SIGNED ROMAN EDGE $k$-DOMINATION IN GRAPHS

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Abstract

Let $k \geq 1$ be an integer, and $G = (V, E)$ be a finite and simple graph. The closed neighborhood $N_G[e]$ of an edge $e$ in a graph $G$ is the set consisting of $e$ and all edges having a common end-vertex with $e$. A signed Roman edge $k$-dominating function (SRE$k$DF) on a graph $G$ is a function $f : E \rightarrow \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge $e$ of $G$, $\sum_{x \in N_G[e]} f(x) \geq k$ and (ii) every edge $e$ for which $f(e) = -1$ is adjacent to at least one edge $e'$ for which $f(e') = 2$. The minimum of the values $\sum_{e \in E} f(e)$, taken over all signed Roman edge $k$-dominating functions $f$ of $G$ is called the signed Roman edge $k$-domination number of $G$, and is denoted by $\gamma'_{sR_k}(G)$. In this paper we initiate the study of the signed Roman edge $k$-domination in graphs and present some (sharp) bounds for this parameter.

Keywords: signed Roman edge $k$-dominating function, signed Roman edge $k$-domination number.

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1. Introduction

In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the open neighborhood $N_G(v) = N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $d_G(v) = d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The open neighborhood $N(e) = N_G(e)$ of an edge $e \in E$ is the set of all edges adjacent to $e$. Its closed neighborhood is $N[e] = N_G[e] = N_G(e) \cup \{e\}$. The degree of an edge $e \in E$ is $d_G(e) = d(e) = |N(e)|$. The minimum and maximum edge degree of a graph $G$ are denoted by $\delta_e = \delta_e(G)$ and $\Delta_e = \Delta_e(G)$, respectively. If $v$ is a vertex, then denote by $E(v)$ the set of edges incident with the vertex $v$. We write $K_n$ for a complete graph, $C_n$ for a cycle, $P_n$ for a path of order $n$ and $K_{1,n}$ for a star of order $n + 1$. A subdivided star, denoted $K_{1,n}^*$, is a star $K_{1,n}$ whose edges are subdivided once, that is each edge is replaced by a path of length 2 by adding a vertex of degree 2. The line graph of a graph $G$, written $L(G)$, is the graph whose vertices are the edges of $G$, with $ee' \in E(L(G))$ when $e = uv$ and $e' = vw$ in $G$. It is easy to see that $L(K_{1,n}) = K_n$, $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

A function $f : E \to \{-1,1\}$ is called a signed edge $k$-dominating function (SE$k$DF) of $G$ if $\sum_{x \in N[e]} f(x) \geq k$ for each edge $e \in E$. The weight of $f$, denoted $\omega(f)$, is defined to be $\omega(f) = \sum_{e \in E} f(e)$. The signed edge $k$-domination number $\gamma'_{sk}(G)$ is defined as $\gamma'_{sk}(G) = \min\{\omega(f) \mid f \text{ is an SE$k$DF of } G\}$. The signed edge $k$-domination number was first defined in [3].

A signed Roman $k$-dominating function (SR$k$DF) on a graph $G$ is a function $f : V \to \{-1,1,2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V$, and (ii) every vertex $u$ for which $f(u) = -1$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an SR$k$DF $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The signed Roman $k$-domination number of $G$, denoted $\gamma'_{skR}$, is the minimum weight of an SR$k$DF in $G$. The signed Roman $k$-domination number was introduced by Henning and Volkmann in [5] and has been studied in [6]. The special case $k = 1$ was introduced and investigated in [1].

A signed Roman edge $k$-dominating function (SRE$k$DF) on a graph $G$ is a function $f : E \to \{-1,1,2\}$ satisfying the conditions that (i) for every edge $e$ of $G$, $\sum_{x \in N[e]} f(x) \geq k$ and (ii) every edge $e$ for which $f(e) = -1$ is adjacent to at least one edge $e'$ for which $f(e') = 2$. The weight of an SRE$k$DF is the sum of its function values over all edges. The signed Roman edge $k$-domination number of $G$, denoted $\gamma'_{sRk}(G)$, is the minimum weight of an SRE$k$DF in $G$. For an edge $e$, we denote $f[e] = f(N[e]) = \sum_{x \in N[e]} f(x)$ for notational convenience. The special case $k = 1$ was introduced by Ahangar et al. [2]. If $G_1, G_2, \ldots, G_s$ are the components of $G$, then
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(1) \[ \gamma'_{sRk}(G) = \sum_{i=1}^{s} \gamma'_{sRk}(G_i). \]

Since assigning a weight 1 to every edge of $G$ produces an SRE$k$DF, we have

(2) \[ \gamma'_{sRk}(G) \leq |E(G)|. \]

The signed Roman edge $k$-domination number exists if $|N_G(e)| \geq k - 1$ for every edge $e \in E$. However, for investigations of the signed Roman edge $k$-domination number it is reasonable to claim that for every edge $e \in E$, $|N_G(e)| \geq k - 1$. Thus we assume throughout this paper that $\delta_e(G) \geq k - 1$.

In this note we initiate the study of the signed Roman edge $k$-domination in graphs and present some (sharp) bounds for this parameter. In addition, we determine the signed Roman edge $k$-domination number of some classes of graphs.

The proof of the following results can be found in [5].

**Proposition 1.** If $k = 1$, then $\gamma^1_{sR}(K_3) = 2$ and $\gamma^1_{sR}(K_n) = 1$ for $n \neq 3$. If $n \geq 2$, then $\gamma^k_{sR}(K_n) = k$.

The case $k = 1$ in Proposition 1 was proved in [1]. A set $S \subseteq V$ is a 2-packing set of $G$ if $N[u] \cap N[v] = \emptyset$ holds for any two distinct vertices $u, v \in S$. The 2-packing number of $G$, denoted $\rho(G)$, is defined as follows:

\[ \rho(G) = \max\{|S| \mid S \text{ is a 2-packing set of } G\}. \]

**Proposition 2.** If $G$ is a graph of order $n$ with $\delta \geq k - 1$, then

\[ \gamma^k_{sR}(G) \geq (\delta + k + 1)\rho(G) - n. \]

**Proposition 3.** \[ \gamma^2_{sR}(P_n) = \begin{cases} n & \text{if } 1 \leq n \leq 7, \\ \left\lceil \frac{2n+5}{3} \right\rceil & \text{if } n \geq 8. \end{cases} \]

**Proposition 4.** For $n \geq 3$, we have $\gamma^2_{sR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$.

The proof of the following result is straightforward and therefore omitted.

