mathcal{C}(G)$ such that $m = p(G)$. For $i = 1, 2, \dots, m$ let $v_i \in V(G) \setminus V(\mathcal{C}_i)$ be any fixed vertex. We prove that $X := S(G) \cup \{v_1, \dots, v_m\}$ is an H-force set which implies $h(G) \leq s(G) + p(G)$. For this purpose, let C be any non-hamiltonian cycle of G.

If there exists a $V(C)$-cycle C' of length $n-1$ in G, then $S(G)$ contains a vertex v such that $\{v\} = V(G) \setminus V(C')$. Hence, $v \notin V(C)$ and, consequently, $X \nsubseteq V(C)$. If there is no $V(C)$-cycle of length $n-1$ in G, then G contains a $V(C)$-cycle $C'' \in \mathcal{C}(G)$. In this case there exists a partition set $\mathcal{C}_i$, $1 \leq i \leq m$, such that $C'' \in \mathcal{C}_i$. Then

$$v_i \in V(G) \setminus V(\mathcal{C}_i) \subseteq V(G) \setminus V(C'') \subseteq V(G) \setminus V(C)$$

implies $X \nsubseteq V(C)$. Thus, every X-cycle is hamiltonian and X is an H-force set.

Assume now that there exists an H-force set X of G with less than $s(G)+p(G)$ vertices. Since, by Theorem 8, $S(G)$ is not an H-force set, there exists a nonempty subset $Y \subseteq V(G) \setminus S(G)$ such that $X = S(G) \cup Y$. Because of the assumption we have $|Y| < p(G)$. Note that every cycle $C \in \mathcal{C}(G)$ is an $S(G)$-cycle because otherwise there would exist an $x \in S(G) \setminus V(C)$ such that $V(G) \setminus \{x\}$ is the vertex set of a cycle C' of length $n-1$ in G with $V(C) \subseteq V(C')$, a contradiction with respect to $C \in \mathcal{C}(G)$. Since, moreover, every X-cycle is hamiltonian, we have that for every $C \in \mathcal{C}(G)$ there exists a vertex $y \in Y$ such that $y \notin V(C)$.

For every $y \in Y$, let us define $\mathcal{D}_y = \{C \in \mathcal{C}(G) \mid y \notin V(C)\}$. Then, we have

$$\mathcal{C}(G) = \bigcup_{y \in Y} \mathcal{D}_y$$

and, because of $\mathcal{C}(G) \neq \emptyset$, there exists a vertex $y_1 \in Y$ such that $\mathcal{D}_{y_1} \neq \emptyset$. Now, we are able to construct an unsaturated partition of $\mathcal{C}(G)$. To this end, let $\mathcal{C}_1 := \mathcal{D}_{y_1}$ and $Y_1 := Y \setminus \{y_1\}$. We may assume that the partition sets $\mathcal{C}_1, \dots, \mathcal{C}_k$ with $k \geq 1$ are already constructed. If Y_k contains a vertex y_{k+1} such that the set

$$\mathcal{D}_{y_{k+1}} \setminus \bigcup_{i=1}^{k} \mathcal{C}_i$$

is not empty, then let

$$\mathcal{C}_{k+1} := \mathcal{D}_{y_{k+1}} \setminus \bigcup_{i=1}^{k} \mathcal{C}_i.$$

This procedure terminates after at most $|Y| - 1$ steps and yields an unsaturated partition $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ with $m < p(G)$ which contradicts the definition of $p(G)$. ■

As an immediate consequence of Theorem 11 we have

Corollary 12. *Let $G \in \mathcal{H}$ be a not weakly cycle extendable graph. Then, the following conditions are equivalent.*

(1) $h(G) = s(G) + 1$,

(2) $(\mathcal{C}(G))$ *is unsaturated.*

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