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GRAPHS WITH LARGE GENERALIZED (EDGE-)CONNECTIVITY

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Abstract

The generalized k-connectivity $\kappa_k(G)$ of a graph G, introduced by Hager in 1985, is a nice generalization of the classical connectivity. Recently, as a natural counterpart, we proposed the concept of generalized k-edge-connectivity $\lambda_k(G)$. In this paper, graphs of order n such that $\kappa_k(G) = n - \frac{k}{2} - 1$ and $\lambda_k(G) = n - \frac{k}{2} - 1$ for even k are characterized.

Keywords: (edge-)connectivity, Steiner tree, internally disjoint trees, edge-disjoint trees, packing, generalized (edge-)connectivity.

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1. Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [3] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$, $\overline{G}$ denote the set of vertices, the set of edges of G and the complement, respectively. Let $d_G(v)$ denote the degree of the vertex v in G. As usual, the *union* of two graphs G and H is the graph, denoted by

$G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let mH be the disjoint union of m copies of a graph H. If M is a subset of edges of a graph G, the subgraph of G induced by M is denoted by $G[M]$, and $G - M$ denotes the subgraph obtained by deleting the edges of M from G. If $M = \{e\}$, we simply write $G - e$ for $G - \{e\}$. If $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained by deleting the vertices of S together with the edges incident with them from G. We denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y. If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$. A subset M of $E(G)$ is called a *matching* of G if the edges of M satisfy that no two of them are adjacent in G. A matching M saturates a vertex v, or v is said to be M*-saturated*, if some edge of M is incident with v; otherwise, v is M*-unsaturated.* If every vertex of G is M-saturated, the matching M is *perfect.* M is a *maximum matching* if G has no matching M' with $|M'| > |M|$.

Connectivity and edge-connectivity are two of the most basic concepts of graph-theoretic subjects, both in a combinatorial sense and an algorithmic sense. As we know, the classical connectivity has two equivalent definitions. The *connectivity* of a graph G, written $\kappa(G)$, is the minimum size of a set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. If $G - S$ is disconnected we call such a set S a *vertex cut-set* for G. We call this definition the 'cut' version definition of connectivity. A well-known Menger's theorem provides an equivalent definition of connectivity, which can be called the 'path' version definition of connectivity. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x, y) \,|\, x, y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G. Similarly, the classical edge-connectivity also has two equivalent definitions. The *edge-connectivity* of G, written $\lambda(G)$, is the minimum size of an edge set $M \subseteq E(G)$ such that $G - M$ is disconnected or has only one vertex. We call this definition the 'cut' version definition of edge-connectivity. Menger's theorem also provides an equivalent definition of edge-connectivity, which can be called the 'path' version definition. For any two distinct vertices x and y in G, the *local edge-connectivity* $\lambda_G(x, y)$ is the maximum number of edge-disjoint paths connecting x and y. Then $\lambda(G) = \min\{\lambda_G(x, y) \,|\, x, y \in V(G),\, x \neq y\}$ is defined to be the *edge-connectivity* of G. For connectivity and edge-connectivity, Oellermann gave a survey paper on this subject, see [34].

Although there are many elegant and powerful results on connectivity in graph theory, the classical connectivity and edge-connectivity also have their defects. So people want some generalizations of both connectivity and edge-connectivity. For the 'cut' version definition of connectivity, we are looking for a minimum vertex-cut with no consideration about the number of components of $G - S$. Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n+1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. Chartrand *et al.* [4] generalized the 'cut' version definition of connectivity. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about k-connectivity, we refer to [4, 6, 35, 36]. The k-edge-connectivity, which is a generalization of the 'cut' version definition of classical edge-connectivity was initially introduced by Boesch and Chen [2] and subsequently studied by Goldsmith in [7, 8] and Goldsmith *et al.* [9]. For more details, we refer to [1, 34].