**Observation 5.** For any nonempty graph $G$ of order $n \geq 2$ and any integer $k \geq 1$,

\[ \gamma'_{sRk}(G) = \gamma^k_{sR}(L(G)). \]

**Observation 6.** Let $G$ be a graph and $f$ be a $\gamma'_{sR2}(G)$-function. If $e = uv$ is a pendant edge in $G$ with $d(v) = 2$ and $w \in N(v) \setminus \{u\}$, then $\min\{f(uv), f(vw)\} \geq 1$. 

Observation 5 and Propositions 1, 2, 3 and 4 lead to

**Corollary 7.** If \( k = 1 \), then \( \gamma'_{sR1}(K_{1,3}) = 2 \) and \( \gamma'_{sR1}(K_{1,n}) = 1 \) for \( n \neq 3 \). If \( n \geq k \geq 2 \), then \( \gamma'_{sRk}(K_{1,n}) = k \).

**Corollary 8.** Let \( G \) be a graph of size \( m \). Then

\[
\gamma'_{sRk}(G) \geq (2\delta + k - 1)\rho(L(G)) - m.
\]

**Corollary 9.** \( \gamma'_{sR2}(P_n) = \begin{cases} 
\frac{n-1}{2} & \text{if } 2 \leq n \leq 8, \\
\left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{if } n \geq 9.
\end{cases} \)

**Corollary 10.** For \( n \geq 3 \), we have \( \gamma'_{sR2}(C_n) = \left\lceil \frac{2n}{3} \right\rceil + \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \).

Next we show that for every two positive integers \( k \) and \( t \), there exists a connected graph \( G \) whose signed Roman edge \( k \)-domination number is at most \(-t\).

**Proposition 11.** For every positive integers \( k \) and \( t \), there exists a connected graph \( G \) such that \( \gamma'_{sRk}(G) \leq -t \).

**Proof.** Let \( n \geq \max\{k + 5, t/3\} \), and let \( G \) be the graph obtained from the complete graph \( K_n \) by adding \( n + 2 \) pendant edges at each vertex of \( K_n \). Define \( f : E(G) \to \{-1, 1, 2\} \) by \( f(e) = 2 \) if \( e \in E(K_n) \) and \( f(e) = -1 \) otherwise. Obviously, \( f \) is an SRE\(k\)DF on \( G \) of weight \(-3n\). This completes the proof. \( \blacksquare \)

We close this section by determining the signed Roman edge \( k \)-domination number of two classes of graphs.

**Example 12.** For \( n \geq 2 \), \( \gamma'_{sR2}(K_{2,n}) = \begin{cases} 
4 & \text{if } n = 2, \\
5 & \text{if } n = 3, 4, \\
6 & \text{otherwise}.
\end{cases} \)

**Proof.** Let \( X = \{u_1, u_2\} \) and \( Y = \{v_1, v_2, \ldots, v_n\} \) be the partite sets of \( K_{2,n} \) and let \( f \) be a \( \gamma'_{sR2}(K_{2,n}) \)-function such that \( r = \min\{\sum_{i=1}^{n} f(u_1 v_i), \sum_{i=1}^{n} f(u_2 v_i)\} \) is as small as possible. Assume that \( r = \sum_{i=1}^{n} f(u_1 v_i) \). The result is immediate for \( n = 2 \) by Corollary 10. Assume that \( n \geq 3 \). Since \( f[u_1 v_1] = f(u_2 v_1) + \sum_{i=1}^{n} f(u_1 v_i) \geq 2 \), we have \( \sum_{i=1}^{n} f(u_1 v_i) \geq 0 \). Consider three cases.

**Case 1.** \( n \geq 5 \). Define \( g : E(K_{2,n}) \to \{-1, 1, 2\} \) by \( g(u_1 v_1) = g(u_2 v_1) = 2 \), \( g(u_1 v_2) = g(u_2 v_1) = 1 \) and \( g(u_1 v_3) = (-1)^i, g(u_2 v_i) = (-1)^{i+1} \) for \( 3 \leq i \leq n \). Obviously, \( g \) is an SRE\(2\)DF of \( K_{2,n} \) of weight 6 and so \( \gamma'_{sR2}(K_{2,n}) \leq 6 \). Now, we show that \( \gamma'_{sR2}(K_{2,n}) = 6 \). If \( r \geq 3 \), then we obtain \( \gamma'_{sR2}(K_{2,n}) = r + \sum_{i=1}^{n} f(u_2 v_i) \geq 6 \) implying that \( \gamma'_{sR2}(K_{2,n}) = 6 \). Assume that \( r \leq 2 \). If \( r = 0 \), then we deduce from \( f[u_1 v_1] = f(u_2 v_1) + \sum_{i=1}^{n} f(u_1 v_i) \geq 2 \) that \( f(u_2 v_i) \geq 2 \) for each \( i \) and hence \( \gamma'_{sR2}(K_{2,n}) = r + \sum_{i=1}^{n} f(u_2 v_i) = 2n > 6 \), a contradiction.
Thus \( r = 1 \) or \( r = 2 \). Then it follows from \( f[u_1v_i] = f(u_2v_i) + \sum_{i=1}^{n} f(u_1v_i) \geq 2 \) that \( f(u_2v_i) \geq 1 \) for each \( i \). Hence, \( \gamma'_s(K_{2,n}) = \sum_{i=1}^{n} f(u_1v_i) + \sum_{i=1}^{n} f(u_2v_i) \geq 1 + n \geq 6 \) that implies \( \gamma'_s(K_{2,n}) = 6 \).

