A FINITE CHARACTERIZATION AND RECOGNITION OF INTERSECTION GRAPHS OF HYPERGRAPHS WITH RANK AT MOST 3 AND MULTIPLICITY AT MOST 2 IN THE CLASS OF THRESHOLD GRAPHS

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Abstract
We characterize the class $L_2^3$ of intersection graphs of hypergraphs with rank at most 3 and multiplicity at most 2 by means of a finite list of forbidden induced subgraphs in the class of threshold graphs. We also give an $O(n)$-time algorithm for the recognition of graphs from $L_2^3$ in the class of threshold graphs, where $n$ is the number of vertices of a tested graph.

Keywords: intersection graph, hypergraph rank, hypergraph multiplicity, forbidden induced subgraph, threshold graph.

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1. Introduction

In this paper, we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively; $N(v) = N_G(v)$ is the neighborhood of a vertex $v$ in $G$ and $\text{deg}(v)$ is the degree of $v$; the subgraph of $G$ induced by a set $X \subseteq V(G)$ is denoted by $G(X)$. A vertex $v$ of a graph $G$ is called dominating if $N(v) \cup \{v\} = V(G)$.

The intersection graph $L(H)$ of a hypergraph $H$ is defined as follows:

1. the vertices of $L(H)$ are in a bijective correspondence with the edges of $H$;
2. two vertices are adjacent in $L(H)$ if and only if the corresponding edges have a non-empty intersection.

The rank of a hypergraph $H$ is the maximum size of its edges. The multiplicity of a pair of vertices $u, v$ of $H$ is the number of edges in $H$ containing both $u$ and $v$; the multiplicity $m(H)$ of $H$ is the maximum multiplicity among all pairs of vertices in $H$ (see for example [15]).

Denote by $L^m_r$ the class of intersection graphs of hypergraphs with rank at most $r$ and multiplicity at most $m$. So, we refer to $L^\infty_r$ as the class of intersection graphs of hypergraphs with rank at most $r$. The class $L^m_r$, where $r \geq 1$, $m \geq 1$ or $m = \infty$, is hereditary (i.e., every induced subgraph of a graph in $L^m_r$ is also in $L^m_r$). Therefore, it can be characterized by means of a list (finite or not) of forbidden induced subgraphs.

A non-trivial characterization of the class $L^m_r$ is known only for $r \leq 2$. These are:

- Beineke’s finite characterization of the class $L^1_2$ of line graphs (i.e., intersection graphs of simple graphs) [1];
- a finite characterization of the class $L^\infty_2$ of intersection graphs of multigraphs by Bermond and Meyer [2];
- a finite characterization of the class $L^m_2$ by Tashkinov [22].

Such finite characterizations of the classes above imply that there exist polynomial algorithms for recognizing graphs from these classes. (For efficient algorithms for recognizing graphs from $L^1_2$ see, e.g., [4, 11, 17, 19].) It is also known that for any $r \geq 3$ and $m$, where $m \geq 1$ or $m = \infty$, there does not exist a finite characterization for the class $L^m_r$ (see [6, 15, 16, 10]).

Poljak, Rödl and Turzik [18] proved that the problem of determining whether a graph belongs to $L^\infty_r$ is NP-complete for an arbitrary $r$. Moreover, they proved that for every fixed $r \geq 4$, the analogous problem remains NP-complete. The question whether or not the class $L^\infty_3$ can be recognized in polynomial time is still open, but recognizing intersection graphs of hypergraphs without multiple edges with rank at most 3 is NP-complete as well [18]. The following result generalizing one from [18] was obtained in [7]: For every fixed $m \geq 1$ and an arbitrary $r$, the problem of determining whether a graph belongs to $L^m_r$ is NP-complete.
Hliněný and Kratochvíl [8] proved that for every fixed \( r \geq 3 \), the problem of determining whether a graph belongs to \( L^1_r \) is NP-complete. The class \( L^1_r \) was studied in different papers, and several graph classes were found, where the problem of recognizing graphs from the class is polynomially solvable or remains NP-complete ([7, 9, 14, 15, 16, 21]).

A graph \( G \) is called **split** [5] if there exists a partition of its vertex set \( V(G) = A \cup B \) into a clique \( A \) and a stable set \( B \) (bipartition \((A, B)\)). It was proved in [12] that for every fixed \( r \), there exists a finite characterization of the graphs from \( L^1_r \) in the class of split graphs. In [13] (see also [7]), this result was generalized to the class \( L^m_r \) for every fixed \( m \).

A split graph with the bipartition \((A, B)\) is called **threshold** [3] if the vertices from \( B \) can be numbered as \( b_1, b_2, \ldots, b_k \) so that \( N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_k) \). In [20], the problem of determining the Krausz dimension of a graph (the minimum \( r \) such that the graph belongs to the class \( L^1_r \)) was solved in the subclass of threshold graphs of the form \( K_n - E(K_p) \).

In Section 2 of this paper, we give some preliminary facts (e.g., a so-called Krausz type characterization of the class \( L^2_3 \) in terms of clique coverings), prove some technical lemmas and formulate Theorem 2 that gives a finite characterization of the class \( L^2_3 \) (consisting of 15 graphs) in the class of threshold graphs. In Sections 3 and 4, we prove the necessity and sufficiency of Theorem 2, respectively. In Section 5 we give an \( O(n) \)-time algorithm for the recognition of graphs from \( L^2_3 \) in the class of threshold graphs, where \( n \) is the number of vertices of a tested graph.

### 2. Some Preliminaries and the Formulation of Theorem 2

A finite family \( \mathcal{C} = (C_1, C_2, \ldots, C_q) \) of cliques of the graph \( G \) is called a covering of \( G \) if every vertex as well as every edge of \( G \) is contained in some \( C_i \). The cliques \( C_i \) are the clusters of \( \mathcal{C} \). For a vertex \( v \in V(G) \), denote by \( \mathcal{C}(v) \) the subfamily of all clusters of \( \mathcal{C} \) that contain \( v \). A covering \( \mathcal{C} \) of the graph \( G \) is called an \((r, m)\)-covering if any vertex of \( G \) belongs to at most \( r \) clusters of \( \mathcal{C} \), and any two clusters of \( \mathcal{C} \) have at most \( m \) vertices in common.

**Theorem 1** [7, 13]. A graph \( G \) belongs to the class \( L^2_3 \) if and only if there exists a \((3, 2)\)-covering of \( G \).

A clique of a graph \( G \) is called maximal if it is not contained in some other clique of \( G \).

