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RADIO GRACEFUL HAMMING GRAPHS

Amanda Niedzialomski

University of Tennessee at Martin
e-mail: aniedzia@utm.edu

Abstract

For $k \in \mathbb{Z}_+$ and G a simple, connected graph, a k-radio labeling $f : V(G) \to \mathbb{Z}_+$ of G requires all pairs of distinct vertices u and v to satisfy $|f(u) - f(v)| \geq k + 1 - d(u, v)$. We consider k-radio labelings of G when $k = \text{diam}(G)$. In this setting, f is injective; if f is also surjective onto $\{1, 2, \ldots, |V(G)|\}$, then f is a *consecutive radio labeling*. Graphs that can be labeled with such a labeling are called *radio graceful*. In this paper, we give two results on the existence of radio graceful Hamming graphs. The main result shows that the Cartesian product of t copies of a complete graph is radio graceful for certain t. Graphs of this form provide infinitely many examples of radio graceful graphs of arbitrary diameter. We also show that these graphs are not radio graceful for large t.

Keywords: radio labeling, radio graceful graph, Hamming graph.

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1. Introduction

Radio labeling has its historical roots in the problem of optimally assigning radio frequencies to transmitters, called the Channel Assignment Problem. In this context, an optimal assignment is one that minimizes interference that can be created when geographically close radio transmitters have an insufficient difference in their radio frequencies. The problem was framed in terms of graph labeling by Hale in 1980 ([9]), and several variations have been defined and studied since. Two notable examples are $L(2, 1)$-labeling and radio labeling, first introduced in [8] and [2], respectively. Both are examples of k-radio labelings, defined by Chartrand and Zhang in [3].

Given a simple, connected graph G with vertex set $V(G)$ and $k \in \mathbb{Z}_+$, a labeling $f : V(G) \to \mathbb{Z}_+$ is a *k-radio labeling* if f satisfies the inequality

$$|f(u) - f(v)| \geq k + 1 - d(u, v)$$

for all distinct $u, v \in V(G)$. Under this definition, 1-radio labeling is equivalent to vertex coloring, and 2-radio labeling (known by several names including $L(2,1)$-labeling) is another related labeling that has likewise been a central focus of study (for a survey, see [21]). In this paper we work with radio labeling, which is k-radio labeling when k is maximized at diameter[1].

Definition. For a simple, connected graph G, a labeling $f : V(G) \to \mathbb{Z}_+$ is a *radio labeling of* G if it satisfies

$$|f(u) - f(v)| \geq \operatorname{diam}(G) + 1 - d(u, v) \tag{1}$$

for all distinct vertices $u, v \in V(G)$. Inequality (1) is called the *radio condition*.

The *span* of a labeling f is the largest element in the range of f; the minimal possible span of any radio labeling for a fixed graph G is called the *radio number of* G ([2]), and is denoted $\operatorname{rn}(G)$. The main objective is to find formulas or bounds for the radio numbers of families of graphs. Two of the first of these results were formulas for paths and cycles (Liu and Zhu [13]), and results for other types of graphs include [1, 4, 10–12, 14–19] and [20]. The computational complexity of k-radio labeling has been studied with partial results (e.g., [5]) and remains unknown in general. In practice, the problem was well-summarized by Liu and Zhu in [13]: "It is surprising that determining the radio number seems a difficult problem even for some basic families of graphs."

Radio labeling differs from all other k-radio labeling in that it is necessarily injective (from which we see $\operatorname{rn}(G) \geq |V(G)|$). We will be interested in graphs G for which a surjective radio labeling $f : V(G) \to \{1, 2, \ldots, n\}$ exists.

Definition. A radio labeling f of a graph G is a *consecutive radio labeling of* G if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. A graph for which a consecutive radio labeling exists is called *radio graceful*.

Equivalently, G is radio graceful if $\operatorname{rn}(G) = |V(G)|$. The term "radio graceful" was introduced by Sooryanarayana and Ranghunath in [19]. We will use the language "G has a consecutive radio labeling" and "G is radio graceful" interchangeably.

Related definitions have been given for some of the other k-radio labelings, including full colorings and no-hole colorings for $L(2,1)$-labeling (Fishburn, Roberts in [6] and [7]). This study has direct connections to consecutive radio labeling when $\operatorname{diam}(G) = 2$, which we do not limit ourselves to here. However, the results

[1] We typically bound k above by $\operatorname{diam}(G)$ because of its natural relationship to distance in G.

[2] We use the convention that the codomain of a radio labeling is $\mathbb{Z}_+$, while some authors use a codomain of $\mathbb{Z}_+ \cup \{0\}$. Radio numbers under the former convention are one greater than those under the latter convention.

about diameter two graphs we establish apply to $L(2,1)$-labeling in addition to radio labeling.