Case 2. \( n = 3 \). Define \( g : E(K_{2,n}) \to \{-1, 1, 2\} \) by \( g(u_1v_2) = 2, g(u_1v_3) = -1, g(u_1v_1) = 1 \) and \( g(u_2v_i) = 1 \) for \( 1 \leq i \leq 3 \). Obviously, \( g \) is an SRE2DF of \( K_{2,3} \) of weight 5 and hence \( \gamma'_s(K_{2,3}) \leq 5 \). Now we show that \( \gamma'_s(K_{2,3}) = 5 \). Since \( \gamma'_s(K_{2,3}) = r + \sum_{i=1}^{3} f(u_2v_i) \leq 5 \), we have \( r \leq 2 \). If \( r = 2 \), then it follows from \( f[u_1v_1] = f(u_2v_2) + r \geq 2 \) that \( f(u_2v_i) \geq 1 \) for each \( i = 1, 2, 3 \). Hence, \( \gamma'_s(K_{2,3}) = r + \sum_{i=1}^{3} f(u_2v_i) \geq 5 \) that implies \( \gamma'_s(K_{2,3}) = 5 \). If \( r = 0 \), then as above we must have \( f(u_2v_i) = 2 \) for each \( i \). But then \( \gamma'_s(K_{2,3}) = r + \sum_{i=1}^{3} f(u_2v_i) = 6 \), a contradiction. Let \( r = 1 \). We may assume without loss of generality, that \( f(u_1v_1) = -1 \) and \( f(u_1v_2) = f(u_1v_3) = 1 \). It follows from \( f[u_1v_1] = f(u_2v_2) + \sum_{i=1}^{3} f(u_1v_i) = f(u_2v_2) + 1 \geq 2 \) that \( f(u_2v_i) \geq 1 \) for each \( i \). Since \( u_1v_1 \) must be adjacent to an edge with label 2, we have \( \sum_{i=1}^{3} f(u_2v_i) \geq 4 \) implying that \( \gamma'_s(K_{2,3}) = 5 \).

Case 3. \( n = 4 \). Define \( g : E(K_{2,4}) \to \{-1, 1, 2\} \) by \( g(u_1v_1) = 2, g(u_1v_2) = g(u_1v_3) = -1 \) and \( g(u_2v_1) = g(u_2v_2) = 1 \) for \( 1 \leq i \leq 4 \). Obviously, \( g \) is an SRE2DF of \( K_{2,4} \) of weight 5 and hence \( \gamma'_s(K_{2,4}) \leq 5 \). Using an argument similar to that described in Case 2, we obtain \( \gamma'_s(K_{2,4}) = 5 \) and the proof is complete.

A leaf of a tree \( T \) is a vertex of degree 1, a support vertex is a vertex adjacent to a leaf. For \( r, s \geq 1 \), a double star \( S(r, s) \) is a tree with exactly two vertices that are not leaves, with one adjacent to \( r \) leaves and the other to \( s \) leaves.

**Example 13.** For positive integers \( r \geq s \geq k-1 \geq 1 \),

\[
\gamma'_s(R_k)(S(r, s)) = \begin{cases} 
3 & \text{if } s = 1, \\
2k - 2 & \text{if } s \geq 2.
\end{cases}
\]

**Proof.** Let \( u \) and \( v \) be the central vertices of \( S(r, s) \) and let \( N(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_r\} \) and \( N(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_s\} \). Suppose that \( f \) is a \( \gamma'_s(R_k)(S(r, s)) \)-function. Consider two cases.

Case 1. \( s = 1 \). By assumption, we have \( k = 2 \). We deduce from \( f[vv_1] = f(vv_1) + f(uv) \geq 2 \) that \( f(vv_1) \geq 1 \). Hence,

\[
\gamma'_s(R_k)(S(r, s)) = f(vv_1) + f(uu_1) = 1 + f(uu_1) \geq 3.
\]

If \( r = 1 \), then define \( f : E(S(r, s)) \to \{-1, 1, 2\} \) by \( f(x) = 1 \) for each \( x \in E(S(r, s)) \). If \( r \) is even, then define \( f : E(S(r, s)) \to \{-1, 1, 2\} \) by \( f(vv_1) = 1, f(uv) = 2 \) and \( f(uu_i) = (-1)^i \) for \( 1 \leq i \leq r \), and if \( r \geq 3 \) is odd, then define \( f : E(S(r, s)) \to \{-1, 1, 2\} \) by \( f(vv_1) = 1, f(uv) = f(uu_1) = 2, f(uu_2) = f(uu_3) = 2 \).
−1 and \( f(uu_i) = (-1)^i \) for \( i \geq 4 \). Clearly, \( f \) is an SRE\( k \)DF of \( S(r, s) \) of weight 3 and so \( \gamma'_{sRk}(S(r, s)) = 3 \).

Case 2. \( s \geq 2 \). We have \( \gamma'_{sRk}(S(r, s)) = f[uu_1]+f[vv_1]−f(uv) \geq 2k−f(uv) \geq 2k−2 \). To prove \( \gamma'_{sRk}(S(r, s)) \leq 2k−2 \), we distinguish the following subcases.

Subcase 2.1. \( r−k+2 \) and \( s−k+2 \) are even. Define \( f : E(S(r, s)) \to \{-1, 1, 2\} \) by \( f(uv) = 2, f(uu_i) = f(vv_i) = 1 \) for \( 1 \leq i \leq k−2 \), \( f(uu_i) = (-1)^i \) for each \( k−1 \leq i \leq r \) and \( f(vv_j) = (-1)^j \) for each \( k−1 \leq j \leq s \). Obviously, \( f \) is an SRE\( k \)DF of \( S(r, s) \) of weight \( 2k−2 \) and so \( \gamma'_{sRk}(S(r, s)) = 2k−2 \).

Subcase 2.2. \( r−k+2 \) and \( s−k+2 \) are odd. Define \( f : E(S(r, s)) \to \{-1, 1, 2\} \) by \( f(uv) = f(uu_1) = f(vv_1) = 2, f(uu_2) = f(vv_2) = −1, f(uu_i) = f(vv_i) = 1 \) for \( 3 \leq i \leq k−1 \), \( f(uu_i) = (-1)^i \) for each \( i \geq k \) and \( f(vv_j) = (-1)^j \) for each \( j \geq k \).

Clearly, \( f \) is an SRE\( k \)DF of \( S(r, s) \) of weight \( 2k−2 \) and so \( \gamma'_{sRk}(S(r, s)) = 2k−2 \). This completes the proof.

