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## ETERNAL DOMINATION: CRITICALITY AND REACHABILITY

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### Abstract

We show that for every minimum eternal dominating set,  $D$ , of a graph  $G$  and every vertex  $v \in D$ , there is a sequence of attacks at the vertices of  $G$  which can be defended in such a way that an eternal dominating set not containing  $v$  is reached. The study of the stronger assertion that such a set can be reached after a single attack is defended leads to the study of graphs which are critical in the sense that deleting any vertex reduces the eternal domination number. Examples of these graphs and tight bounds on connectivity, edge-connectivity and diameter are given. It is also shown that there exist graphs in which deletion of any edge increases the eternal domination number, and graphs in which addition of any edge decreases the eternal domination number.

**Keywords:** dominating set, eternal dominating set, critical graphs.

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## 1. INTRODUCTION

The *eternal domination problem* can be viewed as the dynamic, discrete-time problem where, at each time interval, some specified vertex not in the current dominating set replaces one of its neighbors in the dominating set. Another viewpoint is that of a discrete-time process in which mobile guards defend a graph from an infinite sequence of attacks at vertices. When an attack occurs at vertex  $v$ , a guard located at a neighboring vertex relocates to  $v$  while the other guards remain in place. It is clear that the set of vertices at which the guards are located must be a dominating set, as must each subsequent set arising from a guard moving. In keeping with current practice, we adopt the latter point of view. The paper [10] gives a survey of results on the eternal domination problem and some variants.

A question that arises naturally is that of *reconfiguration*: given two eternal dominating sets  $D_1$  and  $D_2$  of minimum size, is there a sequence of attacks at vertices for which some collection of guards' moves transforms  $D_1$  into  $D_2$ ? Put differently, if  $\mathcal{D}(G)$  is the graph whose vertices are the minimum size eternal dominating sets of  $G$ , and in which two eternal dominating sets are adjacent if and only if they differ by a single guard's move, is  $\mathcal{D}(G)$  connected? Related work on reconfiguration problems for dominating sets can be found in [5] and [7].

The five-cycle  $C_5$  requires three guards to be able to defend any, and every, infinite sequence of attacks. While the graph  $\mathcal{D}(C_5)$  is connected — it is the 5-dimensional cube — it is also true that for any initial configuration of the guards, any infinite sequence of attacks on this graph can be defended in such a way that any specified guard never relocates. Thus another question which arises is whether, for any given configuration of guards, there is an sequence of attacks which can be defended in such a way that any single specified guard relocates. Put differently, if there is a guard at vertex  $v$ , is it possible to defend the graph in such a way that a configuration of guards with no guard at  $v$  is *reachable* by a series of guard moves from the current configuration. A stronger statement is whether, for every minimum eternal dominating set and every guard  $g$  with a neighbor not occupied by a guard, there is a single attack which can be defended by  $g$ .

If any sequence of attacks at the vertices of a graph  $G$  can be defended by the minimum number of guards without the guard located at  $v$  ever relocating, then  $G - v$  can be defended by fewer guards than  $G$ . Thus the study of whether  $G$  can be defended in such a way that an eternal dominating set without a guard at a specified vertex  $v$  is reachable leads to the study of such *eternal domination critical vertices*, and subsequently to eternal domination vertex-critical graphs, that is, graphs in which every vertex is critical.

This paper is organized as follows. Definitions, notation, and other prelimi-

naries are reviewed in the next section. Eternal domination vertex-critical graphs are studied after that. Constructions of these graphs are given, and structural properties like the connectivity, edge-connectivity and diameter are tightly bounded. In the subsequent section, we examine the two reachability questions mentioned above. It is shown that the first question has a positive answer, and two graph classes in which the second question has a positive answer are identified. Finally, the effect of adding, or deleting, an edge on the number of guards needed is explored. This leads to the concept of eternal domination edge-critical graphs.

## 2. PRELIMINARIES

A *dominating set* of graph  $G$  is a set  $D \subseteq V$  such that for each  $u \in V - D$ , there exists  $x \in D$  adjacent to  $u$ . The minimum cardinality amongst all dominating sets of  $G$  is the *domination number*  $\gamma(G)$ . Denote the open and closed neighborhoods of a vertex  $x \in V$  by  $N(x)$  and  $N[x]$ , respectively. That is,  $N(x) = \{v : xv \in E\}$  and  $N[x] = N(x) \cup \{x\}$ . Further, for  $S \subseteq V$ , let  $N(S) = \bigcup_{x \in S} N(x)$ . For any  $X \subseteq V$  and  $x \in X$ , we say that  $v \in V - X$  is an *external private neighbor* of  $x$  with respect to  $X$  if  $v$  is adjacent to  $x$  but to no other vertex in  $X$ . The set of all such vertices  $v$  is denoted  $\text{epn}(x, X)$ .