The generalized connectivity of a graph G, introduced by Hager [12], is a natural and nice generalization of the 'path' version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint Steiner trees connecting S in G. Note that when $|S| = 2$ a minimal Steiner tree connecting S is just a path connecting the two vertices of S. For an integer k with $2 \leq k \leq n$, *generalized k-connectivity* (or *k-tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa(S) \mid S \subseteq V(G), |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. This concept appears to have been introduced by Hager in [12]. It is also studied in [5] for example, where the exact value of the generalized k-connectivity of complete graphs are obtained. Note that the generalized k-connectivity and the k-connectivity of a graph are indeed different. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on the generalized connectivity or tree-connectivity, we refer to [5, 22–31, 37]. Apart from the concept of tree-connectivity, Hager also introduced another tree-connectivity parameter, called the *pendant tree-connectivity* of a graph in [12]. For the tree-connectivity, we only search for edge-disjoint trees which include S and are vertex-disjoint with the exception of the vertices in S. But pendant tree-connectivity further requires the degree of each vertex of S in a Steiner tree connecting S equal to one. Note that it is a special case of the tree-connectivity.

As a natural counterpart of the generalized connectivity, we introduced in [32] the concept of generalized edge-connectivity, which is a generalization of the 'path' version definition of edge-connectivity. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is then defined as $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G)$ and $|S| = k\}$. It is also clear that when $|S| = 2$, $\lambda_2(G)$ is nothing new but the standard edge-connectivity $\lambda(G)$ of G, that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized edge-connectivity of G. Also set $\lambda_k(G) = 0$ when G is disconnected. Results on the generalized edge-connectivity can be found in [28, 29, 32].

In fact, Mader [19] was studying an extension of Menger's theorem to independent sets of three or more vertices. We know from Menger's theorem that if $S = \{u, v\}$ is a set of two independent vertices in a graph G, then the maximum number of internally disjoint u–v paths in G equals the minimum number of vertices that separate u and v. For a set $S = \{u_1, u_2, \ldots, u_k\}$ of k vertices ($k \geq 2$) in a graph G, an *S-path* is defined as a path between a pair of vertices of S that contains no other vertices of S. Two S-paths P_1 and P_2 are said to be *internally disjoint* if they are vertex-disjoint except for their endvertices. If S is a set of independent vertices of a graph G, then a vertex set $U \subseteq V(G)$ with $U \cap S = \emptyset$ is said to *totally separate* S if every two vertices of S belong to different components of $G - U$. Let S be a set of at least three independent vertices in a graph G. Let $\mu(G)$ denote the maximum number of internally disjoint S-paths and $\mu'(G)$ the minimum number of vertices that totally separate S. A natural extension of Menger' s theorem may well be suggested, namely: If S is a set of independent vertices of a graph G and $|S| \geq 3$, then $\mu(S) = \mu'(S)$. However, the statement is not true in general. Take the above graph H_1 for example. For $S = \{v_1, v_2, v_3\}$, $\mu(S) = 1$ but $\mu'(S) = 2$. Mader proved that $\mu(S) \geq \frac{1}{2}\mu'(S)$. Moreover, the bound is sharp. Lovász conjectured an edge analogue of this result and Mader proved this conjecture and established its sharpness. For more details, we refer to [19, 20, 34].

In addition to being natural combinatorial measures, the Steiner Tree Packing Problem (defined as follows) and the generalized edge-connectivity can be motivated by their interesting interpretation in practice as well as theoretical consideration. From a theoretical perspective, both extremes of this problem are fundamental theorems in combinatorics. One extreme of the problem is when we have two terminals. In this case internally (edge-)disjoint trees are just internally (edge-)disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case internally disjoint Steiner trees and edge-disjoint trees are just edge-disjoint spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem.

Theorem 1.1 (Nash-Williams [33], Tutte [39]). *A multigraph G contains a system of ℓ edge-disjoint spanning trees if and only if*

$$\|G/\mathscr{P}\| \geq \ell(|\mathscr{P}| - 1)$$

holds for every partition $\mathscr{P}$ of $V(G)$, where $\|G/\mathscr{P}\|$ denotes the number of crossing edges in G, i.e., edges between distinct parts of $\mathscr{P}$.