Let a threshold graph with the bipartition \((A, B)\) be given, where \( B = \{b_1, b_2, \ldots, b_k\} \) and \( N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_k) \). We denote such a graph by \( G(p, q_1, q_2, \ldots, q_k) \) if \(|A| = p \) and \( \deg(b_i) = q_i \) for any \( i = 1, 2, \ldots, k \). Without loss of generality (W.l.o.g.), we assume below that any threshold graph
with the bipartition \((A, B)\) satisfies the conditions 
\[ A = \{a_1, a_2, \ldots, a_p\}, \quad B = \{b_1, b_2, \ldots, b_k\}, \] 
\[ p > q \] 
and \( N(b_i) = \{a_1, a_2, \ldots, a_{q_i}\} \) for any \( i = 1, 2, \ldots, k \) (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The graph \( G(3, 2, 1) \) and its bipartition \((A, B)\).}
\end{figure}

In this paper, we characterize the class \( L_3^3 \) by means of a finite list of forbidden induced subgraphs in the class of threshold graphs:

**Theorem 2.** A threshold graph \( H \) belongs to the class \( L_3^3 \) if and only if it contains none of the graphs \( K_{1,4}, G(12, 7), G(11, 10), G(10, 9, 5), G(10, 9, 7), G(10, 9, 9), G(10, 7, k), k = 1, 2, \ldots, 7, G(9, 8, 1), G(9, 8, 2) \) as induced subgraphs.

Now we formulate some technical statements that will be used for proving Theorem 2.

A \((3, 2)\)-covering \( \mathcal{C} = (C_1, C_2, \ldots, C_t) \) of a complete graph \( G \) is called a decomposition \((3, 2)\)-covering if \( C_i \neq V(G) \) for any \( i = 1, 2, \ldots, t \).

**Lemma 3.** Let \( \mathcal{C} = (C_1, C_2, \ldots, C_t) \) be a decomposition \((3, 2)\)-covering of a complete graph \( G \). Then the following statements hold:

(i) \( |C_i| \leq 6 \) for any \( i = 1, 2, \ldots, t \).

(ii) If \( C_i \cap C_j \neq \emptyset \) for some \( i, j \in \{1, 2, \ldots, t\} \), then \( |C_j \setminus C_i| \leq 4 \).

(iii) If \( (C_i \cap C_j) \setminus C_k \neq \emptyset \) for some different \( i, j, k \in \{1, 2, \ldots, t\} \), then \( |C_k \setminus (C_i \cup C_j)| \leq 2 \).

**Proof.** (i) Let, to the contrary, \( C_i = \{a_1, a_2, \ldots, a_7, \ldots\} \) for some \( i \in \{1, 2, \ldots, t\} \). Consider a vertex \( v \in V(G) \setminus C_i \). By the definition of a \((3, 2)\)-covering, each cluster of \( \mathcal{C} \) contains at most two edges of \( va_s, s = 1, 2, \ldots, 7 \). Hence, the edges \( va_s, s = 1, 2, \ldots, 7 \), are covered by at least four clusters of \( \mathcal{C} \), and, therefore, the vertex \( v \) is contained in at least four clusters of \( \mathcal{C} \), which is a contradiction to the definition of \( \mathcal{C} \).
(ii) Assume, to the contrary, that for a vertex \( v \in V(G) \), we have \( v \in C_i \setminus C_j \) and \( C_j \setminus C_i = \{a_1, a_2, a_3, a_4, a_5, \ldots \} \). By the definition of a \((3, 2)\)-covering, the edges \( va_s, \ s = 1, 2, \ldots, 5 \), are covered by at least three clusters of \( \mathcal{C} \), different from \( C_i \). So, taking into account the cluster \( C_i \), the vertex \( v \) is contained in at least four clusters of \( \mathcal{C} \), which is a contradiction to the definition of \( \mathcal{C} \).

(iii) Let, instead, \( v \in (C_i \cap C_j) \setminus C_k \neq \emptyset \) and \( C_k \setminus (C_i \cup C_j) = \{a_1, a_2, a_3, \ldots \} \). By the definition of a \((3, 2)\)-covering, the edges \( va_1, va_2, va_3 \) are covered by at least two clusters of \( \mathcal{C} \), different from \( C_i \) and \( C_j \). So, together with the clusters \( C_i, C_j \), the vertex \( v \) is contained in at least four clusters of \( \mathcal{C} \), which is a contradiction.

\[ \blacksquare \]

Lemma 4. Let \( \mathcal{C} = (C_1, C_2, \ldots, C_t) \) be a decomposition \((3, 2)\)-covering of a complete graph \( G \). Then the following statements hold:

(i) If \( G \) has order 11, then it contains no cluster of size at most 2.

(ii) If \( G \) has order 12, then it contains no cluster of size at most 3.

**Proof.** (i) Let \( V(G) = \{a_1, a_2, \ldots, a_{11}\} \), \( C_1 \in \mathcal{C}(a_1) \) and \( |C_1| \leq 2 \). W.l.o.g., assume that \( \{a_3, a_4, \ldots, a_{11}\} \subseteq V(G) \setminus C_1 \). By the definition of \( \mathcal{C} \), there exists a cluster \( C_2 \in \mathcal{C}(a_1) \) of size at least 6 among the clusters covering some of the nine edges \( a_1 a_i, i = 3, 4, \ldots, 11 \). By Lemma 3(i),(ii), \( |C_2| = 6 \) and \( C_1 \subseteq C_2 \). Hence, \( |V(G) \setminus (C_1 \cup C_2)| = 5 \) and there exists a cluster \( C_3 \in \mathcal{C}(a_1) \setminus \{C_1, C_2\} \) of size at least 6 containing the set \( V(G) \setminus (C_1 \cup C_2) \). By Lemma 3(i), \( C_3 = \{a_1\} \cup (V(G) \setminus (C_1 \cup C_2)) \). We have \( |C_2| = |C_3| = 6 \) and \( |C_2 \cap C_3| = 1 \), which is a contradiction to Lemma 3(ii).

The statement (ii) of the lemma follows immediately from the statement (i).

\[ \blacksquare \]

3. Proof of Necessity of Theorem 2

By heredity of the class \( L_3^4 \), one has to show that none of the graphs \( K_{1, 4}, G(12, 7), G(11, 10), G(10, 9, 5), G(10, 9, 7), G(10, 9, 9), G(10, 7, k), k = 1, 2, \ldots, 7, G(9, 8, 1) \) and \( G(9, 8, 2) \) belongs to this class. Obviously, there exists no \((3, 2)\)-covering for the star \( K_{1, 4} \). Therefore, \( K_{1, 4} \not\in L_3^4 \) by Theorem 1.