Let  $D_i \subseteq V, i \geq 1$ , be a set of vertices with one guard located on each vertex of  $D_i$ . In this paper, we allow at most one guard to be located on a vertex at any time. The *eternal domination problem* can be modeled as a two-player game between a *defender* and an *attacker* who alternate turns: the defender goes first and chooses  $D_1$ . On the defender's  $i^{\text{th}}$  turn, it chooses  $D_i, i > 1$ . On its turn, the attacker chooses the location of the  $i^{\text{th}}$  attack,  $r_i$ , which together form the attack sequence  $r_1, r_2, \dots$ . Each  $D_i, i \geq 1$ , has the same cardinality and each is required to be a dominating set,  $r_i \in V$  (assume without loss of generality  $r_i \notin D_i$ ), and  $D_{i+1}$  is obtained from  $D_i$  by moving one guard to  $r_i$  from a vertex  $v \in D_i, v \in N(r_i)$ . We think of each attack as being handled by the defender by choosing the next  $D_i$  subject to it being a dominating set. The defender wins the game if they can successfully defend any sequence of attacks; the attacker wins otherwise. The size of a smallest eternal dominating set for  $G$ , denoted  $\gamma^\infty(G)$ , is the size of a smallest  $D_1$  that enables the defender to win. This problem was first studied in [2]. We observed in the introduction that  $\gamma^\infty(C_5) = 3$ .

We say a vertex set  $D'$  is *reachable* from  $D$  if there exists a sequence  $D_1, D_2, \dots, D_k, k \geq 1$ , with  $D = D_1$  and  $D' = D_k$  such that each pair  $D_j, D_{j+1}, k - 1 \geq j \geq 1$ , satisfies  $D_{j+1} = D_j - u + v, u \in D_j$ , and  $v \in N[u]$ . A vertex is *protected* if there is a guard on the vertex or on an adjacent vertex; an attack at  $v$  is *defended* if we send a guard to  $v$ . More generally, we *defend* a graph by defending all the attacks in an attack sequence. A vertex is *occupied* if there is a guard on it and *unoccupied* otherwise.

An *independent set* of vertices in  $G$  is a set  $I \subseteq V$  such that no two vertices in  $I$  are adjacent. The maximum cardinality amongst all independent sets is the *independence number*, denoted  $\alpha(G)$ . Since the guards located at the vertices of an eternal dominating set must be able to defend a sequence of attacks consisting of the vertices of a maximum independent set of  $G$ , it follows that  $\gamma^\infty(G) \geq \alpha(G)$ .

The *clique covering number*  $\theta(G)$  is the minimum number,  $k$ , of sets in a partition  $V = V_1 \cup V_2 \cup \dots \cup V_k$  of  $V$  such that each  $G[V_i]$  is complete. As one guard can defend each clique in a clique covering, it follows that  $\gamma^\infty(G) \leq \theta(G)$ .

### 3. VERTEX-CRITICALITY

**Theorem 1.** *For any vertex  $v$  of a graph  $G$ ,*

$$\gamma^\infty(G) - 1 \leq \gamma^\infty(G - v) \leq \gamma^\infty(G).$$

**Proof.** We first show the upper inequality. Let  $D$  be a minimum eternal dominating set of  $G$ . If  $v \notin D$  then, starting from the configuration  $D$ , the guards can defend any sequence of attacks at vertices of  $G - v$ . Suppose, then, that  $v \in D$ . We may further assume there is a guard on  $v$  in every minimum eternal dominating set reachable from  $D$ , otherwise the previous argument applies. But then the remaining  $\gamma^\infty(G) - 1$  guards can defend any sequence of attacks at vertices of  $G - v$ .

We now show the lower inequality. Suppose fewer than  $\gamma^\infty(G) - 1$  guards suffice to defend any sequence of attacks in  $G - v$ . Then, using an additional guard who is on  $v$  at all times, fewer than  $\gamma^\infty(G)$  guards can defend  $G$ , a contradiction. ■

A vertex  $v$  of a graph  $G$  is called *eternal domination critical* if  $\gamma^\infty(G - v) = \gamma^\infty(G) - 1$ . If every vertex of  $G$  is eternal domination critical, then we say that  $G$  is an *eternal domination vertex-critical graph*.

Let  $G$  be a graph, and  $v \in V(G)$  be an eternal domination critical vertex of  $G$ . Since  $\gamma^\infty(G - v) = \gamma^\infty(G) - 1$ , the graph  $G$  can be defended by  $\gamma^\infty(G)$  guards in such a way that every configuration of guards that arises has a guard at  $v$ . When this happens, we say that there is a *stationary guard at  $v$* . Note that the converse statement is also true: if  $G$  can be defended using  $\gamma^\infty(G)$  guards, one of which is stationary at  $v$ , then  $v$  is an eternal domination critical vertex of  $G$ .

Odd length cycles with at least five vertices are examples of eternal domination vertex-critical graphs. In fact,  $C_5$  is the smallest non-trivial, connected eternal domination vertex-critical graph. Odd cycles have the property described below, which we will see is useful for finding other examples of eternal domination vertex-critical graphs.

**Observation 2.** *Let  $G$  be a graph for which  $\gamma^\infty(G) = \theta(G)$  and with the property that, for any vertex  $x$ , there is a minimum clique covering in which  $x$  is a clique of size one. Then  $G$  is eternal domination vertex-critical.*

We now describe an infinite family of eternal domination vertex-critical graphs. Let  $k$  and  $t$  be positive integers, and  $n = (k + 1)t + 1$ . Denote by  $C_n^k$  the  $k$ -th power of  $C_n$ , that is, the graph with vertex set  $V(C_n^k) = \mathbb{Z}_n$  and  $xy \in E(C_n^k)$  if and only if  $x - y$  is congruent modulo  $n$  to an element of  $\{\pm 1, \pm 2, \dots, \pm k\}$ . These graphs are known to be domination critical [1]. It is also known that if  $G$  is any power of a cycle, then  $\gamma^\infty(G) = \theta(G)$ , see [9].