The generalized edge-connectivity is related to an important problem, which is called the *Steiner Tree Packing Problem* (definied as follows). For a given graph G and $S \subseteq V(G)$, this problem asks to find a set of maximum number of edge-disjoint Steiner trees connecting S in G. One can see that the Steiner Tree Packing Problem studies local properties of graphs, but the generalized edge-connectivity focuses on global properties of graphs. The generalized edge-connectivity and the Steiner Tree Packing Problem have applications in $VLSI$ circuit design, see [10, 11, 38]. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. Another application, which is our primary focus, arises in the Internet Domain. Imagine that a given graph G represents a network. We choose arbitrary k vertices as nodes. Suppose that one of the nodes in G is a *broadcaster*, and all the other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. So, in essence we need to find the maximum number of Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda(S)$, where S is the set of the k nodes. Clearly, it is a Steiner Tree Packing Problem. Furthermore, if we want to know whether for any k nodes the network G has the above properties, then we need to compute $\lambda_k(G) = \min\{\lambda(S)\}$ in order to prescribe the reliability and the security of the network.

The following two observations are easily seen from the definitions.

Observation 1.2. *Let k, n be two integers with $3 \leq k \leq n$. For a connected graph G of order n, $\kappa_k(G) \leq \lambda_k(G) \leq \delta(G)$.*

Observation 1.3. *Let k, n be two integers with $3 \leq k \leq n$. If H is a spanning subgraph of G of order n, then $\lambda_k(H) \leq \lambda_k(G)$.*

Chartrand *et al.* in [5] got the exact value of the generalized k-connectivity for the complete graph K_n.

Lemma 1.4 [5]**.** *For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil k/2 \rceil$.*

In [32] we obtained some results on the generalized k-edge-connectivity. The following results are restated, which will be used later.

Lemma 1.5 [32]**.** *For every two integers n and k with $2 \leq k \leq n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

Lemma 1.6 [32]**.** *Let k, n be two integers with $3 \leq k \leq n$. For a connected graph G of order n, $1 \leq \kappa_k(G) \leq \lambda_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

We also characterized graphs attaining the upper bound and obtained the following result.

Lemma 1.7 [32]**.** *Let k, n be two integers with $3 \leq k \leq n$. For a connected graph G of order n, $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ or $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for even k; $G = K_n - M$ for odd k, where M is a set of edges such that $0 \leq |M| \leq \frac{k-1}{2}$.*

One may notice that the graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ are the same as the graphs with $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$. Our motivation of this paper is to ask whether the graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil - 1$ are different from the graphs with $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil - 1$. In this paper, graphs of order n such that $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil - 1$ and $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil - 1$ for any even k are characterized.

Theorem 1.8. *Let n and k be two integers such that k is even and $4 \leq k \leq n$, and G be a connected graph of order n. Then $\kappa_k(G) = n - \frac{k}{2} - 1$ if and only if $G = K_n - M$ where M is a set of edges such that $1 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $1 \leq |M| \leq k - 1$.*

The above result can also be established for the generalized k-edge-connectivity, which is stated as follows.

Theorem 1.9. *Let n and k be two integers such that k is even and $4 \leq k \leq n$, and G be a connected graph of order n. Then $\lambda_k(G) = n - \frac{k}{2} - 1$ if and only if $G = K_n - M$ where M is a set of edges satisfying one of the following conditions:*

(1) *$\Delta(K_n[M]) = 1$ and $1 \leq |M| \leq \lfloor \frac{n}{2} \rfloor$;*

(2) *$2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $1 \leq |M| \leq k - 1$.*

2. Main Result

To begin with, we give the following lemmas.