Furthermore, let \( G \) be one of the graphs \( G(12, 7), G(11, 10), G(10, 9, 5), G(10, 9, 7), G(10, 9, 9), G(10, 7, k), k = 1, 2, \ldots, 7, G(9, 8, 1), G(9, 8, 2) \) with the bipartition \((A, B)\). Suppose, to the contrary, that there exists a \((3, 2)\)-covering \( \mathcal{D} = (D_1, D_2, \ldots, D_t) \) of \( G \).

W.l.o.g., we will assume that no cluster of \( \mathcal{D} \) is contained in some other cluster of \( \mathcal{D} \). By Theorem 1, it can be easily seen that \( D_i \neq A \) for any \( i = 1, 2, \ldots, t \), since \( \deg(b_1) \geq 7 \).

Put \( \mathcal{C} = (C_1, C_2, \ldots, C_t) \), where \( C_i = D_i \cap A, i = 1, 2, \ldots, t \). Then \( \mathcal{C} \) is a decomposition \((3, 2)\)-covering of the subgraph \( G(A) \), since \( N(b_i) \neq A \) for each
$b_i \in B$. A cluster $C \in \mathcal{C}$ is called $b_i$-reduced with $b_i \in B$, if $C \cup \{b_i\} \in \mathcal{D}$. A cluster $C \in \mathcal{C}$ is called simply reduced if it is $b_i$-reduced for some $b_i \in B$. By Lemma 3(i), $\mathcal{C}$ contains two or three $b_1$-reduced clusters, since $\deg(b_1) \geq 7$.

**Lemma 5.** The following statements hold:

(i) If $C_1, C_2 \in \mathcal{C}$ are two different $b_i$-reduced clusters with $b_i \in B$, then $|C_1 \cap C_2| \leq 1$.

(ii) If $C_1, C_2 \in \mathcal{C}$ are two different $b_i$-reduced clusters with $b_i \in B$, then $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$.

(iii) If $C_1, C_2, C_3 \in \mathcal{C}$ are three different reduced clusters, then $C_1 \cap C_2 \cap C_3 = \emptyset$.

**Proof.** (i) The validity of the statement follows immediately from the definition of $\mathcal{C}$.

(ii) The statement follows from the above assumption that no cluster of $\mathcal{D}$ is contained in some other cluster of $\mathcal{D}$.

(iii) If, to the contrary, $a \in C_1 \cap C_2 \cap C_3$, then the edge $aa_p$ is not covered by a cluster from $\mathcal{C}(a) = \{C_1, C_2, C_3\}$, which is a contradiction to the definition of $\mathcal{C}$.

We consider the following separate cases and come to a contradiction in each of them.

(1) $G = G(12, 7)$.

(a) Assume that there exist exactly two $b_1$-reduced clusters $C_1, C_2 \in \mathcal{C}$. By Lemma 4(ii), $|C_1| \geq 4$ and $|C_2| \geq 4$. Hence, by Lemma 5(i) and the equality $|C_1 \cup C_2| = 7$, we obtain $|C_1| = |C_2| = 4$ and $|C_1 \cap C_2| = 1$. W.l.o.g., assume that $C_1 \cap C_2 = \{a_1\}$. Consider the cluster $C_3 \in \mathcal{C}(a_1) \setminus \{C_1, C_2\}$. Then $\{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\} \subseteq C_3$. By Lemma 3(i), $C_3 = \{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ (see Figure 2). We have $|C_3 \setminus C_1| = 5$, which is a contradiction to Lemma 3(ii).

![Figure 2. The clusters $C_1$, $C_2$ and $C_3$ of the covering $\mathcal{C}$ in the case (1).](image-url)
(b) Suppose that there exist exactly three $b_1$-reduced clusters $C_1, C_2, C_3 \in \mathcal{C}$.

Taking into account Lemmas 5(i) and 4(ii), we obtain that $|C_1 \cup C_2| \geq 7$ and, therefore, $|C_1 \cup C_2 \cup C_3| \geq 9 > 7 = \deg(b_1)$, which is a contradiction.

(2) $G = G(11, 10)$.

(a) Assume that there exist exactly two $b_1$-reduced clusters $C_1, C_2 \in \mathcal{C}$. By Lemma 5(i), $|C_1 \cap C_2| \leq 1$. By Lemmas 5(ii) and 3(ii), $|C_1 \setminus C_2| \leq 4$ and $|C_2 \setminus C_1| \leq 4$. Therefore, $\deg(b_1) = |C_1 \cup C_2| \leq 9$, which is a contradiction.

(b) Let $\mathcal{C}$ contain three $b_1$-reduced clusters $C_1, C_2$ and $C_3$.

First, we suppose that $C_1, C_2$ and $C_3$ are pairwise disjoint. By Lemmas 5(ii) and 4(i), we have $3 \leq |C_i| \leq 4$ for any $i = 1, 2, 3$. W.l.o.g., assume that $C_1 = \{a_1, a_2, a_3\}$, $C_2 = \{a_4, a_5, a_6\}$, $C_3 = \{a_7, a_8, a_9, a_{10}\}$. By the definition of $\mathcal{C}$ and Lemma 3(i), we have $|\mathcal{C}(a_1)| = 3$, since $|A \setminus C_1| = 8$.

Let $C_4$ and $C_5$ be two clusters in $\mathcal{C}(a_1) \setminus \{C_1\}$. Each of the clusters $C_4$ and $C_5$ has at least one common vertex with any of the clusters $C_2, C_3$. If, for example, $C_4 \cap C_2 = \emptyset$, then $a_1 \in (C_1 \cap C_4) \setminus C_2$ and $|C_2 \setminus (C_1 \cup C_4)| = |C_2| = 3$, which is a contradiction to Lemma 3(iii). Since $C_3 \subseteq C_4 \cup C_5$ by the definition of $\mathcal{C}$ and $|C_3| = 4$, then each of the clusters $C_4$ and $C_5$ has exactly two common vertices with the cluster $C_3$.

The inequalities $|C_4| \geq 5$ and $|C_5| \geq 5$ hold. Otherwise, let, for example, $|C_4| \leq 4$. Then $|C_5| \geq 6$, since $|C_4 \cup C_5| \geq 9$. Hence, by Lemma 3(i), $|C_5| = 6$. Therefore, $C_4 \cap C_5 = \{a_1\}$ and $|C_5 \setminus C_4| = 5$, which is a contradiction to Lemma 3(ii).

W.l.o.g., assume that $\{a_4, a_7, a_8\} \subseteq C_4$, $\{a_6, a_9, a_{10}, a_{11}\} \subseteq C_5$. Since $|C_5 \setminus C_1| \leq 4$ by Lemma 3(ii), then $a_5 \notin C_5$. Hence, $a_5 \in C_4$. We have $a_5 \in (C_2 \cap C_4) \setminus C_5$. By Lemma 3(iii), $|C_5 \setminus (C_2 \cup C_4)| \leq 2$. Then $a_1 \in C_4$ and, by Lemma 3(i), $C_4 = \{a_1, a_4, a_5, a_7, a_8, a_{11}\}$ (see Figure 3). Therefore, $|C_4 \setminus C_1| = 5$, which is a contradiction to Lemma 3(ii).