2. Trees

In this section we first present a lower bound on the signed Roman edge \( k \)-domination number of trees and then we characterize all extremal trees.

**Theorem 14.** Let \( k \geq 2 \) be an integer and \( T \) be a tree of order \( n \geq k \). Then \( \gamma'_{sRk}(T) \geq k \). Moreover, this bound is sharp for stars.

**Proof.** We proceed by induction on \( n \). The base step handles trees with few vertices or diameter 2 and 3. If \( \text{diam}(T) \leq 3 \), then by Corollary 7 and Example 13, we have \( \gamma'_{sRk}(T) \geq k \). Assume that \( T \) is an arbitrary tree of order \( n \) and that the statements holds for all trees of order less than \( n \). We may assume, that \( \text{diam}(T) \geq 4 \). Let \( f \) be a \( \gamma'_{sRk}(T) \)-function.

If \( T \) has a non-pendant edge \( e = u_1u_2 \) with \( f(u_1u_2) = -1 \), then let \( T−u_1u_2 = T_1∪T_2 \) where \( T_i \) is the component of \( T−u_1u_2 \) containing \( u_i \) for \( i = 1, 2 \). It is easy to verify that the function \( f \), restricted to \( T_i \) is an SRE\( k \)DF of \( T_i \) for \( i = 1, 2 \). It follows from the induction hypothesis that

\[
\gamma'_{sRk}(T) = f(E(T_1)) + f(E(T_2)) − 1 \geq \gamma'_{sRk}(T_1) + \gamma'_{sRk}(T_2) − 1 \geq 2k−1 > k.
\]

Henceforth, we may assume that every edge with label \(-1\) is a pendant edge.
Let \( P = u_1u_2 \cdots u_k \) be a diametral path in \( T \) such that \( d_T(u_2) \) is as large as possible. Root \( T \) at \( u_k \). Since \( f[u_1u_2] \geq k \), we have \( d_T(u_2) \geq \lceil \frac{k}{2} \rceil \). By assumption \( f(u_2u_3) \geq 1 \). Let \( T_1 \) and \( T_2 \) be the components of \( T - u_2u_3 \) containing \( u_2 \) and \( u_3 \), respectively. Assume that \( T'_1 \) is the tree obtained from \( T_1 \) by adding a new pendant edge \( u_2w \) and define \( f_1 : E(T'_1) \to \{-1, 1, 2\} \) by \( f_1(u_2w) = f(u_2u_3) \) and \( f_1(x) = f(x) \) otherwise. Clearly, \( f_1 \) is an SRE-kDF of \( T'_1 \) and by the induction hypothesis we have \( \omega(f_1) \geq k \). Consider two cases.

Case 1. \( k = 2 \). Let \( T'_2 \) be the tree obtained from \( T_2 \) by adding a new pendant edge \( u_3w_1 \) and define \( f_2 : E(T'_2) \to \{-1, 1, 2\} \) by \( f_2(u_3w_1) = f(u_2u_3) \) and \( f_2(x) = f(x) \) otherwise. Clearly, \( f_2 \) is an SRE2DF of \( T'_1 \) and by the induction hypothesis we have \( \omega(f_2) \geq 2 \). Since \( \omega(f) = \omega(f_1) + \omega(f_2) - f(u_2u_3) \), we have

\[
\gamma'_{sR_k}(T) = \omega(f_1) + \omega(f_2) - f(u_2u_3) \geq 4 - f(u_2u_3) \geq 2.
\]

Case 2. \( k \geq 3 \). Let \( T'_2 \) be the tree obtained from \( T_2 \) by adding \( \left\lfloor \frac{k-2}{2} \right\rfloor \) new pendant edges \( u_3w_i, \ldots, u_3w_{i+\left\lfloor \frac{k-2}{2} \right\rfloor} \). Clearly, \( |V(T'_2)| < n \). First let \( k \) be odd. Define \( f_2 : E(T'_2) \to \{-1, 1, 2\} \) by \( f_2(u_3w_i) = 2 \) for each \( i \) and \( f_2(x) = f(x) \) otherwise. It is easy to verify that \( f_2 \) is an SREkDF of \( T'_2 \) and by the induction hypothesis we have \( \omega(f_2) \geq k \). Now we have

\[
\gamma'_{sR_k}(T) = \omega(f) = \omega(f_1) + \omega(f_2) - (k - 2) \geq k + (\omega(f_2) - k) + 2 > k.
\]

Now let \( k \) be even. Define \( f_2 : E(T'_2) \to \{-1, 1, 2\} \) by \( f_2(u_3w_4) = f_2(u_3w_5) = 2 \) for each \( i \) and \( f_2(x) = f(x) \) otherwise. It is not hard to see that \( f_2 \) is an SREkDF of \( T'_2 \) and by the induction hypothesis we have \( \omega(f_2) \geq k \). Then

\[
\gamma'_{sR_k}(T) = \omega(f) = \omega(f_1) + \omega(f_2) - (k - 2) - (2 - f(u_3u_4)) \\
\geq k + (\omega(f_2) - k) + f(u_3u_4) > k.
\]

Using Corollary 7, Example 13 and a closer look at the proof of Theorem 14, we obtain the next result.

**Corollary 15.** If \( k \geq 3 \) and \( T \) is a tree of order \( n \geq k \), then \( \gamma'_{sR_k}(T) = k \) if and only if \( T \) is a star.

In what follows, we provide a constructive characterization of all trees \( T \) for which \( \gamma'_{sR_k}(T) = 2 \). To do this, we describe a procedure to build a family \( F \) that attains the bound in Theorem 14 when \( k = 2 \). First we define the following operations. Let \( F \) be the family of trees that:

1. contains \( P_2 \), and
2. is closed under the operations \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \), which extend the tree \( T \) by attaching a tree to the vertex \( y \in V(T) \), called the attacher.
Operation $T_1$. If $T \in \mathcal{F}$, $uv$ is a pendant edge with $d(u) = 1$, and there is a $\gamma'_s\mathcal{R}_2(T)$-function with $f(uv) = 2$ and either no $-1$-edge at $v$ or a $2$-edge at $v$ other than $uv$, then $T_1$ adds a pendant edge $vv'$.