**Proposition 3.** *Let  $k$  and  $t$  be positive integers, and  $n = (k + 1)t + 1$ . The graph  $C_n^k$  is eternal domination vertex-critical.*

**Proof.** By [9] we have  $\gamma^\infty(C_n^k) = \theta(C_n^k) = t + 1$ . Note that any set of  $k$  consecutive vertices in cyclic order induces a maximum clique. Since  $C_n^k$  is vertex-transitive and has a minimum clique cover with a clique of size 1, the result now follows from Observation 2. ■

It is easy to observe that the only eternal domination vertex-critical graph with  $\gamma^\infty = 2$  is  $\overline{K_2}$ . The eternal domination vertex-critical graphs with  $\gamma^\infty = 3$  can also be determined.

**Proposition 4.** *The eternal domination vertex-critical graphs with  $\gamma^\infty = 3$  are exactly the complements of odd cycles.*

**Proof.** Let  $G$  be the complement of an odd cycle. Then  $\gamma^\infty(G) = \theta(G) = 3$ , see [9]. Since,  $G$  is vertex-transitive and has a minimum clique cover with a clique of size 1, it is eternal domination vertex-critical by Observation 2.

Now let  $G$  be an eternal domination vertex-critical graph with  $\gamma^\infty(G) = 3$ . Then, for any vertex  $v$  we have  $\gamma^\infty(G - v) = 2$ . Since  $G$  is not complete, we also have  $\alpha(G - v) = 2$ . By [9], Theorem 5, we must also have  $\theta(G - v) = 2$ . It follows that  $\theta(G) = 3$ . Hence, the graph  $\overline{G}$  has chromatic number 3, and for any vertex  $v$ , the chromatic number of  $G - v$  equals 2. Therefore,  $\overline{G}$  is 3-(color)-critical, that is, an odd cycle. ■

For any positive integer  $k$ , when  $t = 2$ , the graphs  $C_n^k$  in Proposition 3 can be seen to be the complements of  $C_{2k+3}$ .

In the following, we describe two constructions of eternal domination vertex-critical graphs. The first of these is easy to observe and the second is a commonly used construction that gives domination critical graphs of various types (for example, see [1, 4]).

**Observation 5.** *Let  $G$  and  $H$  be disjoint graphs. The graph  $G \cup H$  is eternal domination vertex-critical if and only if both  $G$  and  $H$  are eternal domination vertex-critical.*

Let  $G$  and  $H$  be disjoint graphs. For vertices  $x \in V(G)$  and  $y \in V(H)$ , the *coalescence of  $G$  and  $H$  with respect to  $x$  and  $y$*  is the graph  $G \cdot_{xy} H$  constructed from  $G \cup H$  by identifying the vertices  $x$  and  $y$ . It has vertex set  $V(G) \cup (V(H) - \{y\})$ , and edge set  $E(G) \cup E(H - y) \cup \{xz : yz \in E(H)\}$ .

**Proposition 6.** *Let  $G$  and  $H$  be disjoint graphs. If  $x$  and  $y$  are eternal domination critical vertices of  $G$  and  $H$ , respectively, then*

$$\gamma^\infty(G \cdot_{xy} H) = \gamma^\infty(G) + \gamma^\infty(H) - 1.$$

**Proof.** It is clear that  $\gamma^\infty(G) + \gamma^\infty(H) - 1$  guards suffice and  $\gamma^\infty(G)$  guards can defend the copy of  $G$  in  $G \cdot_{xy} H$  and, since  $y$  is an eternal domination critical vertex of  $H$ ,  $\gamma^\infty(H) - 1$  guards can defend the copy of  $H - y$  in  $G \cdot_{xy} H$ .

To see that this many guards are necessary, suppose that fewer than  $\gamma^\infty(G) + \gamma^\infty(H) - 1$  guards are located at vertices of  $G \cdot_{xy} H$ . We may assume that there is a guard at  $x$ . Then, either there are fewer than  $\gamma^\infty(G) - 1$  guards located at vertices of the copy of  $G - x$ , or fewer than  $\gamma^\infty(H) - 1$  guards located at vertices of the copy of  $H - y$ .

Suppose there are fewer than  $\gamma^\infty(G) - 1$  guards located at vertices of the copy of  $G - x$ . In order for the guards to be able to defend all attacks at vertices of  $G - x$ , there must be  $\gamma^\infty(G) - 2$  guards located at vertices of the copy of  $G - x$  and a sequence of attacks at vertices of  $G - x$  that result in the guard located at  $x$  moving to a vertex of  $G - x$ . But now there are fewer than  $\gamma^\infty(H) - 1$  guards located at vertices of the copy of  $H - y$  and no guard at  $x$ . Hence there is a sequence of attacks at vertices of  $H - y$  that cannot be defended.