Now, w.l.o.g., assume that $a_1 \in C_1 \cap C_2$. By Lemma 5(i), $C_1 \cap C_2 = \{a_1\}$. By Lemmas 5(ii) and 3(ii), $|C_1| \leq 5$ and $|C_2| \leq 5$. Each of the clusters $C_1, C_2$ has size at least 4. If not, then $a_1 \in (C_1 \cap C_2) \setminus C_3$ by Lemma 5(iii), and $|C_3 \setminus (C_1 \cup C_2)| \geq 10 - (3 + 5 - 1) = 3$, which is a contradiction to Lemma 3(iii).

Furthermore, assume that at least one of the clusters $C_1, C_2$, say $C_1$, has size 5. Then $|C_1 \setminus C_3| \leq 4$ by Lemmas 5(ii) and 3(ii), and so $|C_1 \cap C_3| = 1$ by Lemma 5(i). Let $C_1 \cap C_3 = \{a_2\}$. Then $a_2 \in (C_1 \cap C_3) \setminus C_2$ by Lemma 5(iii). We obtain that $|C_2 \setminus (C_1 \cup C_3)| \leq 2$ by Lemma 3(iii). Therefore, $|C_2 \cap C_3| = 1$. Let $C_2 \cap C_3 = \{a_3\}$. We have $a_3 \in (C_2 \cap C_3) \setminus C_1$ and $|C_1 \setminus (C_2 \cup C_3)| = 3$, contradicting Lemma 3(iii).

Thus, $|C_1| = |C_2| = 4$. Let, w.l.o.g., $C_1 = \{a_1, a_2, a_3, a_4\}$, $C_2 = \{a_1, a_5, a_6, a_7\}$. Then $\{a_8, a_9, a_{10}\} \subseteq C_3$, since $\{a_1, a_2, \ldots, a_{10}\} = N(b_1)$. By Lemma 5(iii), $a_1 \in (C_1 \cap C_2) \setminus C_3$. However, then $|C_3 \setminus (C_1 \cup C_2)| = 3$, which is a contradiction to Lemma 3(iii).
(3) \( G = G(10, 9, 5) \).

Each vertex \( a_i \), where \( i = 1, 2, \ldots, 5 \), belongs to one \( b_1 \)- and one \( b_2 \)-reduced clusters. Therefore, by Lemma 5(iii), each two of the \( b_2 \)-reduced clusters have no common vertices. By Lemma 5(iii), if a vertex belongs to two of the \( b_1 \)-reduced clusters, then this vertex belongs to the set \( \{a_6, a_7, a_8, a_9\} \).

(a) Let \( \mathcal{C} \) contain exactly two \( b_1 \)-reduced clusters \( C_1, C_2 \). Since \( |C_1 \cup C_2| = 9 \), we get \( |C_1 \cap C_2| = 1 \) and \( |C_1| = |C_2| = 5 \) by Lemmas 5(i),(ii) and 3(ii). Let, w.l.o.g., \( C_1 \cap C_2 = \{a_9\} \). By the definition of \( \mathcal{C} \), any vertex \( a_i \), where \( i = 1, 2, \ldots, 8 \), belongs to exactly two clusters from \( \mathcal{C}(a_i) \setminus \{C_1, C_2\} \). Moreover, it is easy to obtain that, for any vertex \( a_i \), where \( i = 1, 2, \ldots, 8 \), each cluster \( C \in \mathcal{C}(a_i) \setminus \{C_1, C_2\} \) satisfies the equalities \( |C \cap (C_1 \setminus C_2)| = 2 \) and \( |C \cap (C_2 \setminus C_1)| = 2 \). Since every \( b_2 \)-reduced cluster is a subset of \( (C_1 \setminus C_2) \cup (C_2 \setminus C_1) \) and belongs to \( \mathcal{C}(a_i) \setminus \{C_1, C_2\} \), it has size 4, which is a contradiction.

(b) Let \( \mathcal{C} \) contain three pairwise non-intersecting \( b_1 \)-reduced clusters \( C_1, C_2 \) and \( C_3 \). By Lemma 3(ii), \( |C_i| \leq 4 \) for every \( i = 1, 2, 3 \).

(b1) First, suppose that \( |C_1| = 1 \), \( |C_2| = 4 \) and \( |C_3| = 4 \). Put \( C_1 = \{a_1\} \). Consider the clusters \( C_4, C_5 \in \mathcal{C}(a_1) \setminus \{C_1\} \). By the definition of \( \mathcal{C} \), \( |C_i \cap C_i| = 2 \) for any \( i = 2, 3 \) and \( j = 4, 5 \). In particular, \( (C_4 \cap C_5) \cap (C_2 \cup C_3) = \emptyset \). Since \( (C_2 \cap C_4) \cap C_5 \neq \emptyset \), then \( |C_5 \cap (C_2 \cup C_4)| \leq 2 \) by Lemma 3(iii). Similarly, \( |C_4 \cap (C_2 \cup C_5)| \leq 2 \). Therefore, \( a_{10} \in C_4 \cap C_5 \). We obtain that there does not exist a \( b_2 \)-reduced cluster in \( \mathcal{C}(a_1) \), which is a contradiction.

Now, let \( C_1 \subset \{a_6, a_7, a_8, a_9\} \). W.l.o.g., put \( C_1 = \{a_9\} \). Note that each \( b_2 \)-reduced cluster \( C \) in \( \mathcal{C} \) has size at most 4. If not (i.e., \( C = \text{deg}(b_2) = 5 \)), then the inclusion \( C \subseteq C_2 \cup C_3 \) implies that \( |C \cap C_2| \geq 3 \) or \( |C \cap C_3| \geq 3 \), which is
a contradiction to the definition of $C$. Let $C_4$ be a $b_2$-reduced cluster in $C$ with size at most 2. Let $a_1 \in C_4 \cap C_2$. Consider the cluster $C_5 \in C(a_1) \setminus \{C_2, C_4\}$. By the definition of $C$, we have $|C_4| < 2$ and $C_4 \cap C_2 = \emptyset$, we have $|C_3 \setminus C_4| \geq 3$. Therefore, $|C_3 \cap C_5| \geq 3$, which is a contradiction.