The complete graphs K_n are trivial examples of radio graceful graphs (any injective labeling with consecutive integers satisfies the radio condition for K_n), and the Petersen graph is another well-known example. Higher diameter examples are desirable because they are more specialized. Observe that if $V(G) = \{v_1, v_2, \ldots, v_n\}$, then G is radio graceful if and only if there exists an ordering $x_1, x_2, \ldots, x_n$ of its vertices such that

$$d(x_i, x_{i+\Delta}) \geq \operatorname{diam}(G) - \Delta + 1 \tag{2}$$

for all $\Delta \in \{1, 2, \ldots, \operatorname{diam}(G)\}$, $i \in \{1, 2, \ldots, n - \Delta\}$. In particular, when $\operatorname{diam}(G) = 2$, G is radio graceful if and only if the complement of G has a Hamiltonian path. The larger the diameter, the more values of Δ must be considered when checking that an ordering satisfies (2).

The main result of this paper establishes the existence of radio graceful graphs of arbitrary diameter, a fact previously unknown. It employs the Cartesian product of graphs. The Cartesian product of graphs G and H, denoted $G \square H$, has vertex set $V(G) \times V(H)$, and has edges defined by the following property. Vertices $(u, v), (u', v') \in V(G \square H)$ are adjacent if $u = u'$ and v is adjacent to v' in H, or if $v = v'$ and u is adjacent to u' in G. The Cartesian product of t copies of a graph G is denoted G^t.

Theorem 1. *Let $n \in \mathbb{Z}$, $n \geq 3$, and $t \in \{1, 2, \ldots, n\}$. Then K_n^t is radio graceful.*[3]

We remind the reader that $\operatorname{diam}(G \square H) = \operatorname{diam}(G) + \operatorname{diam}(H)$ and therefore $\operatorname{diam}(K_n^t) = t$. By choosing $t = n$, for example, this theorem shows the existence of radio graceful graphs of arbitrary diameter, as advertised. In fact, the theorem gives infinitely many examples of radio graceful graphs of any specified diameter.

These graphs K_n^t show up as Hamming graphs, which are of interest in coding theory.

Definition. A *Hamming graph* is a graph of the form $K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}$ where $n_1, n_2, \ldots, n_d$ are (not necessarily distinct) integers with both $d \geq 2$ and $n_i \geq 2$ for all i.

Another result we give states that a Hamming graph with $n_1, n_2, \ldots, n_d$ relatively prime is radio graceful.

2. Preliminaries

Graphs are assumed simple and connected unless otherwise stated. We denote the distance between x and y in G by $d_G(x, y)$, or, where no ambiguity is created,

[3] The $n \geq 3$ condition is required; it is easily checked that K_2^2 is not radio graceful.

by $d(x, y)$. We use the convention throughout that $a \pmod n \in \{1, 2, \ldots, n\}$ for all $a \in \mathbb{Z}$.

Given an ordering $x_1, x_2, \ldots, x_n$ of the vertices of G, we can always construct a radio labeling f with the property that $f(x_i) < f(x_j)$ for all $i < j$. The radio labeling of minimal span that satisfies this property is called the radio labeling *induced* by the ordering. In particular, if the ordering $x_1, x_2, \ldots, x_n$ of $V(G)$ induces a consecutive radio labeling of G, then $f(x_i) = i$ is that induced labeling.

We will apply the following strategy for proving that a graph G is radio graceful: (1) Give a list of vertices of G, (2) prove that the given list is an ordering (i.e., no repetition, no exclusion) of $V(G)$, and (3) prove that this ordering induces a consecutive radio labeling. As vertices of a Cartesian product of t graphs are represented by t-tuples, it will be useful (particularly in the third step) to keep track of the number of instances where two vertices have identical entries. Thus, we define a function $\pi(x_i, x_j)$ which counts the number of coordinates over which vertices x_i and x_j agree.

Definition. Let $G = G_1 \Box G_2 \Box \cdots \Box G_t$. For $x_i \in V(G)$, let the coordinate representation of x_i be $x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_t})$. Then we define

$$\pi(x_i, x_j) : V(G) \times V(G) \to \{0, 1, \ldots, t\}$$

by $\pi(x_i, x_j) = \sum_{k=1}^{t} \pi_k(x_i, x_j)$ where $\pi_k(x_i, x_j) = \begin{cases} 1 & \text{if } x_{i_k} = x_{j_k} \\ 0 & \text{otherwise} \end{cases}$.