Operation $T_2$. If $T \in \mathcal{F}$, $uv$ is a pendant edge with $d(u) = 1$, and there is a $\gamma'_s\mathcal{R}_2(T)$-function with $f(uv) = 1$, then $T_2$ adds a pendant edge $vv_1$.

Operation $T_3$. If $T \in \mathcal{F}$, $uv \in E(T)$, and there is a $\gamma'_s\mathcal{R}_2(T)$-function with $f(uv) = 2$, then $T_3$ adds two pendant edges $vv_1, vw_2$.

Lemma 16. If $T \in \mathcal{F}$, then $\gamma'_s\mathcal{R}_2(T) = 2$.

Proof. Let $T \in \mathcal{F}$ be obtained from a path $P_2$ by successive operations $T^1, T^2, \ldots, T^m$, where $T^i \in \{T_1, T_2, T_3\}$ if $m \geq 1$ and $T = P_2$ if $m = 0$. The proof is by induction on $m$. If $m = 0$, then clearly the statement is true. Let $m \geq 1$ and assume that the statement holds for all trees which are obtained from $P_2$ by applying at most $m - 1$ operations. Let $T_{m-1}$ be the tree obtained from $P_2$ by the first $m - 1$ operations $T^1, T^2, \ldots, T^{m-1}$. We consider the following cases.

Case 1. $T^m = T_1$. Assume that $uv \in T_{m-1}$ is a pendant edge with $d(u) = 1$, $f$ a $\gamma'_s\mathcal{R}_2(T)$-function with $f(uv) = 2$ such that either no $-1$-edge at $v$ or a $2$-edge at $v$ other than $uv$, and $T^m$ adds a pendant edge $vv'$. Define $g : E(T) \to \{-1, 1, 2\}$ by $g(uv) = g(vv') = 1$ and $g(x) = f(x)$ otherwise. Obviously, $g$ is an SRE2DF of $T = T_m$ of weight 2 and so $\gamma'_s\mathcal{R}_2(T) = 2$ by Theorem 14.

Case 2. $T^m = T_2$. Let $uv \in T_{m-1}$ be a pendant edge with $d(u) = 1$, $f$ a $\gamma'_s\mathcal{R}_2(T)$-function with $f(uv) = 1$, and $T^m$ adds a pendant edge $vw_1$. Then the function $g : E(T) \to \{-1, 1, 2\}$ defined by $g(uv) = 2, g(vw_1) = -1$ and $g(x) = f(x)$ otherwise, is an SRE2DF of $T = T_m$ of weight 2 that implies $\gamma'_s\mathcal{R}_2(T) = 2$ by Theorem 14.

Case 3. $T^m = T_3$. Let $uv \in T_{m-1}$, $f$ be a $\gamma'_s\mathcal{R}_2(T)$-function with $f(uv) = 2$, and $T^m$ adds two pendant edges $vw_1, vw_2$. Define $g : E(T) \to \{-1, 1, 2\}$ by $g(vw_1) = 1, g(vw_2) = -1$ and $g(x) = f(x)$ otherwise. Obviously, $g$ is an SRE2DF of $T = T_m$ of weight 2 implying that $\gamma'_s\mathcal{R}_2(T) = 2$. This completes the proof.

Theorem 17. Let $T$ be a tree of order $n \geq 2$. Then $\gamma'_s\mathcal{R}_2(T) = 2$ if and only if $T \in \mathcal{F}$.

Proof. By Lemma 16, we only need to prove that every tree $T$ with $\gamma'_s\mathcal{R}_2(T) = 2$ is in $\mathcal{F}$. We prove this by induction on $n$. If $n = 2$, then the only tree $T$ of order 2 and $\gamma'_s\mathcal{R}_2(T) = 2$ is $P_2 \in \mathcal{F}$. If $\text{diam}(T) = 2$, then $T$ is a star and obviously $T$ can be obtained from $P_2$ by applying Operations $T_1$ and $T_2$. Let $n \geq 4$ and assume that the statement holds for every tree of order less than $n$ with $\gamma'_s\mathcal{R}_2(T) = 2$. Let $T$ be a tree of order $n$ and $\gamma'_s\mathcal{R}_2(T) = 2$. We may assume that $\text{diam}(T) \geq 3$. 


Suppose $f$ is a $\gamma'_{sR2}(T)$-function. Then $f(v) = \sum_{e \in E(v)} f(e) \geq 2$ for every support vertex $v$.

**Claim 1.** $T$ has no non-pendant edge $e = u_1u_2$ with $f(u_1u_2) = -1$.

**Proof.** Assume, to the contrary, that $T$ has a non-pendant edge $e = u_1u_2$ such that $f(u_1u_2) = -1$. Assume $T - e = T_{u_1} \cup T_{u_2}$, where $T_{u_i}$ is the component of $T - e$ containing $u_i$, for $i = 1, 2$. Obviously, $\gamma'_{sR2}(T) = f(E(T_{u_1})) - 1 + f(E(T_{u_2}))$ and the function $f$, restricted to $T_{u_i}$ is an SRE2DF and hence $\gamma'_{sR2}(T_{u_i}) \leq f(E(T))$ for $i = 1, 2$. By Theorem 14, we get

$$\gamma'_{sR2}(T) \geq \gamma'_{sR2}(T_{u_1}) + \gamma'_{sR2}(T_{u_2}) - 1 \geq 3,$$

a contradiction. 

**Claim 2.** $T$ has no non-pendant edge with label 1.

**Proof.** Assume, to the contrary, that $T$ has a non-pendant edge $e = u_1u_2$ such that $f(u_1u_2) = 1$. Let $T_{u_1}$ and $T_{u_2}$ be the components of $T - e$ containing $u_1$ and $u_2$, respectively, and let $T_{u_i}'$ be the tree obtained from $T_{u_i}$ by adding a new pendant edge $u_iu_i'$ for $i = 1, 2$. Define $f_i : E(T_i') \rightarrow \{-1, 1, 2\}$ by $f_i(u_iu_i') = 1$ and $f_i(e) = f(e)$ if $e \in E(T_i)$, for $i = 1, 2$. Clearly, $f_i$ is an SRE2DF of $T_i'$ for each $i$, and $\omega(f) = \omega(f_1) + \omega(f_2) - 1$. Similar to Case 2, we can get the contradiction $\gamma'_{sR2}(T) = \omega(f_1) + \omega(f_2) - 1 \geq 3$.