(b2) Suppose that $|C_1| = 2$, $|C_2| = 3$ and $|C_3| = 4$. Let $a \in C_1$, where

$\{a, a_2, \ldots, a_9\}$. Consider the clusters $C_4, C_5 \in C(a) \setminus \{C_1\}$. By the definition of $C$, we have $|C_4| \leq 2$ and $|C_4 \cap C_3| = 2$ for any $i = 1, 2, 3$. Moreover, at least one of the clusters $C_4, C_5$, say $C_5$, has exactly two common vertices with $C_2$. Clearly, 

$(C_4 \cap C_5) \cap C_3 = \emptyset$ and $|C_4 \cap C_5 \cap C_2| = 1$. If $a_{10} \in C_5$, then $|C_5| = 6$ by Lemma 3(i). We have $C_1 \setminus C_5 = \emptyset$ and $|C_5 \cap C_1| = 5 > 4$, which is a contradiction to Lemma 3(ii). Therefore, $a_{10} \in C_4 \setminus C_5$. By Lemma 3(i), at least one vertex $a'$ of the set $C_5 \cap C_2$ does not belong to $C_4$. We obtain that $a' \in (C_2 \cap C_5) \setminus C_4$ and $|C_4 \setminus (C_2 \cup C_5)| \geq 3$, which is a contradiction to Lemma 3(iii).

(b3) Let $|C_1| = |C_2| = |C_3| = 3$. Assume that there exists a $b_2$-reduced cluster in $C$ with size at most 2. Therefore, this cluster does not intersect with some of the clusters $C_1, C_2$ and $C_3$, which is a contradiction to the definition of $C$.

Now, let $C_4 = N(b_2)$ be the only $b_2$-reduced cluster in $C$. W.l.o.g., assume that $C_1 = \{a_1, a_6, a_7\}$, $C_2 = \{a_2, a_3, a_8\}$ and $C_3 = \{a_4, a_5, a_9\}$. Consider the clusters $C' \in C(a_2) \setminus \{C_2, C_4\}$ and $C'' \in C(a_3) \setminus \{C_2, C_4\}$. By the definition of $C$, we have $a_6, a_7, a_9, a_{10} \in C' \cap C''$. Therefore, $C' = C''$. Put $C_5 = C'$. Then $C_3 \cap C_5 = \emptyset$ and $|C_5 \setminus C_3| = 5 > 4$, which is a contradiction to Lemma 3(ii).

(c) Let $C$ contain three $b_1$-reduced clusters $C_1, C_2, C_3$ and $C_1 \cap C_2 \neq \emptyset$. W.l.o.g., assume that $|C_1| \geq |C_2|$. By Lemma 5(iii), we obtain that $C_1 \cap C_2 \cap C_3 = \emptyset$. It follows from Lemma 3(iii) that $|C_3 \setminus (C_1 \cup C_2)| \leq 2$. Hence, $|C_1 \cup C_2| \geq 7$. Then $|C_1| \geq 4$. Moreover, by Lemmas 5(ii) and 3(ii), we have $|C_1| \leq 5$.

(c1) Let $|C_1| = 5$. Then $C_1 \cap C_3 \neq \emptyset$ by Lemmas 5(ii) and 3(ii). Furthermore, $C_2 \cap C_3 = \emptyset$ by Lemma 5(iii) and 3(iii). Since $(C_1 \cap C_3) \setminus C_2$ \neq \emptyset and, by Lemma 3(iii), $|C_2 \setminus (C_1 \cup C_3)| \leq 2$, we have $|C_1| = 5$, $|C_2| = 3$ and $|C_3| = 3$. Recall that $C_1 \cap C_2, C_1 \cap C_3 \subseteq \{a_6, a_7, a_8, a_9\}$. W.l.o.g., assume that $C_1 \cap C_2 = \{a_8\}$, $C_1 \cap C_3 = \{a_9\}$. Consider the clusters $C_4 \in C(a_8) \setminus \{C_1, C_2\}$ and $C_5 \in C(a_9) \setminus \{C_1, C_3\}$. By the definition of $C$, we have $|C_4 \cap (C_1 \setminus \{a_8, a_9\})| \leq 1$ and $|C_5 \cap (C_1 \setminus \{a_8, a_9\})| \leq 1$. Note that $C_1 \cap (C_2 \setminus \{a_8\}) = \emptyset$. If, to the contrary, $a \in C_1 \cap (C_2 \setminus \{a_8\})$, then $C(a) = \{C_2, C_4, C_5\}$ and some vertex of the set $C_1 \setminus \{a_8, a_9\}$ does not belong to the set $C_2 \cup C_4 \cup C_5$, contradicting the definition of $C$. Analogously, $C_5 \cap (C_3 \setminus \{a_9\}) = \emptyset$. At least one of the clusters $C_2, C_3$, say $C_3$, contains a vertex $a' \in \{a_1, a_2, \ldots, a_5\}$, since $|\{a_1, a_2, \ldots, a_5\} \cap C_1| \leq 3$. Let $a''$ be another vertex in the set $C_3 \setminus \{a_9\}$. Consider the clusters $C'' \in C(a'') \setminus \{C_3, C_4\}$ and $C'' \in C(a'') \setminus \{C_3, C_4\}$. Each of them contains the set $C_2 \setminus \{a_8\} \cup (C_1 \setminus (C_3 \cup C_4))$ of size at least 4. Therefore, $C_3 \cap C'' = C_5$ is a cluster of $C$ of size at least 6. By Lemma 3(i), $|C_6| = 6$ (see Figure 4). Since $a' \in \{a_1, a_2, \ldots, a_5\}$, then $C_6$ is a $b_2$-reduced cluster in $C$, which is a contradiction.
(c2) Now, let $|C_1| = 4$. Then, taking into consideration the inequalities $|C_1 \cup C_2| \geq 7$ and $|C_1| \geq |C_2|$, we have $|C_2| = 4$.

Let $C_3$ intersect with $C_1$ or $C_2$. Then, by Lemma 3(iii), $C_3$ intersects with both $C_1$ and $C_2$. By Lemma 5(i),(iii), we can assume, w.l.o.g., that $C_1 = \{a_1, a_2, a_7, a_8\}$, $C_2 = \{a_3, a_4, a_7, a_9\}$ and $C_3 = \{a_5, a_6, a_8, a_9\}$. Consider the cluster $C_4 \in C(a_7) \setminus \{C_1, C_2\}$. By the definition of $\mathcal{C}$, we have $a_5, a_6, a_{10} \in C_4$. Initially, let $C_4 = \{a_5, a_6, a_7, a_{10}\}$. Consider the cluster $C_5 \in C(a_5) \setminus \{C_3, C_4\}$.