Using this definition we see that, for $t \in \mathbb{Z}_+$,

$$d_{G^t}(x_i, x_j) = \sum_{k=1}^{t} d_G(x_{i_k}, x_{j_k}) \leq \operatorname{diam}(G)(t - \pi(x_i, x_j)), \tag{3}$$

and for a Hamming graph $H = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_t}$,

$$d_H(x_i, x_j) = t - \pi(x_i, x_j). \tag{4}$$

Proposition 2. *Let G be a graph of order n, let $t \in \mathbb{Z}_+$, and let $x_1, x_2, \ldots, x_{n^t}$ be an ordering of $V(G^t)$ that induces a consecutive radio labeling of G^t. Then $\pi(x_i, x_j) \leq \dfrac{|i-j| - 1}{\operatorname{diam}(G)}$ for all $i, j \in \{1, 2, \ldots, n^t\}$.*

Proof. As $x_1, x_2, \ldots, x_{n^t}$ is an ordering of $V(G^t)$ that induces a consecutive radio labeling of G^t, $f : V(G^t) \to \mathbb{Z}_+$ defined by $f(x_i) = i$ must satisfy the radio condition:

$$d(x_i, x_j) \geq \operatorname{diam}(G^t) - |f(x_i) - f(x_j)| + 1 = t \cdot \operatorname{diam}(G) - |i - j| + 1.$$

Combining this bound on $d(x_i, x_j)$ with the one given in (3),

$$t \cdot \operatorname{diam}(G) - |i-j| + 1 \leq \operatorname{diam}(G)(t - \pi(x_i, x_j)).$$

Thus, $\pi(x_i, x_j) \leq \dfrac{|i-j|-1}{\operatorname{diam}(G)}$. ∎

Proposition 3. *Let H be the Hamming graph $K_{n_1} \square K_{n_2} \square \cdots \square K_{n_t}$ of order N. An ordering $x_1, x_2, \ldots x_N$ of $V(H)$ induces a consecutive radio labeling of H if and only if $\pi(x_i, x_j) \leq |i-j| - 1$ for all $i, j \in \{1, 2, \ldots, N\}$.*

Proof. ($\Rightarrow$) The proof is nearly identical to that of Proposition 2, using (4) rather than (3), and noting that $\operatorname{diam}(K_{n_i}) = 1$ for all $i \in \{1, 2, \ldots, t\}$.

($\Leftarrow$) Let $\pi(x_i, x_j) \leq |i-j| - 1$ for all $i, j \in \{1, 2, \ldots, N\}$. Then

$$t - |i-j| + 1 \leq t - \pi(x_i, x_j).$$

Since $t = \operatorname{diam}(H)$ and $t - \pi(x_i, x_j) = d(x_i, x_j)$, we have

$$\operatorname{diam}(H) - |i-j| + 1 \leq d(x_i, x_j).$$

Thus, $\operatorname{diam}(H) + 1 - d(x_i, x_j) \leq |i-j| = |f(x_i) - f(x_j)|$.

Therefore, the labeling $f : V(H) \to \mathbb{Z}_+$ defined by $f(x_i) = i$ satisfies the radio condition. ∎

3. K_n^t Is Radio Graceful for Some t

In this section we establish our main result, starting with step one: defining what will end up being an ordering of $V(K_n^t)$ that induces a consecutive radio labeling of K_n^t.

3.1. Definition of $x_1, x_2, \ldots, x_{n^t}$

Let $n \geq 3$, and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Consider K_n^t where $t \in \{1, 2, \ldots, n\}$. We describe a list of the vertices of K_n^t in groups of n vertices at a time, organized as $n \times t$ matrices. The first n vertices are given by the rows of the matrix $A^{(1)} = \left[a_{i,j}^{(1)}\right]$ defined by

$$A^{(1)} = \begin{bmatrix} v_1 & v_1 & \cdots & v_1 \\ v_2 & v_2 & \cdots & v_2 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & \cdots & v_n \end{bmatrix}.$$

We will create a total of n^{t-1} matrices: $A^{(1)}, A^{(2)}, \ldots, A^{(n^{t-1})}$. To produce $A^{(k)}$, $2 \leq k \leq n^{t-1}$, we first determine p, the largest integer such that $k \equiv 1 \pmod{n^p}$. (Note that $p = 0$ will satisfy the equivalence for all k.) Then $A^{(k)}$ is the $n \times t$ matrix made up of entries

$$a_{i,j}^{(k)} = \begin{cases} \sigma\left(a_{i,j}^{(k-1)}\right) & \text{if } j = t - p, \\ a_{i,j}^{(k-1)} & \text{otherwise,} \end{cases}$$

where $\sigma \in S_{V(K_n)}$ is the n-cycle $(v_1 v_2 \cdots v_n)$. Observe that, for each $k \in \{2, 3, \ldots, n^{t-1}\}$, $A^{(k)}$ is identical to $A^{(k-1)}$ except for a single column which differs by an application of σ. If $2 \leq k \leq n^{t-1}$, and $k \equiv 1 \pmod{n^p}$, then $p \leq t - 2$. Consequently, σ is never applied to the first column during the construction of these matrices. (We note now, as it will be important later, that this implies all of the matrices $A^{(1)}, A^{(2)}, \ldots, A^{(n^{t-1})}$ have the same first column.)