Thus, all $-1$-edges and 1-edges are pendant edges and hence all non-pendant edges are 2-edges.

Let $v_1v_2 \cdots v_D$ be a diametral path in $T$ and root $T$ at $v_D$. Obviously, $d(v_1) = d(v_D) = 1$.

**Claim 3.** $d(v_2) \geq 3$.

**Proof.** Assume, to the contrary, that $d(v_2) = 2$. By Observation 6, we have $f(v_1v_2) \geq 1$. If there is a pendant $-1$-edge at $v_3$, then let $T' = T - v_1$. It is easy to see that the function $h = f|_{E(T')} \geq f|_{E(T')} \geq \omega(f)$, and it follows from Theorem 14 that $\gamma'_{sR2}(T) = \omega(f) \geq \omega(f|_{E(T)}) \geq \gamma'_{sR2}(T') \geq 2$. Assume that there is no pendant $-1$-edge at $v_3$. Let $T' = T - v_1$. Since $f(v_1v_2) \geq 1$, we have $\omega(f) \geq \omega(f|_{E(T)}) + 1$ and the function $f$ restricted to $T'$ is an SRE2DF of $T'$. This implies $\gamma'_{sR2}(T) > 2$ which is a contradiction.

Now we consider three cases.

**Case 1.** $T$ has two pendant edges $v_2u_1$ and $v_2u_2$ with $f(v_2u_1) = 1$ and $f(v_2u_2) = -1$. Assume $T' = T - \{u_1, u_2\}$. Clearly, the function $f$ restricted to $T'$ is an SRE2DF on $T'$. So $\gamma'_{sR2}(T') = 2$ and by the induction hypothesis $T' \in \mathcal{F}$. Obviously $T$ can be obtained from $T'$ by operation $\Sigma_3$. Thus $T \in \mathcal{F}$.  


Case 2. \( T \) has two pendant edges \( v_2u_1 \) and \( v_2u_2 \) with \( f(v_2u_1) = 2 \) and \( f(v_2u_2) = -1 \). Since \( T \) is not a star, we deduce that there is an edge \( v_2v_3 \) such that \( f(v_2v_3) = 2 \) and \( v_3 \neq u_1 \). Assume that \( T' = T - \{u_1\} \) and define \( g : E(T') \rightarrow \{-1, 1, 2\} \) by \( f(v_2u_2) = 1 \) and \( g(e) = f(e) \) for \( e \in E(T') \setminus \{v_2u_2\} \). Obviously, \( g \) is an SRE2DF on \( T' \) of weight 2 and by the induction hypothesis we have \( T' \in \mathcal{F} \). Clearly, \( T \) can be obtained from \( T' \) by operation \( \Sigma_2 \). This implies \( T \in \mathcal{F} \).

Case 3. \( T \) has two pendant edges \( v_2u_1 \) and \( v_2u_2 \) with \( f(v_2u_1) = f(v_2u_2) = 1 \). Assume \( T' = T - \{u_1\} \) and define \( g : E(T') \rightarrow \{-1, 1, 2\} \) by \( g(v_2u_2) = 2 \) and \( g(e) = f(e) \) for \( e \in E(T') \setminus \{v_2u_2\} \). Obviously, \( g \) is an SRE2DF on \( T' \) of weight 2 and by the induction hypothesis we have \( T' \in \mathcal{F} \). Then \( T \) can be obtained from \( T' \) by operation \( \Sigma_1 \). Thus \( T \in \mathcal{F} \) and the proof is complete. \( \blacksquare \)

3. Bounds on the Signed Roman Edge \( k \)-Domination

In this section we establish some sharp bounds on the signed Roman edge \( k \)-domination number and we characterize all connected graphs whose signed Roman edge \( k \)-domination number is equal to their size.

Proposition 18. If \( G \) is a graph of size \( m \), then

\[
\gamma'_{sRk}(G) \geq k + \Delta + \delta - m - 1.
\]

This bound is sharp for stars \( K_{1,r} \) with \( r \neq 3 \) when \( k = 1 \).

Proof. Let \( f \) be a \( \gamma'_{sRk}(G) \)-function, \( v \) a vertex of maximum degree \( \Delta \) and \( u \in N(v) \). By definition \( f[uv] \geq k \) and the least possible weight for \( f \) will now be achieved if \( f(e') = -1 \) for each \( e' \in E(G) \setminus N[uv] \). Thus \( \gamma'_{sRk}(G) \geq k - [m - (d(u) + d(v) - 1)] \geq k - m + \Delta + \delta - 1 \).

Theorem 19. Let \( G \) be a graph of size \( m \). Then

\[
\gamma'_{sRk}(G) \geq \frac{m(2(\delta - \Delta) + k)}{2\Delta - 1}.
\]

Proof. Assume that \( g \) is a \( \gamma'_{sRk}(G) \)-function. Define \( f : E(G) \rightarrow \{0, 2, 3\} \) by \( f(e) = g(e) + 1 \) for each \( e \in E \). We have

\[
\sum_{e \in E(G)} f(N[e]) = \sum_{e = uv \in E(G)} (g(N[e]) + d(u) + d(v) - 1) \geq \sum_{e = uv \in E(G)} (g(N[e]) - 1) + 2m\delta = m(2\delta + k - 1).
\]


On the other hand,

\[ \sum_{e \in E(G)} f(N[e]) = \sum_{e=uv \in E(G)} (d(u) + d(v) - 1) f(e) \]

(4)

\[ \leq \sum_{e \in E(G)} (2\Delta - 1) f(e) = (2\Delta - 1) f(E(G)). \]

By (3) and (4), we have

\[ f(E(G)) \geq m \left( 2\delta + k - 1 \right) \frac{2\Delta - 1}{2\Delta - 1} - m, \]

as desired.

**Corollary 20.** For any \( r \)-regular graph \( G \), \( \gamma'_{sRk}(G) \geq \frac{km}{2r-1} \).

The special case \( k = 1 \) of Theorem 19 and Corollary 20 can be found in [2].

Corollary 10 shows that Corollary 20 is sharp for \( k = 2 \) and \( m \equiv 0 \pmod{3} \).