By the definition of $\mathcal{C}$, we have $a_1, a_2, a_3, a_4 \in C_5$. Both clusters $C_3, C_4$ are not $b_2$-reduced since each of them contains at least one of the vertices $a_6, a_7, a_8, a_9, a_{10}$. Hence $C_5$ is a $b_2$-reduced cluster. It follows from the inclusion $N(b_2) \subseteq C_5$ that $C_5 = N(b_2) = \{a_1, a_2, \ldots, a_5\}$. Consider the cluster $C_6 \in \mathcal{C}(a_6) \setminus \{C_3, C_4\}$. By the definition of $\mathcal{C}$, we have $a_1, a_2, a_3, a_4 \in C_6$. Thus $C_6 \neq C_5$ and $|C_6 \cap C_5| \geq 4 > 2$, which is a contradiction to the definition of $\mathcal{C}$. If the cluster $C_4$ has a non-empty intersection with the set $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$, for example $a_1 \in C_4$, then at least one of the vertices $a_3, a_4$ also belongs to $C_4$. Otherwise, by the definition of $\mathcal{C}$, the cluster $C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\}$ contains the vertices $a_3, a_4$ and $a_9$. We obtain that $C_5 \neq C_2$ and $|C_5 \cap C_2| \geq 3 > 2$, which is a contradiction. Let $a_3 \in C_4$ and $C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\}$. Then $a_4, a_9 \in C_5$. We obtain that none of the clusters $C_1, C_4, C_5 \in \mathcal{C}(a_1)$ is $b_2$-reduced, which is a contradiction.

Assume that the cluster $C_5$ does not intersect with $C_1$ and $C_2$. Then $|C_3| = 2$. One of the vertices $a_6, a_7, a_8$ and $a_9$, say $a_9$, belongs to $C_1 \cap C_2$. Consider the cluster $C_4 \in \mathcal{C}(a_9) \setminus \{C_1, C_2\}$. Clearly, $C_3 \cup \{a_9\} \subseteq C_4$. We show that $|C_4 \cap
(C_1 \setminus C_2) = 1 and |C_4 \cap (C_2 \setminus C_1)| = 1. Indeed, if C_4 has no common vertices
with one of the sets C_1 \setminus C_2 or C_2 \setminus C_1, say with C_1 \setminus C_2, then
(C_4 \cap C_1) \setminus C_1 \neq \emptyset and |C_1 \setminus (C_3 \cup C_4)| = 3, contradicting Lemma 3(iii).
Let C_3 = \{a', a''\}. Consider the clusters C' \in \mathcal{C}(a') \setminus \{C_3, C_4\} and C'' \in \mathcal{C}(a'') \setminus \{C_3, C_4\}. We have
(C_1 \setminus C_4) \cup (C_2 \setminus C_4) \subseteq C' \cap C'''. Since |(C_1 \setminus C_4) \cup (C_2 \setminus C_4)| = 4, we obtain that
C' = C''' by the definition of \mathcal{C}. Denote the cluster C' by C_5. It can be easily
obtained by the definition of \mathcal{C} that there are two clusters C_6 and C_7 in \mathcal{C} such
that ((C_1 \setminus C_2) \cap C_4) \cup ((C_2 \setminus C_5) \cup \{a_{10}\}) \subseteq C_6 and ((C_2 \setminus C_1) \cap C_4) \cup (C_1 \cap C_5) \cup \{a_{10}\} \subseteq C_7. Each vertex from the set
(C_1 \setminus C_2) \cup (C_2 \setminus C_1) belongs to exactly three of the non-b_{2}\reduced clusters
C_1, C_2, C_4, C_5, C_6, C_7. Clearly, at least three vertices a_1, a_2, . . . , a_5 belong to the set
(C_1 \setminus C_2) \cup (C_2 \setminus C_1), which is a contradiction.

(4) We can come to a contradiction for each of the graphs G = G(10, 9, 9)
and G = G(10, 9, 7) analogously to the graph G = G(10, 9, 5).

(5) G = G(10, 7, k), k = 1, 2, . . . , 7.

(a) First, assume that 4 \leq k \leq 7. For any i = 1, 2, 3, 4, denote by C_{1i} and C_{2i},
respectively, b_{1}\reduced clusters from \mathcal{C}(a_i). Consider the cluster C_{3i} \in
\mathcal{C}(a_i) \setminus \{C_{1i}, C_{2i}\}. Since C_{1i}, C_{2i} \subseteq \{a_1, a_2, . . . , a_7\}, we have \{a_8, a_9, a_{10}\} \subseteq C_{3i}
for any i = 1, 2, 3, 4. By the definition of \mathcal{C}, we obtain C_{13} = C_{23} = C_{33} = C_{3i}
and \{a_1, a_2, a_3, a_4, a_6, a_9, a_{10}\} \subseteq C_{3i} for any i = 1, 2, 3, 4, which is a contradiction
to Lemma 3(i).

(b) Put k = 1. Let C_1 and C_2, respectively, be b_{1}\reduced clusters
from \mathcal{C}(a_1). Then C_1 \subseteq \{a_1, a_2, . . . , a_7\}, C_2 = \{a_1\}. By Lemma 3(i), |C_1| \leq 6.
Consider the cluster C_3 \in \mathcal{C}(a_1) \setminus \{C_1, C_2\}. The equality C_1 \cup C_2 \cup C_3 = C_1 \cup C_3 = A
implies that |C_1| \geq 5 by Lemma 3(i).

W.l.o.g., assume that C_1 = \{a_1, a_2, . . . , a_5\}. Then C_3 = \{a_1, a_6, a_7, . . . , a_{10}\}
by Lemma 3(i). We obtain C_3 \setminus C_1 \neq \emptyset and |C_3 \setminus C_1| = 5, contradicting
Lemma 3(ii). Now, w.l.o.g. put C_1 = \{a_1, a_2, . . . , a_6\}. Then \{a_1, a_7, a_8, a_9, a_{10}\} \subseteq
C_3. By Lemma 3(ii), |C_1 \setminus C_3| \geq 4. Therefore, one of the vertices a_2, a_3, . . . , a_6, say
a_2, belongs to C_3. By Lemma 3(i), C_3 = \{a_1, a_2, a_7, a_8, a_9, a_{10}\}. Let C_4 be a b_{1}\reduced
cluster from \mathcal{C}(a_7). We get C_3 \neq C_4, since C_3 \not\subseteq N(b_1). By Lemma 5(i),
|C_4 \cap C_1| \leq 1. We obtain that a_7 \in (C_3 \setminus C_4) \setminus C_1 and |C_1 \setminus (C_3 \cup C_4)| \geq 3, which
is a contradiction to Lemma 3(iii).