The list $x_1, x_2, \ldots, x_{n^t}$ is given by the rows of the matrices in the natural way: if $i = bn + c$, $c \in \{1, 2, \ldots, n\}$, then

$$x_i = x_{bn+c} = \left(a_{c,1}^{(b+1)}, a_{c,2}^{(b+1)}, \ldots, a_{c,t}^{(b+1)}\right). \tag{5}$$

For example, the list of vertices $x_1, x_2, \ldots, x_{27}$ for K_3^3 is given in Table 1.

$x_1 : (v_1, v_1, v_1)$	$x_{10} : (v_1, v_2, v_3)$	$x_{19} : (v_1, v_3, v_2)$
$x_2 : (v_2, v_2, v_2)$	$x_{11} : (v_2, v_3, v_1)$	$x_{20} : (v_2, v_1, v_3)$
$x_3 : (v_3, v_3, v_3)$	$x_{12} : (v_3, v_1, v_2)$	$x_{21} : (v_3, v_2, v_1)$
$x_4 : (v_1, v_1, v_2)$	$x_{13} : (v_1, v_2, v_1)$	$x_{22} : (v_1, v_3, v_3)$
$x_5 : (v_2, v_2, v_3)$	$x_{14} : (v_2, v_3, v_2)$	$x_{23} : (v_2, v_1, v_1)$
$x_6 : (v_3, v_3, v_1)$	$x_{15} : (v_3, v_1, v_3)$	$x_{24} : (v_3, v_2, v_2)$
$x_7 : (v_1, v_1, v_3)$	$x_{16} : (v_1, v_2, v_2)$	$x_{25} : (v_1, v_3, v_1)$
$x_8 : (v_2, v_2, v_1)$	$x_{17} : (v_2, v_3, v_3)$	$x_{26} : (v_2, v_1, v_2)$
$x_9 : (v_3, v_3, v_2)$	$x_{18} : (v_3, v_1, v_1)$	$x_{27} : (v_3, v_2, v_3)$

Table 1. List of vertices for K_3^3.

3.2. The list is an ordering of $V(K_n^t)$

Our goal now is to show that this definition of $x_1, x_2, \ldots, x_{n^t}$ is an ordering of the vertices of K_n^t by proving $x_i \neq x_j$ for all $i \neq j$. Notice that each matrix in the definition inherits some structure from $A^{(1)}$: $a_{i,j}^{(k)} = \sigma\left(a_{i-1,j}^{(k)}\right)$ for all $i \in \{2, 3, \ldots, n\}$, $j \in \{1, 2, \ldots, t\}$, $k \in \{1, 2, \ldots, n^{t-1}\}$. We conclude $A^{(k)}$ has no duplicate rows for all $k \in \{1, 2, \ldots, n^{t-1}\}$. The last observation, along with our earlier one that all of the matrices $A^{(1)}, A^{(2)}, \ldots, A^{(n^{t-1})}$ have the same first column, allow us to reduce the problem. We only need to prove that no two matrices in the set $\left\{A^{(1)}, A^{(2)}, \ldots, A^{(n^{t-1})}\right\}$ have the same first row. So we form a new matrix A from these first rows and prove that A's rows are all distinct.

Let $A = [a_{i,j}]$ be the $n^{t-1} \times t$ matrix defined by $a_{i,j} = a^{(i)}_{1,j}$. Equivalently, let p be the largest integer such that $i \equiv 1 \pmod{n^p}$. Then

$$a_{i,j} = \begin{cases} v_1 & \text{if } i = 1, \\ \sigma(a_{i-1,j}) & \text{if } j = t - p, \\ a_{i-1,j} & \text{otherwise} \end{cases}$$

defines A. We now make notes on the repetitive structure of A by partitioning each column into uniform *blocks*.

Definition. A *j-block* is any one of the vectors

$$\begin{bmatrix} a_{1,j} \\ \vdots \\ a_{n^{t-j},j} \end{bmatrix}, \begin{bmatrix} a_{n^{t-j}+1,j} \\ \vdots \\ a_{2n^{t-j},j} \end{bmatrix}, \ldots, \begin{bmatrix} a_{(n^{j-1}-1)n^{t-j}+1,j} \\ \vdots \\ a_{n^{t-1},j} \end{bmatrix}.$$

We will call $\begin{bmatrix} a_{(c-1)n^{t-j}+1,j} \\ \vdots \\ a_{cn^{t-j},j} \end{bmatrix}$ the c^{th} *j-block*, and denote it $\beta(c, j)$.