Lemma 2.1. *If G is a graph obtained from the complete graph K_n by deleting a set of edges M such that $\Delta(K_n[M]) \geq r$, then $\lambda_k(G) \leq n-1-r$.*

Proof. Since $\Delta(K_n[M]) \geq r$, there exists at least one vertex, say v, such that $d_{K_n[M]}(v) \geq r$. Then $d_G(v) = n-1-d_{K_n[M]}(v) \leq n-1-r$. So $\delta(G) \leq d_G(v) \leq n-1-r$. From Observation 1.2, $\lambda_k(G) \leq \delta(G) \leq n-1-r$. ■

Corollary 2.2. *For every two integers n and k with $4 \leq k \leq n$, if k is even and M is a set of edges in the complete graph K_n such that $\Delta(K_n[M]) \geq \frac{k}{2}+1$, then $\kappa_k(K_n - M) \leq \lambda_k(K_n - M) < n - \frac{k}{2} - 1$.*

Remark 2.1. From Corollary 2.2, if $\kappa_k(K_n - M) = n - \frac{k}{2} - 1$ or $\lambda_k(K_n - M) = n - \frac{k}{2} - 1$ for k even, then $\Delta(K_n[M]) \leq \frac{k}{2}$.

In [32], we stated a useful lemma for general k.

Let $S \subseteq V(G)$ be such that $|S| = k$, and $\mathscr{T}$ be a maximum set of edge-disjoint S-Steiner trees in G. Let $\mathscr{T}_1$ be the set of trees in $\mathscr{T}$ whose edges belong to $E(G[S])$, and $\mathscr{T}_2$ be the set of S-Steiner trees containing at least one edge of $E_G[S, \bar{S}]$, where $\bar{S} = V(G) - S$. Thus, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. (Throughout this paper, $\mathscr{T}$, $\mathscr{T}_1$, $\mathscr{T}_2$ are defined in this way.)

Lemma 2.3 [32]**.** *Let G be a connected graph of order n, and $S \subseteq V(G)$ with $|S| = k$ $(3 \leq k \leq n)$ and let T be an S-Steiner tree. If $T \in \mathscr{T}_1$, then T contains exactly $k-1$ edges of $E(G[S])$. If $T \in \mathscr{T}_2$, then T contains at least k edges of $E(G[S]) \cup E_G[S, \bar{S}]$.*

Lemma 2.4. *For every two integers n and k with $4 \leq k \leq n$, if k is even and M is a set of edges of the complete graph K_n such that $|M| \geq k$ and $\Delta(K_n[M]) \geq 2$, then $\lambda_k(K_n - M) < n - \frac{k}{2} - 1$.*

Proof. Set $G = K_n - M$. We claim that there is an $S \subseteq V(G)$ with $|S| = k$ such that $|M \cap \big(E(K_n[S]) \cup E_{K_n}[S, \bar{S}]\big)| \geq k$ and $|M \cap E(K_n[S])| \geq 1$. Choose a subset M' of M such that $|M'| = k$. Suppose that $K_n[M']$ contains s independent edges and r connected components $C_1, \ldots, C_r$ such that $\Delta(C_i) \geq 2$ $(1 \leq i \leq r)$. Set $|V(C_i)| = n_i$ and $|E(C_i)| = m_i$. Then $m_i \geq n_i - 1$. For each C_i $(1 \leq i \leq r)$, we select one of the vertices having maximum degree, say u_i. Set $X_i = V(C_i) - u_i$.

If there exists some X_j such that $|E(K_n[X_j])| \geq 1$, then we choose $X_i \subseteq S$ for all $1 \leq i \leq r$. Since $|V(C_i)| = n_i$ and $X_i = V(C_i) - u_i$, we have $|X_i| = n_i - 1$. By such a choosing, the number of the vertices belonging to S is $\sum_{i=1}^{r} |X_i| = \sum_{i=1}^{r} (n_i - 1) \leq \sum_{i=1}^{r} m_i \leq k - s$. In addition, we select one endvertex of each independent edge into S. Till now, the total number of the vertices belonging to S is $\sum_{i=1}^{r} |X_i| + s \leq (k-s) + s = k$. Note that if $\sum_{i=1}^{r} |X_i| + s < k$, then we can add some other vertices in G into S such that $|S| = k$. Thus all edges of $E(C_i)$ and the s independent edges are put into $E(K_n[S]) \cup E_{K_n}[S, \bar{S}]$, that is, all edges of M' belong to $E(K_n[S]) \cup E_{K_n}[S,\bar{S}]$. So $|M \cap \big(E(K_n[S]) \cup E_{K_n}[S,\bar{S}]\big)| \geq k$, as desired. Since $|E(K_n[X_j])| \geq 1$, it follows that $|M \cap E(K_n[S])| \geq 1$, as desired.