The other case is handled by a similar argument. ■

**Theorem 7.** *For disjoint graphs  $G$  and  $H$ , and vertices  $x \in V(G)$  and  $y \in V(H)$ , the graph  $G \cdot_{xy} H$  is eternal domination vertex-critical if and only if both  $G$  and  $H$  are eternal domination vertex-critical.*

**Proof.** Suppose both  $G$  and  $H$  are eternal domination vertex-critical. Then, by Proposition 6,  $\gamma^\infty(G \cdot_{xy} H) = \gamma^\infty(G) + \gamma^\infty(H) - 1$ . Let  $z \in V(G \cdot_{xy} H)$ . If  $z = x$ , then  $\gamma^\infty(G) - 1$  guards can defend  $G - x$ , and  $\gamma^\infty(H) - 1$  guards can defend  $H - y$ . Otherwise, without loss of generality,  $z$  is a vertex of the copy of  $G$ . Since  $G$  is eternal domination vertex-critical,  $\gamma^\infty(G) - 1$  guards can defend the copy  $G - z$ . Since this subgraph includes the vertex  $x$ , a further  $\gamma^\infty(H) - 1$  guards can defend  $H - y$ . Therefore  $G \cdot_{xy} H$  is eternal domination vertex-critical.

Now suppose  $G \cdot_{xy} H$  is eternal domination vertex-critical. Then, by Proposition 6, for any vertex  $z \in V(G \cdot_{xy} H)$  the graph  $(G \cdot_{xy} H) - z$  can be defended

by  $\gamma^\infty(G) + \gamma^\infty(H) - 2$  guards. If  $z = x$  then there must be  $\gamma^\infty(G) - 1$  guards on vertices of  $G - x$ , and  $\gamma^\infty(H) - 1$  guards on vertices of  $H - y$ , otherwise there is a sequence of attacks that cannot be defended. It follows that  $x$  and  $y$  are eternal domination critical vertices of  $G$  and  $H$ , respectively. Otherwise, suppose that  $z$  is a vertex of the copy of  $G$ . If it is not an eternal domination critical vertex, then there is a sequence of attacks at vertices of the copy of  $G - z$  which require  $\gamma^\infty(G)$  guards to be located at the vertices of this subgraph. Thus, at most  $\gamma^\infty(H) - 2$  guards are located at vertices of the copy of  $H$  and, by Proposition 1, not all attacks at vertices of the copy of  $H - y$  can be defended. It follows that  $z$  must be an eternal domination critical vertex of  $G$ .

The argument is identical if  $z$  is a vertex of the copy of  $H$ . ■

**Corollary 8.** *A graph is eternal domination vertex-critical if and only if each of its blocks is eternal domination vertex-critical.*

**Corollary 9.** *If  $G$  is an eternal domination vertex-critical graph with blocks  $B_1, B_2, \dots, B_k$ , then*

$$\gamma^\infty(G) = \gamma^\infty(B_1) + \gamma^\infty(B_2) + \dots + \gamma^\infty(B_k) - (k - 1).$$

The coalescence construction shows that it is possible to have eternal domination vertex-critical graphs with a cut-vertex. We now show that it is not possible for such graphs to have a cut-edge.

**Proposition 10.** *If  $G$  is a connected eternal domination vertex-critical graph with at least one edge, then  $\kappa'(G) \geq 2$ .*

**Proof.** Since the only eternal domination vertex-critical graph on two vertices is  $\overline{K_2}$ , the graph  $G$  has at least three vertices. Suppose, for contradiction, that  $xy$  is a cut-edge of  $G$ . Let  $G_x$  and  $G_y$  be the components of  $G - xy$  containing  $x$  and  $y$ , respectively. Without loss of generality,  $G_y$  has at least two vertices.

Since  $G$  is eternal domination vertex-critical, it can be defended by  $\gamma^\infty(G)$  guards with a stationary guard at  $x$ . If the condition that the guard be stationary is relaxed, this guard can defend all attacks at  $x$  or  $y$ . Therefore,  $\gamma^\infty(G) \leq \gamma^\infty(G_x) + \gamma^\infty(G_y) - 1$ .

But, the graph  $G - y = G_x \cup (G_y - y)$ . By Theorem 1 it requires  $\gamma^\infty(G_x) + \gamma^\infty(G_y - y) \geq \gamma^\infty(G_x) + \gamma^\infty(G_y) - 1$  guards. Therefore  $y$  is not an eternal domination critical vertex of  $G$ , a contradiction. ■

Our next goal is to give a tight bound on the diameter of an eternal domination vertex-critical graph. Odd cycles show that the diameter of an eternal domination vertex-critical graph can be at least  $\gamma^\infty - 1$ . We show that this is the maximum possible value. The first part of the argument uses a method introduced in [6].

**Proposition 11.** *A vertex  $x$  of a graph  $G$  for which there exists  $y$  such that  $N[x] \supseteq N[y]$  is not an eternal domination critical vertex of  $G$ .*

**Proof.** Any collection of guards that defends  $G - x$  also defends  $G$ , as an attack at  $x$  can be defended by the same guard that would defend it if it were at  $y$ . ■

**Theorem 12.** *If  $G$  is a connected eternal domination vertex-critical graph with at least two vertices, then  $\text{diam}(G) \leq \gamma^\infty(G) - 1$ .*

**Proof.** Let  $x$  be an end-vertex of a diametrical path in  $G$ . For  $i = 0, 1, \dots, \text{diam}(G)$ , define the level-set  $X_i = \{y : d(x, y) = i\}$ , the set  $U_i = X_0 \cup X_1 \cup \dots \cup X_i$ , and  $H_i$  to be the subgraph of  $G$  induced by  $U_i$ . Since  $G$  has at least two vertices,  $\text{diam}(G) \geq 1$ .