Note that, for each $j \in \{1, 2, \ldots, t\}$, there are n^{j-1} j-blocks, each of dimension n^{t-j}.

We will keep track of the rows associated to each j-block with the next definition.

Definition. The *scope of* $\beta(c, j)$ is the set of consecutive integers $\{(c-1)n^{t-j} + 1, (c-1)n^{t-j} + 2, \ldots, cn^{t-j}\}$. The scope of multiple j-blocks is the union of the scopes of the individual j-blocks.

Proposition 4. *Let $j \in \{1, 2, \ldots, t\}$, and $c \in \{1, 2, \ldots, n^{j-1}\}$. The vector $\beta(c, j)$ has identical entries.*

Proof. The $j = t$ case is immediate since every block has only a single entry. Let $j \in \{1, 2, \ldots, t-1\}$, $c \in \{1, 2, \ldots, n^{j-1}\}$. Based on the definition of A, $a_{i,j} = a_{i-1,j}$ unless $j = t - p$ where p is the largest integer such that $i \equiv 1 \pmod{n^p}$. However, $i \not\equiv 1 \pmod{n^{t-j}}$ for all $i \in \{(c-1)n^{t-j} + 2, (c-1)n^{t-j} + 3, \ldots, cn^{t-j}\}$, and therefore $j \neq t - p$. Hence $a_{(c-1)n^{t-j}+1,j} = a_{(c-1)n^{t-j}+2,j} = \cdots = a_{cn^{t-j},j}$. ∎

One conclusion to draw from this proposition (in light of the definition of A) is that $\beta(c,j)$ and $\beta(c+1,j)$ have two possible relationships: either $\beta(c,j) = \beta(c+1,j)$, or $\sigma(\beta(c,j)) = \beta(c+1,j)$.

Proposition 5. *Let $j \in \{1,2,\ldots,t\}$, and $c \in \{1,2,\ldots,n^{j-1}-1\}$. Consecutive j-blocks $\beta(c,j)$ and $\beta(c+1,j)$ are identical if and only if n divides c.*

Proof. It follows from Proposition 4 that the blocks are identical if and only if $a_{cn^{t-j},j} = a_{cn^{t-j}+1,j}$, or equivalently if $j \neq t-p$, where p is computed for $i = cn^{t-j}+1$. Since $cn^{t-j}+1 \equiv 1 \pmod{n^{t-j}}$, $p \neq t-j$ if and only if $p > t-j$, which is equivalent to requiring that n divides c. ■

Now we examine how the structure of the different columns relate. It happens that the scope of a $(j-1)$-block is equal to the scope of n consecutive j-blocks. This is immediate because of the size of the blocks. A $(j-1)$-block has n^{t-j+1} entries, and j-blocks have n^{t-j} entries. For every block in column $j-1$, there are $n^{t-j+1}/n^{t-j} = n$ blocks in column j. Precisely, the scope of $\beta(c, j-1)$ is equal to the scope of $\{\beta((c-1)n+1,j), \beta((c-1)n+2,j), \ldots, \beta(cn,j)\}$. We show that this is a collection of distinct j-blocks.

Proposition 6. *Let $j \in \{2,3,\ldots,t\}$. Then $\beta(x,j) \neq \beta(y,j)$ for all distinct $x, y \in \{(c-1)n+1, (c-1)n+2, \ldots, cn\}$.*

Proof. Let x, y be distinct elements of $\{(c-1)n+1, (c-1)n+2, \ldots, cn\}$. Without loss of generality, say $y = x + \Delta$. Since n does not divide any element of $\{(c-1)n+1, (c-1)n+2, \ldots, cn-1\}$, it follows from Proposition 5 that $\beta(cn,j) = \sigma(\beta(cn-1,j)) = \sigma^2(\beta(cn-2,j)) = \cdots = \sigma^{n-1}(\beta((c-1)n+1,j))$. In particular, $\beta(y,j) = \beta(x+\Delta,j) = \sigma^{\Delta}(\beta(x,j))$. Therefore $\beta(x,j) \neq \beta(y,j)$. ■

Lemma 7. *The matrix A has no identical rows.*

Proof. We begin by proving the following claim using an induction argument.

Claim. *If two rows share their first k entries, then they both belong to the scope of the same k-block.*

Proof. The $k = 1$ case is trivial as there is exactly one 1-block. Suppose the claim is true for k, and let the x^{th} and y^{th} rows share their first $k+1$ entries. Then they must share their first k entries, so by assumption rows x and y belong to the scope of a single k-block. By Proposition 6, this scope is equal to that of a collection of n distinct $(k+1)$-blocks, and since rows x and y also share the $(k+1)$st entry, they can only be in the scope of one of those $(k+1)$-blocks. This proves the claim. □

Now, if the x^{th} and y^{th} rows in A share all t of their entries, then the preceding claim asserts that they belong to the scope of the same t-block. A t-block consists of $n^{t-t} = 1$ entry, so $x = y$, and the lemma is established. ■

With this, the objective of this section has been achieved.