**Theorem 21.** Let \( G \) be a connected graph of size \( m \geq 2 \). Then

\[ \gamma'_{sRk}(G) \leq \frac{\gamma'_{sk}(G) + m}{2}. \]

**Proof.** Let \( f \) be a \( \gamma'_{sk}(G) \)-function, and let \( P = \{ e \mid f(e) = 1 \} \) and \( M = \{ e \mid f(e) = -1 \} = \{ e_1, e_2, \ldots, e_{|M|} \} \). Suppose \( e'_i \in P \) is an edge adjacent to \( e_i \) for each \( i \). Define \( g : E(G) \rightarrow \{-1, 1, 2\} \) by \( g(e'_i) = 2 \) for \( 1 \leq i \leq |M| \) and \( g(e) = f(e) \) otherwise. It is easy to see that \( g \) is an SREkDF on \( G \) of weight at most \( \gamma'_{sk}(G) + |M| \). It follows from \( \gamma'_{sk}(G) = |P| - |M| \) and \( m = |P| + |M| \) that

\[ |P| = \frac{\gamma'_{sk}(G) + m}{2} \]

and hence

\[ \gamma'_{sRk}(G) \leq \omega(g) \leq \gamma'_{sk}(G) + |M| = |P| = \frac{\gamma'_{sk}(G) + m}{2}, \]

as desired.

**Theorem 22.** Let \( G \) be a connected graph of order \( n \geq 3 \) and size \( m \). Then

\[ \gamma'_{sR2}(G) \geq 2(n - m). \]

Furthermore, this bound is sharp.

**Proof.** Let \( p \) be the number of cycles of \( G \). The proof is by induction on \( p \). The statement is true for \( p = 0 \) by Theorem 14. Assume the statement is true for all simple connected graphs \( G \) for which the number of cycles is less than \( p \), where \( p \geq 1 \). Let \( G \) be a simple connected graph with \( p \) cycles. Assume that \( f \) is a
\(\gamma'_{sR2}(G)\)-function and let \(e = uv\) be a non-cut edge. If \(f(e) = -1\), then obviously \(f|_{G-e}\) is an SRE2DF for \(G - e\) and by the induction hypothesis, we have

\[
2(n - m) < 2(n - (m - 1)) - 1 \leq f(E(G - e)) - 1 = f(E(G)) = \gamma'_{sR2}(G).
\]

Thus, we may assume that all non-cut edges are assigned 1 or 2 by \(f\). We consider two cases.

Case 1. \(f(uv) = 1\). Consider two subcases.

Subcase 1.1. \(f(E(u)) \leq 1\) (the case \(f(E(v)) \leq 1\) is similar). Then \(u\) has at least one neighbor \(u'\) such that \(f(uu') = -1\). Assume that \(G'\) is the graph obtained from \(G - \{uv, uu'\}\) by adding a new pendant edge \(vv'\). Define \(g : E(G') \to \{-1, 1, 2\}\) by \(g(vv') = 1, g(a) = f(a)\) for \(a \in E(G) \setminus \{uv, uu'\}\). Clearly, \(g\) is an SRE2DF for \(G'\) and it follows from the induction hypothesis and (1) that

\[
\omega(f) = -1 + \omega(g) \geq -1 + 2(n(G') - m(G')) = -1 + 2(n - (m - 1)) > 2(n - m).
\]

Subcase 1.2. \(f(E(u)) \geq 2\) and \(f(E(v)) \geq 2\). Let \(G'\) be the graph obtained from \(G - \{e\}\) by adding two new pendant edges \(vv'\) and \(uu'\) and define \(g : E(G') \to \{-1, 1, 2\}\) by \(g(vv') = g(uu') = 1\) and \(g(a) = f(a)\) otherwise. Clearly, \(g\) is an SRE2DF for \(G'\). It follows from the induction hypothesis that

\[
\omega(f) = -1 + \omega(g) \geq -1 + 2(n(G') - m(G')) = -1 + 2(n + 2 - (m + 1)) > 2(n - m).
\]

By Case 1, we may assume that all non-cut edges are assigned 2 by \(f\).

Case 2. \(f(uv) = 2\). Consider two subcases.

Subcase 2.1. \(f(E(u)) \leq 2\) (the case \(f(E(v)) \leq 2\) is similar). Then clearly \(f(E(v)) \geq 2\). Since all non-cut edges are assigned 2 by \(f\) (by assumption) and since \(uv\) belongs to a cycle in \(G\), it follows from \(f(E(u)) \leq 2\) that there are two \(-1\)-edges at \(u\), say \(e', e''\). Assume that \(G'\) is the graph obtained from \(G - \{e, e', e''\}\) by adding a new pendant edge \(vv'\) at \(v\). Define \(g : E(G') \to \{-1, 1, 2\}\) by \(g(vv') = 2\) and \(g(a) = f(a)\) otherwise. It is easy to see that \(g\) is an SRE2DF of \(G'\) and we deduce from the induction hypothesis and (1) that

\[
\omega(f) = -2 + \omega(g) \geq -2 + 2(n(G') - m(G')) = -2 + 2(n - 1 - (m - 2)) = 2(n - m).
\]

Subcase 2.2. \(f(E(u)) \geq 3\) and \(f(E(v)) \geq 3\). Let \(G'\) be the graph obtained from \(G - \{e\}\) by adding two new pendant edges \(vv'\) and \(uu'\). Define \(g : E(G') \to \{-1, 1, 2\}\) by \(g(vv') = g(uu') = 2\) and \(g(a) = f(a)\) otherwise. Clearly, \(g\) is an SRE2DF for \(G'\) and by the induction hypothesis, we obtain

\[
\omega(f) = -2 + \omega(g) \geq -2 + 2(n(G') - m(G')) = -2 + 2(n + 2 - (m + 1)) = 2(n - m).
\]

\[\blacksquare\]
Theorem 23. Let $k \geq 1$ be an integer, and let $G$ be a graph of size $m$ and minimum degree $\delta$. If $2\delta - k \geq 3$, then $\gamma_{sRk}'(G) \leq m - 1$.