We observe a useful inequality chain. For  $i = 1, 2, \dots, \text{diam}(G) - 1$ , we have

$$(1) \quad \gamma^\infty(H_i) + \gamma^\infty(G - U_i) \geq \gamma^\infty(G) \geq \gamma^\infty(H_{i-1}) + 1 + \gamma^\infty(G - U_i).$$

The left-hand inequality is clear. The right-hand inequality follows from the fact that  $G$  can be defended in such a way that there is a stationary guard located at any particular vertex, due to the fact that  $G$  is an eternal domination vertex-critical graph.

By Proposition 11, the subgraph of  $G$  induced by  $X_1$  is not complete. Thus, since  $x$  is an eternal domination critical vertex, there exists an eternal dominating set of  $G$  containing  $x$  and at least two vertices of  $X_1$ . It can be obtained from an eternal dominating set with a stationary guard at  $x$  by attacking two non-adjacent vertices in  $X_1$ .

Let  $m$  be the largest integer for which there exists an eternal dominating set that contains  $m + 2$  vertices of  $U_m$ . By the argument above, there is an eternal dominating set containing at least three vertices of  $U_1$ ; hence  $m \geq 0$ . If  $m = \text{diam}(G) - 1$ , then the statement to be proved holds, hence we may assume  $m < \text{diam}(G) - 1$ . Thus,  $\text{diam}(G) > m + 1$ , so  $X_{m+2} \neq \emptyset$ .

Let  $D$  be an eternal dominating set which contains  $m + 2$  vertices of  $U_m$ . By definition of  $m$ , there is no eternal dominating set which contains  $m + 3$  vertices of  $U_{m+1}$ . Therefore,  $D \cap X_{m+1} = \emptyset$ . Attacking a vertex of  $X_{m+2}$  yields an eternal dominating set  $D'$  that contains the same vertices of  $U_{m+1}$  as  $D$ , and also contains a vertex of  $X_{m+2}$ . Since  $D'$  contains  $m + 3$  vertices of  $U_{m+2}$ , the maximality of  $m$  implies that there is no sequence of attacks at vertices of  $G - U_{m+1}$  which results in a second guard being located at a vertex of  $X_{m+2}$ .

Since  $D' \cap X_{m+1} = \emptyset$ , it now follows that any sequence of attacks at the vertices of  $X_{m+2}$  are defended by the one guard located there (i.e., the subgraph of  $G$  induced by  $X_{m+2}$  is complete).

We claim that  $\gamma^\infty(H_{m+1}) = m + 2$ . Since  $D'$  is an eternal dominating set, it is at most  $m + 2$ . This follows from the maximality of  $m$ . Suppose it is less.

Since  $\gamma^\infty(G) \leq \gamma^\infty(H_{m+1}) + \gamma^\infty(G - U_{m+1})$ , the eternal dominating set  $D'$  has fewer than  $\gamma^\infty(G - U_{m+1})$  guards located at vertices of  $G - U_{m+1}$ . Hence, there is a sequence of attacks at vertices of  $G - U_{m+1}$  that cannot be defended by the guards located there. Since  $D' \cap X_{m+1} = \emptyset$ , it is not possible for guards located at vertices of  $U_{m+1}$  to be used to defend any of these attacks. Thus there is a sequence of attacks that cannot be defended starting from the configuration  $D'$ , a contradiction. This proves the claim.

We now show by induction that  $\gamma^\infty(H_k) = k + 1$  for all  $k \geq m + 1$ . It then follows that  $\gamma^\infty(G) = 1 + \text{diam}(G)$ , which completes the proof of the theorem.

The base case, that  $\gamma^\infty(H_{m+1}) = m + 2$ , is proved above. We also have that  $\text{diam}(G) > m + 1$ . Suppose, for some  $k$  such that  $m + 2 \leq k < \text{diam}(G)$ , that  $\gamma^\infty(H_k) = k + 1$ . Consider  $H_{k+1}$ . There are two cases.

If  $\text{diam}(G) = k + 1$ , then since  $G$  can be dominated with a stationary guard at any vertex of  $X_{k+1}$  and  $\gamma^\infty(H_k) = k + 1$ , we have  $\gamma^\infty(G) \geq 1 + \gamma^\infty(H_k) = k + 2$  (in fact, equality holds by the definition of  $m$ ) and the result follows.

Otherwise,  $\text{diam}(G) \geq k + 2$ . Then, by inequality (1),

$$\gamma^\infty(H_{k+1}) + \gamma^\infty(G - U_{k+1}) \geq \gamma^\infty(H_k) + 1 + \gamma^\infty(G - U_{k+1}).$$

Consequently,  $\gamma^\infty(H_{k+1}) \geq \gamma^\infty(H_k) + 1 = k + 2$ . The maximality of  $m$  then implies that equality holds. This completes the proof. ■

Recall that the *block-cutpoint graph* of a graph  $G$  is the bipartite graph  $\mathcal{BC}(G)$  whose vertices are the blocks and cut-vertices of  $G$ , with block  $B$  adjacent to cut-vertex  $x$  if and only if  $x$  is a vertex of  $B$ . It is easy to observe that if  $G$  is an eternal domination vertex-critical graph with maximum diameter and a cut-vertex, then  $\mathcal{BC}(G)$  is a path, each block of  $G$  is an eternal domination vertex-critical graph with maximum diameter, and the cut-vertices of  $G$  are antipodal vertices of the blocks to which they belong.