Theorem 8. *Let $n \in \mathbb{Z}$, $n \geq 3$, and $t \in \{1, 2, \ldots, n\}$. As defined in* (5)*, $x_1, x_2, \ldots, x_{n^t}$ is an ordering of the vertices of K_n^t.*

3.3. Proof of Theorem 1

We now prove that the ordering defined in the previous section induces a consecutive radio labeling. Let $x_1, x_2, \ldots, x_{n^t}$ be the ordering of $V(K_n^t)$ from Theorem 8, and define $f : V(K_n^t) \to \mathbb{Z}_+$ by $f(x_i) = i$ for all $i \in \{1, 2, \ldots, n^t\}$. We will prove that f satisfies the radio condition by showing

$$\pi(x_i, x_{i+\Delta}) \leq \Delta - 1 \tag{6}$$

for all $\Delta \in \{1, 2, \ldots, n^t - 1\}$, $i \in \{1, 2, \ldots, n^t - \Delta\}$ (Proposition 3). By Theorem 8, $\pi(x_i, x_{i+\Delta}) \leq t-1$; it remains to show (6) for $\Delta \in \{1, 2, \ldots, t-1\}$. We will do this in three cases, and we will again utilize the matrix $A^{(k)}$, defined in Section 3.1.

Let x_i be a row in $A^{(k)}$, and let $\Delta \leq t-1$. Since $\Delta < n$, the row associated with $x_{i+\Delta}$ must lie in either $A^{(k)}$ or $A^{(k+1)}$.

Case 1. $x_{i+\Delta}$ in $A^{(k)}$. Suppose $x_{i+\Delta}$ is also a row in $A^{(k)}$. Because $a_{i,j}^{(k)} = \sigma\left(a_{i-1,j}^{(k)}\right)$, $\pi(x_i, x_{i+\Delta}) = 0 \leq \Delta - 1$.

Case 2. $x_{i+\Delta}$ in $A^{(k+1)}$, $\Delta \neq 1$. Suppose $x_{i+\Delta}$ is a row in $A^{(k+1)}$, and let $\Delta > 1$. Recall that the matrices $A^{(k)}$ and $A^{(k+1)}$ are identical except for one column which differs by an application of σ. This, along with the assumption that $\Delta < n$, implies $\pi(x_i, x_{i+\Delta}) \leq 1 \leq \Delta - 1$.

Case 3. $x_{i+\Delta}$ in $A^{(k+1)}$, $\Delta = 1$. Let $x_{i+\Delta}$ be a row in $A^{(k+1)}$ and let $\Delta = 1$. That is, $x_i = \left(a_{n,1}^{(k)}, \ldots, a_{n,t}^{(k)}\right)$ is the n^{th} row of $A^{(k)}$, and $x_{i+\Delta} = \left(a_{1,1}^{(k+1)}, \ldots, a_{1,t}^{(k+1)}\right)$ is the first row of $A^{(k+1)}$. Let $j_* = t - p$ where p is the largest integer such that $k + 1 \equiv 1 \pmod{n^p}$. Then, by definition of $A^{(k+1)}$,

$$x_{i+\Delta} = \left(a_{1,1}^{(k)}, \ldots, a_{1,j_*-1}^{(k)}, \sigma\left(a_{1,j_*}^{(k)}\right), a_{1,j_*+1}^{(k)}, \ldots, a_{1,t}^{(k)}\right).$$

The vertex x_i is also easily expressed in terms of the first row of $A^{(k)}$:

$$x_i = \left(\sigma^{n-1}\left(a_{1,1}^{(k)}\right), \ldots, \sigma^{n-1}\left(a_{1,j_*}^{(k)}\right), \ldots, \sigma^{n-1}\left(a_{1,t}^{(k)}\right)\right).$$

Given that σ is an n-cycle with $n \geq 3$, it is apparent $\pi(x_i, x_{i+\Delta}) = 0 \leq \Delta - 1$. This proves Theorem 1.

4. Other Hamming Graphs

The complete graphs K_n immediately show that there are graphs of any order that have consecutive radio labelings. This section gives a partial answer to a question that logically follows that observation: Are there *nontrivial* radio graceful graphs of arbitrary order?