**Proof.** Let $v \in V(G)$ be an arbitrary vertex, and let $u_1, u_2, \ldots, u_p$ be the neighbors of $v$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(uv_1) = -1$, $f(uv_2) = 2$ and $f(x) = 1$ otherwise. If $e = wz$ is an arbitrary edge, then $f[wz] \geq d(w) + d(z) - 3 \geq 2\delta - 3 \geq k$. Therefore $f$ is an SREkDF on $G$ of weight $m - 1$ and so $\gamma_{sRk}'(G) \leq m - 1$. 

Theorem 24. Let $k \geq 1$ be an integer, and let $G$ be a graph of size $m$ and minimum degree $\delta$. If $2\delta - k \geq 5$, then

$$\gamma_{sRk}'(G) \leq m - 2 \left\lfloor \frac{2\delta - k}{2} \right\rfloor + 1.$$ 

**Proof.** Let $t = \left\lfloor \frac{2\delta - k}{2} \right\rfloor$, and let $v \in V(G)$ be an arbitrary vertex. Now let $A = \{u_1, u_2, \ldots, u_t\}$ be a set of $t$ neighbors of $v$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(uv_i) = -1$ for $1 \leq i \leq t$, $f(uv_{t+1}) = 2$ and $f(x) = 1$ otherwise. Then $f[wz] \geq t + 1 + (d(v) - t) + (d(u) - 1) \geq 2\delta - 2t \geq k$ for $1 \leq i \leq d(v)$. If $e = wz$ is an edge different from $vu_i$, then $f[wz] \geq d(w) + d(z) - 5 \geq 2\delta - 5 \geq k$. Therefore $f$ is an SREkDF on $G$ of weight $m - 2t + 1$ and so $\gamma_{sRk}'(G) \leq m - 2t + 1$.

Theorem 25. Let $k \geq 1$ be an integer, and let $G$ be a graph of size $m$, minimum degree $\delta$ and maximum matching $M$. If $2\delta - k \geq 5$, then $\gamma_{sRk}'(G) \leq m - |M|$.

**Proof.** Let $M = \{e_1, e_2, \ldots, e_{|M|}\}$ be a maximum matching, and let $x_1, x_2, \ldots, x_t$ be a minimum edge set such that each $e_i$ is adjacent to an edge $x_j$ for $1 \leq i \leq |M|$ and $1 \leq j \leq t$. Then $t \leq |M|$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(e_i) = -1$ for $1 \leq i \leq |M|$, $f(x_j) = 2$ for $1 \leq j \leq t$ and $f(x) = 1$ otherwise. If $e = uv$ is an arbitrary edge of $G$, then $f[e] \geq d(u) + d(v) - 5 \geq 2\delta - 5 \geq k$. Therefore $f$ is an SREkDF on $G$ of weight $m - 2|M| + t \leq m - |M|$ and so $\gamma_{sRk}'(G) \leq m - |M|$. 

In what follows, we characterize all connected graphs attaining the bound in (2).

Theorem 26. Let $G$ be a connected graph of size $m \geq 2$. Then $\gamma_{sR2}'(G) = m$ if and only if $G = C_4$, $G = C_5$, $G = P_n$ ($3 \leq n \leq 8$) or $G$ is a subdivided star $K_{1,r}$ ($r \geq 1$).

**Proof.** If $G = C_4$, $G = C_5$, $G = P_n$ ($3 \leq n \leq 7$) or $G$ is a subdivided star $K_{1,r}$ ($r \geq 1$), then the result is immediate by Corollary 9 and Observation 6. Let $\gamma_{sR2}'(G) = m$. If $\Delta \leq 2$, then it follows from Corollaries 9 and 10 that $G = P_n$ ($3 \leq n \leq 8$) or $G = C_4$ or $G = C_5$. Assume that $\Delta \geq 3$.

Claim 1. $G$ has no support vertex of degree at least 3.
Proof. Let $G$ have a support vertex $u$ with $d(u) \geq 3$ and let $v, w \in N(u)$ where $d(v) = 1$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(uv) = -1$, $f(uw) = 2$ and $f(x) = 1$ for $x \in E(G) \setminus \{uv, uw\}$. Obviously, $f$ is an SRE2DF of weight less than $m$, a contradiction. 
\hfill \Box

Claim 2. $G$ is acyclic.

Proof. Let $C_g = (v_1v_2 \cdots v_g)$ be a cycle of $G$ of length $g = \text{girth}(G)$. Since $\Delta \geq 3$, we observe that $G \neq C_g$. By Claim 1, $v_i$ is not a support vertex for each $1 \leq i \leq g$. Since $G \neq C_g$, we may assume that $d(v_1) \geq 3$ and $u \in N(v_1) \setminus \{v_2, v_g\}$. Then the function $f : E(G) \to \{-1, 1, 2\}$ defined by $f(v_1v_2) = -1, f(v_2v_3) = 2$ and $f(x) = 1$ otherwise, is an SRE2DF of weight less than $m$, a contradiction. 
\hfill \Box

Claim 3. For each non pendant edge $e = uv$, $\min\{d(u), d(v)\} = 2$.

Proof. Let $e = uv$ be a non pendant edge of $G$ such that $\min\{d(u), d(v)\} \geq 3$. By Claim 1, both $u$ and $v$ are not support vertices. Let $v_1 \in N(v) \setminus \{u\}$ and define $f : E(G) \to \{-1, 1, 2\}$ by $f(vv_1) = 2, f(uv) = -1$ and $f(x) = 1$ otherwise. Clearly, $f$ is an SRE2DF of weight $m - 1$, a contradiction. 
\hfill \Box

Let $v$ be a vertex of maximum degree $\Delta$ and let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. By Claims 1 and 3, we deduce that $d(v_i) = 2$ for each $i$. If $v_i$ is a support vertex for each $i$, then $G = K^*_\Delta$ and we are done. Assume that $v_1$ is not a support vertex. Let $u \in N(v_1) \setminus \{v\}$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(vv_1) = -1$, $f(uv_1) = 2$ and $f(x) = 1$ otherwise. Clearly, $f$ is an SRE2DF of weight $m - 1$, a contradiction. This completes the proof.

We conclude this paper with an open problem.

Problem 27. Characterize all connected graphs $G$ of order $n$ and size $m$ attaining the bound of Theorem 22.

References


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