Suppose that $|E(K_n[X_i])| = 0$ for all $1 \leq i \leq r$. Then each C_i must be a star such that $|E(C_i)| \geq 2$. Recall that u_i is one of the vertices having maximum degree in C_i. Select one vertex from $V(C_i) - u_i$, say v_i. Put all the vertices of $Y_i = V(C_i) - v_i$ into S, that is, $Y_i \subseteq S$. Thus $|Y_i| = n_i - 1$. In addition, we choose one endvertex of each independent edge into S. By such a choosing, the total number of the vertices belonging to S is $\sum_{i=1}^{r} |Y_i| + s = \sum_{i=1}^{r}(n_i - 1) + s \leq \sum_{i=1}^{r} m_i + s \leq (k-s) + s = k$. Note that if $\sum_{i=1}^{r} |X_i| + s < k$, then we can add some other vertices in G into S such that $|S| = k$. Thus all edges of $E(C_i)$ and the s independent edges are put into $E(K_n[S]) \cup E_{K_n}[S,\bar{S}]$, that is, and all edges of M' belong to $E(K_n[S]) \cup E_{K_n}[S,\bar{S}]$. So $|M \cap \big(E(K_n[S]) \cup E_{K_n}[S,\bar{S}]\big)| \geq k$, as desired. Since $|E(C_i)| \geq 2$, it follows that there is an edge $u_iw_i \in M \cap K_n[S]$ where $w_i \in V(C_i) - \{u_i, v_i\}$, which implies that $|M \cap E(K_n[S])| \geq 1$, as desired.

From the above arguments, we conclude that there exists an $S \subseteq V(G)$ with $|S| = k$ such that $|M \cap \big(E(K_n[S]) \cup E_{K_n}[S,\bar{S}]\big)| \geq k$ and $|M \cap E(K_n[S])| \geq 1$. Since each tree $T \in \mathscr{T}_1$ uses $k-1$ edges in $E(G[S]) \cup E_G[S,\bar{S}]$, it follows that $|\mathscr{T}_1| \leq \left(\binom{k}{2} - 1\right)/(k-1) = \frac{k}{2} - \frac{1}{k-1}$, which results in $|\mathscr{T}_1| \leq \frac{k}{2} - 1$ since $|\mathscr{T}_1|$ is an integer. From Lemma 2.3, each tree $T \in \mathscr{T}_2$ uses at least k edges of $E(G[S]) \cup E_G[S,\bar{S}]$. Thus $|\mathscr{T}_1|(k-1) + |\mathscr{T}_2|k \leq |E(G[S])| + |E_G[S,\bar{S}]|$, that is, $|\mathscr{T}_1|k + |\mathscr{T}_2|k \leq |\mathscr{T}_1| + \binom{k}{2} + k(n-k) - k$. So $\lambda_k(G) = |\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2| \leq n - \frac{k}{2} - 1 - \frac{1}{k} < n - \frac{k}{2} - 1$. ■

Remark 2.2. From Lemmas 1.7, 2.4 and Remark 2.1, if $\kappa_k(K_n - M) = n - \frac{k}{2} - 1$ or $\lambda_k(K_n - M) = n - \frac{k}{2} - 1$ for k even and $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$, then $1 \leq |M| \leq k-1$, where $M \subseteq E(K_n)$.