Theorem 9. *Let the Hamming graph $H = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_s}$ have the property that $n_1, n_2, \ldots, n_s$ are relatively prime. Then H is radio graceful.*

Proof. For each $i \in \{1, 2, \ldots, s\}$, let $V(K_{n_i}) = \{v^i_1, v^i_2, \ldots, v^i_{n_i}\}$. Then, to simplify notation, we refer to an arbitrary vertex of H, $(v^1_{a_1}, v^2_{a_2}, \ldots, v^s_{a_s})$, by the s-tuple $(a_1, a_2, \ldots, a_s)$, an element of $\{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\} \times \cdots \times \{1, 2, \ldots, n_s\}$.

Let $N = n_1 n_2 \cdots n_s$. For $k \in \{1, 2, \ldots, N\}$, define

$$x_k = \Big(k \pmod{n_1},\ k \pmod{n_2}, \ldots, k \pmod{n_s}\Big).$$

We show that $x_1, x_2, \ldots, x_N$ is an ordering of the vertices of H by proving $x_j \neq x_k$ whenever $j \neq k$.

Suppose $x_j = x_{j+\Delta}$ for some $\Delta \in \{0, 1, \ldots, N-1\}$, $j \in \{1, 2, \ldots, N-\Delta\}$. Then

$$\begin{aligned}&\Big(j \pmod{n_1},\ j \pmod{n_2}, \ldots, j \pmod{n_s}\Big)\\ &= \Big(j+\Delta \pmod{n_1},\ j+\Delta \pmod{n_2}, \ldots, j+\Delta \pmod{n_s}\Big).\end{aligned}$$

Since $n_1, n_2, \ldots, n_s$ are relatively prime, N divides $\Delta \in \{0, 1, \ldots, N-1\}$. Therefore, $\Delta = 0$ and $x_j = x_{j+\Delta}$. This shows that $x_1, x_2, \ldots, x_N$ is an ordering of $V(H)$.

Let $f : V(H) \to \mathbb{Z}_+$ be defined by $f(x_k) = k$. We show f is a radio labeling of H by using Proposition 3. That is, by proving that

$$\pi(x_k, x_{k+\Delta}) \leq \Delta - 1 \tag{7}$$

for all $\Delta \in \{1, 2, \ldots, N-1\}$, $k \in \{1, 2, \ldots, N-\Delta\}$. Since $x_k \neq x_{k+\Delta}$ when $\Delta \neq 0$, (7) is satisfied for $\Delta \geq s$.

Let $\Delta \in \{1, 2, \ldots, s-1\}$. From the definition of $\pi(x_k, x_{k+\Delta})$, it follows that the set $\mathbb{I}$ of all values of i such that $k \equiv k + \Delta \pmod{n_i}$ has order $\pi(x_k, x_{k+\Delta})$. Then n_i divides Δ for all $i \in \mathbb{I}$. As Δ can have at most $\Delta - 1$ prime divisors, we have $\pi(x_k, x_{k+\Delta}) \leq \Delta - 1$, and f gives a consecutive radio labeling of H. ■

Corollary 10. *If $n \in \mathbb{Z}_+$ has at least s distinct prime divisors, then there is a radio graceful graph with n vertices and diameter s.*

Proof. If n has $s = 1$ distinct prime divisors, then K_n is a graph with n vertices and diameter s that has a consecutive radio labeling. Consider $s > 1$, and let n have a prime factorization of

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$$

where $t \geq s$. Set $n_i = p_i^{\alpha_i}$ for $i \in \{1, 2, \ldots, s-1\}$, and let $n_s = p_s^{\alpha_s} p_{s+1}^{\alpha_{s+1}} \cdots p_t^{\alpha_t}$. By Theorem 9, $H = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_s}$ is a radio graceful graph of order n and diameter s. ■

5. G^t Is Not Radio Graceful for Some t

In light of Section 3, it is natural to wonder if K_n^t might have a consecutive radio labeling for all $t \in \mathbb{Z}_+$. Theorem 11 gives the negative answer not only for K_n but for any graph.

Theorem 11. *Given a graph G, there is an integer s such that G^t does not have a consecutive radio labeling for any $t \geq s$. In particular, if G has n vertices,*

$$s = 1 + \sum_{k=\operatorname{diam}(G)}^{n-1} (n-k) \left\lfloor \frac{k}{\operatorname{diam}(G)} \right\rfloor$$

is such a value.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $s = 1 + \sum_{k=\operatorname{diam}(G)}^{n-1} (n-k) \lfloor \frac{k}{\operatorname{diam}(G)} \rfloor$, and let $t \in \mathbb{Z}$, $t \geq s$. In search of contradiction, suppose $x_1, x_2, \ldots, x_{n^t}$ is an ordering of the vertices such that $f : V(G^t) \to \mathbb{Z}_+$ defined by $f(x_i) = i$ is a radio labeling.