The next two propositions bound the number of vertices in an eternal domination vertex critical graph. Similar results appear in [1] (also see [6, 4]).

**Proposition 13.** *Let  $G$  be an eternal domination vertex-critical graph with  $n$  vertices and no isolated vertices. Then  $n \leq (\gamma^\infty(G) - 1)(\Delta + 1) - 1$ .*

**Proof.** Let  $v \in V(G)$ . We can assume that there is a stationary guard on  $v$ . By Proposition 11, the subgraph induced by  $N(v)$  is not complete. Attacking two independent vertices in  $N(v)$  yields an eternal dominating set containing two vertices of  $N(v)$ . These two vertices dominate at most  $\Delta$  vertices in  $V(G) - \{v\}$ . Therefore,  $n \leq 1 + 2\Delta + (\gamma^\infty - 3)(\Delta + 1) = (\gamma^\infty(G) - 1)(\Delta + 1) - 1$ . ■

The bound in the above proposition is achieved by any graph which is the disjoint union of copies of  $C_5$ . A slightly better bound can be obtained by taking

into account the quantity  $p_2(G)$ , the maximum size of a 2-packing — a set of vertices whose closed neighborhoods are pairwise disjoint.

**Proposition 14.** *Let  $G$  be an eternal domination vertex-critical graph with  $n$  vertices and no isolated vertices. Then  $n \leq (\gamma^\infty(G) - 1)(\Delta + 1) - p_2(G)$ .*

*Proof.* Let  $P$  be a maximum size 2-packing of  $G$ . By Proposition 11, for each  $v \in P$ , the subgraph induced by  $N(v)$  is not complete. Since  $P$  is an independent set, we can assume there is a guard located at each vertex of  $P$ . We can assume further that some vertex  $s \in P$  holds a stationary guard and, as above, that there are two guards at vertices of  $N(s)$ . Since no vertex in an eternal dominating set can have two independent private neighbors, for each  $x \in P - \{s\}$ , at least one vertex in  $N(x)$  is not a private neighbor of  $x$ . Therefore,

$$n \leq 1 + 2\Delta + (\gamma^\infty - 3)(\Delta + 1) - (p_2(G) - 1) = (\gamma^\infty(G) - 1)(\Delta + 1) - p_2(G). \quad \blacksquare$$

Equality can be seen to hold for  $C_5, C_7$  and  $C_9$ , and any graph in which each component is one of these.

We conclude this section with an observation that bears a formal resemblance to the colouring number bound for the chromatic number of a graph. Let  $\pi = x_1, x_2, \dots, x_n$  be an enumeration of the vertices of  $G$ . For  $i = 1, 2, \dots, n$ , let  $G_i$  be the subgraph of  $G$  induced by  $x_1, x_2, \dots, x_i$ . Let  $X_\pi$  be the number of subscripts  $i$  such that  $x_i$  is an eternal domination critical vertex of  $G_i$ , where  $x_1$  is deemed to be an eternal domination critical vertex of  $G_1$ . Since deleting a vertex which is not eternal domination critical leaves a subgraph with the same eternal number, we have that  $\gamma^\infty(G) \leq X_\pi$ . Consequently,  $\gamma^\infty(G) \leq \min_\pi X_\pi$ . Further, there exists an enumeration  $\pi$  for which equality holds: working from  $n$  down to 1, add  $v$  to the list if it is not an eternal domination critical vertex of the subgraph induced by the vertices not listed, and add any vertex to the list if no such  $v$  exists. It is expected that testing whether a graph is vertex-critical is difficult in general, so the enumeration is not expected to be easy for arbitrary graphs. However, it may be very easy to find for special classes like chordal graphs: by Proposition 11 no cover of a simplicial vertex  $x$  (a vertex  $y$  such that  $N[y] \supseteq N[x]$ ) can be an eternal domination critical vertex.

#### 4. REACHABILITY

We begin this section by showing that any guard can move eventually. That is, for every minimum eternal dominating set with a guard on  $v$ , it is possible to reach a minimum eternal dominating set with no guard on  $v$ .

**Theorem 15.** *Let  $G$  be a graph with no isolated vertices and  $D$  a minimum eternal dominating set of  $G$ . For every vertex  $u \in D$ , there exists a minimum eternal dominating set  $D'$  of  $G$  such that (i)  $u \notin D'$  and (ii)  $D'$  is reachable from  $D$ .*

**Proof.** Suppose that  $u$  belongs to every eternal dominating set reachable from  $D$ . Then  $u$  is an eternal domination critical vertex, and  $D - u$  is an eternal dominating set of  $G - u$ .

Let  $x \in N(u)$ . Since  $D_x = (D - \{u\}) \cup \{x\}$  is not an eternal dominating set, there is a sequence of attacks,  $S = v_1, v_2, \dots, v_k$ , which cannot be defended. We will derive a contradiction.