Lemma 2.5. *For every two integers n and k with $4 \leq k \leq n$, if k is even and M is a set of edges in the complete graph K_n such that $|M| \geq k$ and $\Delta(K_n[M]) = 1$, then $\kappa_k(K_n - M) < n - \frac{k}{2} - 1$.*

Proof. Let $G = K_n - M$. Since $\Delta(K_n[M]) = 1$, it follows that M is a matching in K_n. Since $|M| \geq k$, we can choose $M_1 \subseteq M$ such that $|M_1| = k$. Let $M_1 = \{u_iw_i \,|\, 1 \leq i \leq k\}$. Choose $S = \{u_1, u_2, \ldots, u_k\}$. We will show that $\kappa(S) < n - \frac{k}{2} - 1$. Clearly, $|\bar{S}| = n - k$, and let $\bar{S} = \{w_1, w_2, \ldots, w_{n-k}\}$. Since each tree in $\mathscr{T}_2$ contains at least one vertex of $\bar{S}$, it follows that $|\mathscr{T}_2| \leq n - k$. By the definition of $\mathscr{T}_1$, we have $|\mathscr{T}_1| \leq \frac{k}{2}$. If $|\mathscr{T}_1| \leq \frac{k}{2} - 2$, then $\kappa(S) \leq \lambda(S) = |\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2| \leq (\frac{k}{2} - 2) + (n-k) = n - \frac{k}{2} - 2 < n - \frac{k}{2} - 1$, as desired. Let us assume $\frac{k}{2} - 1 \leq |\mathscr{T}_1| \leq \frac{k}{2}$.

Consider the case $|\mathscr{T}_1| = \frac{k}{2} - 1$. Recall that $|\mathscr{T}_2| \leq n - k$. Furthermore, we claim that $|\mathscr{T}_2| \leq n - k - 1$. Assume, to the contrary, that $|\mathscr{T}_2| = n - k$. Let $T_1, T_2, \ldots, T_{n-k}$ be the $n-k$ edge-disjoint S-Steiner trees in $\mathscr{T}_2$. For each tree T_i $(1 \le i \le n-k)$, this tree only occupies one vertex of $\bar{S}$, say w_i. Since $u_iw_i \in M_1$ $(1 \le i \le k)$, namely, $u_iw_i \notin E(G)$, and each T_i $(1 \le i \le k)$ is an S-Steiner tree in $\mathscr{T}_2$, it follows that this tree T_i must contain at least one edge in $G[S] = K_k$. So the trees $T_1, T_2, \ldots, T_k$ must use at least k edges in $G[S]$, and $|\mathscr{T}_1| = \frac{\binom{k}{2}-k}{k-1} = \frac{k-2}{2} - \frac{1}{k-1}$. Since $|\mathscr{T}_1|$ is an integer, we have $|\mathscr{T}_1| < \frac{k-2}{2}$, a contradiction. We conclude that $|\mathscr{T}_2| \le n-k-1$, and hence $\kappa(S) \le \lambda(S) = |\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2| \le (\frac{k}{2}-1) + (n-k-1) = n - \frac{k}{2} - 2 < n - \frac{k}{2} - 1$, as desired.

Consider the case $|\mathscr{T}_1| = \frac{k}{2}$. We claim that $|\mathscr{T}_2| \le n-k-2$. Assume, to the contrary, that $n-k-1 \le |\mathscr{T}_2| \le n-k$. Since $|\mathscr{T}_1| = \frac{k}{2}$, it follows that each edge of $G[S]$ is occupied by some tree in $\mathscr{T}_1$, which implies that each tree in $\mathscr{T}_2$ only uses the edges of $E_G[S,\bar{S}] \cup E(G[\bar{S}])$. Suppose that T_1 is a tree in $\mathscr{T}_2$ occupying w_1. Since $u_1w_1 \notin E(G)$, if T_1 contains three vertices of $\bar{S}$, then the remaining $n-k-3$ vertices in $\bar{S}$ must be contained in at most $n-k-3$ trees in $\mathscr{T}_2$, which results in $|\mathscr{T}_2| \le (n-k-3)+1 = n-k-2$, a contradiction. So we assume that the tree T_1 contains another vertex of $\bar{S}$ except w_1, say w_2. Recall that $k \ge 4$. Then $|\bar{S}| \ge k \ge 4$. By the same reason, there is another tree T_2 containing two vertices of $\bar{S}$, say w_3, w_4. Furthermore, the remaining $n-k-4$ vertices in $\bar{S}$ must be contained in at most $n-k-4$ trees in $\mathscr{T}_2$, which results in $|\mathscr{T}_2| \le (n-k-4)+2 = n-k-2$, a contradiction. We conclude that $|\mathscr{T}_2| \le n-k-2$. Since $|\mathscr{T}_1| = \frac{k}{2}$, we have $\kappa(S) \le \lambda(S) = |\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2| \le \frac{k}{2} + (n-k-2) = n - \frac{k}{2} - 2 < n - \frac{k}{2} - 1$, as desired. ■