Our reasoning will go as follows. We consider the first $n+1$ vertices in the ordering, $x_1, x_2, \ldots, x_{n+1}$. Assuming the known bounds for $\pi(x_i, x_j)$, we count the maximum possible occurrences of any of these vertices agreeing in some coordinate. We show that this number is less than the number of coordinates t, which means that there must be at least one coordinate for which none of the vertices $x_1, x_2, \ldots, x_{n+1}$ agree. Since these vertices have only n choices for entries, this is not possible, and we will have our desired contradiction.

From the definition, $\pi(x_i, x_j) \leq t - 1$, and by Proposition 2, $\pi(x_i, x_j) \leq \frac{|i-j|-1}{\operatorname{diam}(G)}$. Define

$$\Pi(x_i, x_j) = \min \left\{ t - 1, \left\lfloor \frac{|i-j|-1}{\operatorname{diam}(G)} \right\rfloor \right\}.$$

The maximum number of coordinates that x_i can have in common with any prior vertex is $\sum_{j=1}^{i-1} \Pi(x_i, x_j)$. Then the total number of coordinates in which any of the first $n+1$ vertices in the ordering can agree is

$$\sum_{j=1}^{n}\Pi(x_{n+1},x_j)+\sum_{j=1}^{n-1}\Pi(x_n,x_j)+\cdots+\sum_{j=1}^{1}\Pi(x_2,x_j)=\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}\Pi(x_i,x_j).$$

We can obtain with some computation that

$$\begin{aligned}\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}\Pi(x_i,x_j) &\le \sum_{i=2}^{n+1}\sum_{j=1}^{i-1}\left\lfloor\frac{|i-j|-1}{\operatorname{diam}(G)}\right\rfloor=\sum_{j=1}^{1}\left\lfloor\frac{|2-j|-1}{\operatorname{diam}(G)}\right\rfloor+\sum_{i=3}^{n+1}\sum_{j=1}^{i-1}\left\lfloor\frac{|i-j|-1}{\operatorname{diam}(G)}\right\rfloor\\
&=\sum_{j=1}^{2}\left\lfloor\frac{|3-j|-1}{\operatorname{diam}(G)}\right\rfloor+\sum_{j=1}^{3}\left\lfloor\frac{|4-j|-1}{\operatorname{diam}(G)}\right\rfloor+\cdots+\sum_{j=1}^{n}\left\lfloor\frac{|n+1-j|-1}{\operatorname{diam}(G)}\right\rfloor\\
&=\sum_{j=1}^{2}\left\lfloor\frac{2-j}{\operatorname{diam}(G)}\right\rfloor+\sum_{j=1}^{3}\left\lfloor\frac{3-j}{\operatorname{diam}(G)}\right\rfloor+\cdots+\sum_{j=1}^{n}\left\lfloor\frac{n-j}{\operatorname{diam}(G)}\right\rfloor\\
&=\sum_{k=1}^{1}\left\lfloor\frac{k}{\operatorname{diam}(G)}\right\rfloor+\sum_{k=1}^{2}\left\lfloor\frac{k}{\operatorname{diam}(G)}\right\rfloor+\cdots+\sum_{k=1}^{n-1}\left\lfloor\frac{k}{\operatorname{diam}(G)}\right\rfloor\\
&=(n-1)\left\lfloor\frac{1}{\operatorname{diam}(G)}\right\rfloor+(n-2)\left\lfloor\frac{2}{\operatorname{diam}(G)}\right\rfloor+\cdots+\left\lfloor\frac{n-1}{\operatorname{diam}(G)}\right\rfloor\\
&=\sum_{k=1}^{n-1}(n-k)\left\lfloor\frac{k}{\operatorname{diam}(G)}\right\rfloor=\sum_{k=\operatorname{diam}(G)}^{n-1}(n-k)\left\lfloor\frac{k}{\operatorname{diam}(G)}\right\rfloor=s-1<t.\end{aligned}$$

Since t is larger than the number of coordinates with this property, there is at least one coordinate in which none of the vertices $x_1,x_2,\ldots,x_{n+1}$ agree. This is impossible to accomplish, however, as we have only n possible entries for each coordinate: $v_1,v_2,\ldots,v_n$. Consequently, there is no ordering of the n^t vertices of G^t that induces a consecutive radio labeling. ■

Corollary 12. *K_n^t does not have a consecutive labeling for any $t\ge 1+\frac{n(n^2-1)}{6}$.*

Remark 13. The t^{th} Cartesian power of K_n is an object that has surfaced repeatedly throughout the course of this work. Together, Theorem 1 and Corollary 12 give us a lot of information about K_n^t. If $1\le t\le n$, then K_n^t is radio graceful, while if $t\ge 1+\frac{n(n^2-1)}{6}$ then K_n^t is not. In the future, it would be great to be able to say, for any n and t, whether K_n^t is radio graceful.

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