(c) Put k = 2. Let C_1 and C_2, respectively, be b_{1}\reduced clusters
from \mathcal{C}(a_1). Taking into account the case (b), we can assume that C_2 = \{a_1, a_2\}. Then
we can proceed analogously to the case (b).

(d) Finally, we assume that k = 3. For any i = 1, 2, 3, denote by C_{i1}
and C_{i2}, respectively, b_{1}\reduced clusters from \mathcal{C}(a_i). Taking into account
the cases (b) and (c), we can assume that C_{12} = \{a_1, a_2, a_3\}. Consider the cluster
C_{33} \in \mathcal{C}(a_1) \setminus \{C_{11}, C_{12}\}. Since C_{11}, C_{12} \subseteq \{a_1, a_2, . . . , a_7\}, we have
\{a_8, a_9, a_{10}\} \subseteq C_{3i} for any i = 1, 2, 3. By the definition of \mathcal{C}, C_{13} = C_{23} = C_{33}
and \( \{a_1, a_2, a_3, a_8, a_9, a_{10}\} \subseteq C_{i3} \) for any \( i = 1, 2, 3 \). By Lemma 3(i), \( C_{i3} = \{a_1, a_2, a_3, a_8, a_9, a_{10}\} \). We obtain that \( C_{12} \neq C_{i3} \) and \( |C_{12} \cap C_{i3}| = 3 \), which is a contradiction to the definition of \( \mathcal{C} \).

(6) \( G = G(9, 8, 1) \).

(a) Assume that there exist exactly two \( b_1 \)-reduced clusters \( C_1, C_2 \in \mathcal{C} \). Clearly, \( \mathcal{C} \) contains a unique \( b_2 \)-reduced cluster \( C_3 = \{a_1\} \). If \( C_1 \cap C_2 = \emptyset \), then \( |C_1| = |C_2| = 4 \) by Lemma 3(ii). W.l.o.g., assume that \( a_1 \in C_1 \). Thus, \( a_1 \in (C_1 \cap C_3) \setminus C_2 \) and \( |C_2 \setminus (C_1 \cup C_3)| = 4 > 2 \), which is a contradiction to Lemma 3(iii).

Let \( C_1 \cap C_2 \neq \emptyset \). It follows from Lemma 5(i) that \( |C_1 \cap C_2| = 1 \). Then \( C_1 \cap C_2 \neq \{a_1\} \) by Lemma 5(iii). Let \( C_1 \cap C_2 = \{a_2\} \) and \( a_1 \in C_1 \). Since \( C_1 \nsubseteq C_2 \) and \( C_2 \nsubseteq C_1 \), we have \( |C_1| \leq 5 \) and \( |C_2| \leq 5 \) by Lemma 3(ii). The equality \( |C_1 \cup C_2| = 8 \) implies \( |C_1| \geq 4 \) and \( |C_2| \geq 4 \). We have \( a_1 \in (C_1 \cap C_3) \setminus C_2 \) and \( |C_2 \setminus (C_1 \cup C_3)| \geq 3 \), which is a contradiction to Lemma 3(iii).

(b) Now, let \( C_1, C_2, C_3 \) and \( C_4 = \{a_1\} \), respectively, be three \( b_1 \)- and a unique \( b_2 \)-reduced clusters in \( \mathcal{C} \). W.l.o.g., assume that \( a_1 \in C_1 \). By Lemma 5(iii), \( C_1 \cap C_i \neq \{a_1\} \) for any \( i = 2, 3 \).

Furthermore, we have \( |C_1| \geq 5 \). Otherwise, \( |A \setminus C_1| \geq 5 \) and, by the definition of \( \mathcal{C} \), there exists a cluster \( C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\} \) such that \( (A \setminus C_1) \cup \{a_1\} \subseteq C_5 \).

By Lemma 3(i), it follows that \( |A \setminus C_1| = 5 \), i.e., \( |C_1| = 4 \). We have \( C_1 \setminus C_5 \neq \emptyset \) and \( |C_5 \setminus C_1| \geq 5 \), which is a contradiction to Lemma 3(ii). Therefore, by the same lemma, \( C_1 \cap C_2 \neq \emptyset \) and \( C_1 \cap C_3 \neq \emptyset \). By Lemmas 5(ii) and 3(ii), we have \( |C_1 \setminus C_2| \leq 4 \) and, consequently, \( |C_1| = 5 \).

The equality \( C_2 \cap C_3 = \emptyset \) holds. Otherwise, by Lemma 5(i) and (iii), we have \( (C_2 \cap C_3) \setminus C_1 \neq \emptyset \) and \( |C_1 \setminus (C_2 \cup C_3)| = 3 \), which is a contradiction to Lemma 3(iii).

Let \( C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\} \). Since \( A \setminus (C_1 \cup C_4) \subseteq C_5 \), we have \( |C_5| \geq 5 \).

Since \( |C_1 \cap C_i| = 1 \) for any \( i = 2, 3 \), \( C_2 \cap C_3 = \emptyset \) and \( |(C_2 \cup C_3) \setminus C_1| = 3 \), one of the clusters \( C_2, C_3 \), say \( C_2 \), has size 2. So, we have \( (C_2 \cap C_5) \setminus C_1 \neq \emptyset \) and \( |C_1 \setminus (C_2 \cup C_5)| \geq 3 \) both in the case \( |C_5| = 6 \) (since \( C_2 \subseteq C_5 \) by Lemma 3(ii)) and in the case \( |C_5| = 5 \), which is a contradiction to Lemma 3(iii).

(7) We can come to a contradiction for the graph \( G = G(9, 8, 2) \) analogously to the graph \( G = G(9, 8, 1) \).

4. Proof of Sufficiency of Theorem 2

Let a threshold graph \( H = G(p, q_1, q_2, \ldots, q_k) \) with the bipartition \((A, B)\) not contain any of the graphs \( K_{1,4}, G(12, 7), G(11, 10), G(10, 9, 9), G(10, 9, 7), G(10, 9, 5), G(10, 7, k), k = 1, 2, \ldots, 7, G(9, 8, 2), G(9, 8, 1) \) as an induced subgraph. By Theorem 1, we have to show that there exists a \((3, 2)\)-covering of \( H \).

W.l.o.g., assume that \( H \) is a connected non-complete graph. Therefore, \( H \)}
has a dominating vertex by the definition of \( H \). Furthermore, \(|B| \leq 2\), since \( H \) does not contain \( K_{1,4} \) as an induced subgraph. Thus, we have \( H = G(p,q_1) \) or \( H = G(p,q_1,q_2) \).