Let  $S_{x,u}$  be the subsequence of  $S$  with all attacks at  $x$  and  $u$  deleted. Since the elements of  $S_{x,u}$  are vertices of  $G - u$  and  $D_x - u \supseteq D - u$  is an eternal dominating set of  $G - u$ , this sequence of attacks can be defended without ever moving the guard from  $x$ . But then  $S$  can also be defended — any attack at  $u$  or  $x$  can be defended by the guard initially located at  $x$  — a contradiction. ■

A fundamental question is whether, for every minimum eternal dominating set and for each guard with an unoccupied neighbor, there is a single attack which can be defended by that guard. We state this as a conjecture, and then identify two graph classes for which the conjecture holds.

**Conjecture 16.** *Let  $G$  be a graph with no isolated vertices. Let  $D$  be a minimum eternal dominating set of  $G$ . For every vertex  $u \in D$  with an unoccupied neighbor, there exists an eternal dominating set  $D'$  with  $|D| = |D'|$  such that  $D' = (D - \{u\}) \cup \{v\}$ , where  $v \notin D$  and  $v \in N(u)$ .*

If Conjecture 16 is true, it would follow that the eviction number of any graph is less than or equal to  $\gamma^\infty(G)$ , which is a problem stated in [8, 10]. The *eviction number* is the analog of the eternal domination number in which attacks occur at vertices with guards and the guard at an attacked vertex must move to a neighboring vertex containing no guard, if one exists (otherwise the guard need not move).

**Lemma 17.** *Let  $D$  be an eternal dominating set of the graph  $G$  and  $u \in D$ . If  $N(u) \subseteq N(x)$  for all  $x \in D - \{u\}$ , then any attack at a neighbor of  $u$  can be defended by the guard at  $u$ , and the resulting configuration of guards is an eternal dominating set.*

**Proof.** Any configuration of guards reachable subsequent to the attack being defended by a guard located at  $x \in D - \{u\}$  is reachable if it is defended by the guard located at  $u$ . ■

**Theorem 18.** *Let  $G$  be a graph with no isolated vertices and  $\alpha(G) = 2$ . Let  $D$  be a minimum eternal dominating set of  $G$ . For every vertex  $u \in D$  with an unoccupied neighbor, there exists an eternal dominating set  $D'$  with  $|D| = |D'|$  such that  $D' = (D - \{u\}) \cup \{v\}$ , where  $v \notin D$  and  $v \in N(u)$ .*

**Proof.** Since  $\alpha(G) = 2$ , we know that  $2 \leq \gamma^\infty(G) \leq 3$ , see [3] or [10]. Thus there are two cases to consider.

Suppose first  $\gamma^\infty(G) = 2$ . Let  $D = \{u, x\}$ ; place guards on these two vertices. Suppose to the contrary that the guard on  $u$  cannot move in response to an attack on one of its neighbors. Then  $u$  has no external private neighbors, as any attack on one of them forces  $u$  to move. Let  $W = N(u) \cap N(x)$ . Since we are assuming  $u$  has no external private neighbors, in fact,  $W = N(u)$ . The result then follows from Lemma 17.

Now suppose  $\gamma^\infty(G) = 3$ . Let  $D = \{u, x_1, x_2\}$ ; place guards on these three vertices. Suppose to the contrary that the guard on  $u$  cannot move in response to an attack on one of its neighbors. As above, the vertex  $u$  has no external private neighbors.

Suppose  $D$  does not induce a  $K_3$ . If  $x_1$  and  $x_2$  are independent, then the guard at  $u$  can move to any adjacent attacked vertex and the resulting configuration of guards is an eternal dominating set, via the strategy given in Theorem 4 of [3] which maintains guards on three vertices that do not induce a  $K_3$ . So suppose  $x_1x_2 \in E$ . Then using the same strategy from [3], the guard at  $u$  can defend an attack at any vertex  $y$  such that  $y$  is not adjacent to both of  $x_1$  and  $x_2$ . If no such  $y$  exists, then both  $N(x_1)$  and  $N(x_2)$  contain  $N(u)$  and the result follows from Lemma 17.

Suppose  $D$  induces a  $K_3$ . If  $u$  has a neighbor  $y$  such that  $y \notin N(x_1)$  or  $y \notin N(x_2)$  then by moving the guard from  $u$  to  $y$ , the configuration used in Theorem 4 of [3] can be achieved and we are done, since any such configuration is an eternal dominating set (noting that any independent set of size two is a dominating set). If there is no such  $y$ , the result follows from Lemma 17. ■

**Proposition 19.** *Let  $G$  be a graph with no isolated vertices and  $\gamma^\infty(G) = \theta(G)$ . Then Conjecture 16 holds for  $G$ .*

**Proof.** When  $\gamma^\infty(G) = \theta(G)$ , a strategy to defend the graph is to use exactly one guard in each clique from a minimum clique covering. Since  $G$  has no isolated vertices, if there exists a clique in a minimum clique covering  $C$  consisting of a single vertex  $v$ , then there exists a minimum clique covering  $C'$  in which  $v$  is in a clique with more than one vertex. Thus using  $C'$  to define our guard strategy, if  $v$  is occupied, there is an attack, namely the other vertex in the same clique as  $v$ , which will allow the guard from  $v$  to move. ■

Many graph classes with  $\gamma^\infty(G) = \theta(G)$  are known, for example perfect graphs and series-parallel graphs; see [10].

## 5. EDGE-CRITICALITY

In the final section of the paper, we consider the effect on the eternal domination number of adding (or deleting) an edge.

**Proposition 20.** *For any edge  $e$  of a graph  $G$ ,*

$$\gamma^\infty(G) \leq \gamma^\infty(G - e) \leq \gamma^\infty(G) + 1.$$

**Proof.** The left-hand inequality is clear. We prove the right-hand inequality.

Let  $e = xy$ . We claim there is an eternal dominating set of size  $\gamma^\infty(G)$  for which there is a guard on  $x$  or  $y$ , but not both. Otherwise,  $G$  can be defended in such a way that there is always a guard on  $x$  and a guard on  $y$ . But then one fewer guard suffices: a single guard can defend all attacks that occur at  $x$  or  $y$ , and the remaining guards can defend all attacks on the rest of  $G$ , a contradiction.

Hence, without loss of generality, let  $D$  be an eternal dominating set of  $G$  for which there is a guard on  $x$  and no guard on  $y$ . The graph  $G - e$  can be defended by  $1 + |D|$  guards using the following strategy. The initial configuration of these guards is  $D \cup \{y\}$ . The guards not on  $y$  move as if defending  $G$ , and the guard on  $y$  remains stationary. Since we can assume  $y$  is never attacked in  $G - e$  (it has a guard), the edge  $e$  would never be used in defending the sequence of attacks in  $G$ . It follows that all sequences of attacks in  $G - e$  can be defended. ■

An edge  $e$  of a graph  $G$  is called *eternal domination critical* if  $\gamma^\infty(G - e) = \gamma^\infty(G) + 1$ . If every edge of  $G$  is eternal domination critical, then we say that  $G$  is *eternal domination critical graph with respect to edge deletion*.

Any graph in which every component is a clique is eternal domination critical graph with respect to edge deletion. It is easy to see that these are the only ones with  $\gamma^\infty = \theta$ : any edge with ends in different cliques of a minimum clique cover cannot be an eternal domination critical edge. That is, choose a minimum clique cover for graph  $G$  that is eternal domination critical graph with respect to edge deletion. An edge not in this minimum clique cover is not critical as its deletion does not increase  $\theta$ . Thus every edge is in one of the cliques. It follows right away that  $G$  is a union of cliques.

Characterizing the graphs that are eternal domination critical graph with respect to edge deletion seems challenging. We show that such graphs exist besides disjoint unions of cliques. Let  $G$  be a non-empty graph such that  $\gamma^\infty(G) < \theta(G)$ . Such graphs exist [3]. Then, the graph  $G$  has a spanning subgraph,  $H$ , which is critical with respect to edge removal and has the same eternal domination

number as  $G$ : keep deleting edges of  $G$  that are not critical until eventually there are no more. This situation must arise because each such edge deletion does not change the eternal domination number. The graph  $H$  is eternal domination critical with respect to edge deletion and satisfies

$$\gamma^\infty(H) = \gamma^\infty(G) < \theta(G) \leq \theta(H),$$

so it cannot be a disjoint union of cliques.

**Corollary 21.** *For any non-adjacent vertices  $x$  and  $y$  of a graph  $G$ ,*

$$\gamma^\infty(G) - 1 \leq \gamma^\infty(G + xy) \leq \gamma^\infty(G).$$

**Proof.** The right hand inequality is clear. The left-hand inequality follows from the upper inequality in Proposition 20. ■

**Proposition 22.** *Let  $e = xy$  be an eternal domination critical edge of the graph  $G$ . Then neither  $x$  nor  $y$  is an eternal domination critical vertex of  $G$ .*

**Proof.** Without loss of generality  $x$  is an eternal domination critical vertex. Then any sequence of attacks at vertices of  $G$  can be defended in such a way that a guard remains stationary at  $x$ . Since  $e$  is never used in a guard move, the same strategy defends any sequence of attacks at vertices of  $G - e$ . Therefore  $\gamma^\infty(G - e) = \gamma^\infty(G)$ . ■

**Corollary 23.** *There is no graph that is both eternal domination vertex-critical and eternal domination critical with respect to edge removal.*

If  $\gamma^\infty(G + xy) = \gamma^\infty(G) - 1$  then the pair of vertices  $\{x, y\}$  is called an *eternal domination critical pair* of  $G$ . If every two non-adjacent vertices are an eternal domination critical pair, then we say that  $G$  is *eternal domination critical graph with respect to edge addition*.

**Observation 24.** *An edge  $e = xy$  is an eternal domination critical edge of  $G$  if and only if  $\{x, y\}$  is an eternal domination critical pair of  $G - xy$ .*

The following are examples of graphs which are eternal domination critical with respect to edge addition:  $K_n - e$ , ( $n \geq 2$ ), stars with at least three vertices, (more generally) the join of a complete graph and an independent set of size at least 2, and the join of a complete graph and the complement of an odd cycle. (Recall that the *join* of the disjoint graphs  $G_1$  and  $G_2$  is the graph constructed from  $G_1 \cup G_2$  by adding all possible edges with one end in  $V(G_1)$  and the other in  $V(G_2)$ .) Each of these examples has the property that  $\gamma^\infty = \theta$  and the addition of any edge decreases  $\theta$ . That is, they are graphs with  $\gamma^\infty = \theta$  which are complements of graphs in which the deletion of any edge increases the chromatic number.

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