First, we suppose that \(|A| = p \geq 14\). Then \( q_1 \leq 6\), since \( H \) does not contain any of the graphs \( G(11, 10) \) and \( G(12, 7) \) as an induced subgraph. For any vertex \( b \in B \), partition the set \( N(b) \) into \( n_b \leq 3 \) pairwise disjoint cliques \( C_i^b \) each having size at most 2. Obviously, the list of cliques \( \{C_i^b : b \in B_i\} \) together with the clique \( A \) gives a desired \((3, 2)\)-covering of \( H \).

If \(|A| \leq 7\), then \( q_1 \leq 6\) by the maximality of the clique \( A \). Therefore, a desired \((3, 2)\)-covering of \( H \) can be constructed as above.

Now, let \( 8 \leq |A| \leq 13 \). Taking into account the above considerations, we can assume that \( q_1 \geq 7 \).

Let \( H = G(p,q_1) \). Since \( H \) does not contain any of the graphs \( G(12,7) \) and \( G(11,10) \) as an induced subgraph, it is isomorphic to one of the graphs \( G(13,9), G(12,9), G(12,8), G(11,9), G(11,8), G(10,9), G(10,8), G(10,7), G(9,8), G(9,7), G(8,7) \). Clearly, the set of cliques

\[
\mathcal{C} = \{\{a_1, a_2, a_3, a_4, a_5, b_1\}, \{a_1, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_{10}, a_{11}, a_{12}, a_{13}\},
\{a_2, a_3, a_6, a_7, a_{10}, a_{11}\}, \{a_2, a_3, a_8, a_9, a_{12}, a_{13}\}, \{a_4, a_5, a_6, a_7, a_{12}, a_{13}\},
\{a_4, a_5, a_8, a_9, a_{10}, a_{11}\}\}
\]

of the graph \( G(13,9) \) is one of its \((3, 2)\)-coverings. Each of the graphs \( G(12,9), G(12,8), G(11,9), G(11,8), G(10,9), G(10,8), G(10,7), G(9,8), G(9,7) \) and \( G(8,7) \) is an induced subgraph of \( G(13,9) \). Therefore, a desired \((3, 2)\)-covering for each of these graphs can be obtained from the covering \( \mathcal{C} \).

Now, let \( H = G(p,q_1,q_2) \). Since \( H \) does not contain any of the graphs \( G(12,7), G(11,10), G(10,9,9), G(10,9,7), G(10,9,5), G(10,7, k), k = 1, 2, \ldots, 7, G(9,8,2) \) and \( G(9,8,1) \) as an induced subgraph, it is isomorphic to one of the graphs \( G(11,9,8), G(11,9,6), G(11,9,4), G(10,9,8), G(10,9,6), G(10,9,4), G(10,8,8), G(10,8,7), G(10,8,6), G(10,8,5), G(10,8,4), G(10,8,3), G(9,8,8), G(9,8,7), G(9,8,6), G(9,8,4), G(9,8,3), G(9,7,7), G(9,7,6), G(9,7,5), G(9,7,4), G(9,7,3), G(9,7,2), G(9,7,1), G(8,7,7), G(8,7,6), G(8,7,5), G(8,7,4), G(8,7,3), G(8,7,2), G(8,7,1) \). Some of the desired \((3, 2)\)-coverings \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \) for the graphs \( G(11,9,8), G(11,9,6), G(11,9,4), G(9,7,1) \), respectively, are given below:

\[
\mathcal{C}_1 = \{\{a_1, a_2, a_3, a_4, a_9, b_1\}, \{a_5, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_2, a_7, a_8, b_2\},
\{a_3, a_4, a_5, a_6, b_2\}, \{a_1, a_2, a_5, a_6, a_{10}, a_{11}\}, \{a_3, a_4, a_7, a_8, a_{10}, a_{11}\},
\{a_9, a_{10}, a_{11}\}\}
\]

\[
\mathcal{C}_2 = \{\{a_1, a_2, a_7, a_9, b_1\}, \{a_3, a_4, a_7, a_8, b_1\}, \{a_5, a_6, a_8, a_9, b_1\},
\{a_1, a_2, a_3, a_4, a_5, a_6, b_2\}, \{a_5, a_6, a_7, a_{10}, a_{11}\}, \{a_1, a_2, a_8, a_{10}, a_{11}\},
\{a_3, a_4, a_9, a_{10}, a_{11}\}\}
\]
Each of the remaining graphs $G(10, 9, 8)$, $G(10, 9, 6)$, $G(10, 9, 4)$, $G(10, 8, 7)$, $G(10, 8, 6)$, $G(10, 8, 5)$, $G(10, 8, 4)$, $G(10, 8, 3)$, $G(9, 8, 8)$, $G(9, 8, 7)$, $G(9, 8, 6)$, $G(9, 8, 5)$, $G(9, 8, 4)$, $G(9, 7, 7)$, $G(9, 7, 6)$, $G(9, 7, 5)$, $G(9, 7, 4)$, $G(9, 7, 3)$, $G(9, 7, 2)$, $G(8, 7, 7)$, $G(8, 7, 6)$, $G(8, 7, 5)$, $G(8, 7, 4)$, $G(8, 7, 3)$, $G(8, 7, 2)$, $G(8, 7, 1)$ is an induced subgraph for some of the graphs $G(11, 9, 8)$, $G(11, 9, 6)$, $G(11, 9, 4)$, $G(9, 7, 1)$. Therefore, a desired $(3, 2)$-covering for each of the remaining graphs can be obtained from one of the coverings $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$.

5. Recognition Algorithm

The proof of sufficiency of Theorem 2 implies the following linear algorithm for recognizing graphs from $L_3^2$ in the class of threshold graphs.

**Algorithm**

**Input**: a connected threshold graph $H$ with bipartition $(A, B)$, where $A$ is a maximal clique in $H$.

**Output**: 1 if $H \in L_3^2$, and 0 otherwise.

1. **begin**
2. if $B = \emptyset$, i.e., the graph $H$ is complete,
3. return 1;
4. if $|B| \geq 3$
5. return 0;
6. if $\deg(b) \leq 6$ for every $b \in B$
7. return 1;
8. if $|A| \geq 14$
9. return 0;
10. if $H$ contains some of the graphs $G(12, 7)$, $G(11, 10)$, $G(10, 9, 9)$, $G(10, 9, 7)$, $G(10, 9, 5)$, $G(10, 7, k)$, $k = 1, 2, \ldots, 7$, $G(9, 8, 2)$, $G(9, 8, 1)$ as an induced subgraph
11. return 0;
12. return 1;
13. end.
The complexity of the algorithm in lines 1–9 is at most $O(n)$, where $n = |V(H)|$. Since the order of the graph $H$ in line 10 is at most 13, this line takes $O(1)$ time.

So, the total complexity of the recognition algorithm is $O(n)$.

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References


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