Lemma 2.6. *If n $(n \ge 4)$ is even and M is a set of edges in the complete graph K_n such that $1 \le |M| \le n-1$ and $1 \le \Delta(K_n[M]) \le \frac{n}{2}$, then $G = K_n - M$ contains $\frac{n-2}{2}$ edge-disjoint spanning trees.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ $(1 \le i \le p)$, and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in G. It suffices to show that $|\mathcal{E}_p| \ge \frac{n-2}{2}(|\mathscr{P}|-1)$ so that we can use Theorem 1.1.

The case $p = 1$ is trivial by Theorem 1.1, thus we assume $p \ge 2$. For $p = 2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Clearly, $|V_2| = n - n_1$. Since $\Delta(K_n[M]) \le \frac{n}{2}$, it follows that $\delta(G) = n - 1 - \Delta(K_n[M]) \ge n - 1 - \frac{n}{2} = \frac{n-2}{2}$. Therefore, if $n_1 = 1$ then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \ge \frac{n-2}{2}$. Suppose $n_1 \ge 2$. Then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \ge \binom{n}{2} - (n-1) - \binom{n_1}{2} - \binom{n-n_1}{2} = -n_1^2 + nn_1 - n + 1$. Since $2 \le n_1 \le n-2$, one can see that $|\mathcal{E}_2|$ achives its minimum value when $n_1 = 2$ or $n_1 = n-2$. Thus $|\mathcal{E}_2| \ge n - 3 \ge \frac{n-2}{2}$ since $n \ge 4$. The result follows from Theorem 1.1.

Let us consider the remaining cases for p, namely, for $3 \le p \le n$. Since $|\mathcal{E}_p| \ge \binom{n}{2} - |M| - \sum_{i=1}^{p} \binom{n_i}{2} \ge \binom{n}{2} - (n-1) - \sum_{i=1}^{p} \binom{n_i}{2} = \binom{n-1}{2} - \sum_{i=1}^{p} \binom{n_i}{2}$, we only need to show $\binom{n-1}{2} - \sum_{i=1}^{p} \binom{n_i}{2} \ge \frac{n-2}{2}(p-1)$, that is, $(n-p)\frac{n-2}{2} \ge \sum_{i=1}^{p} \binom{n_i}{2}$. Because $\sum_{i=1}^{p} \binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$

and $n_p = n-p+1$, we need inequality $(n-p)\frac{n-2}{2} \geq \binom{1}{2}(p-1)+\binom{n-p+1}{2}$, namely, $(n-p)\frac{p-3}{2} \geq 0$. It is easy to see that the inequality holds since $3 \leq p \leq n$. Thus, $|\mathcal{E}_p| \geq \binom{n}{2} - |M| - \sum_{i=1}^{p}\binom{n_i}{2} \geq \frac{n-2}{2}(p-1)$.

From Theorem 1.1, there exist $\frac{n-2}{2}$ edge-disjoint spanning trees in G, as desired. ■

Lemma 2.7. *Let k, n be two integers with $4 \leq k \leq n$, and M be an edge set of the complete graph K_n satisfying $\Delta(K_n[M]) = 1$. Then*

(1) *If $|M| = k-1$, then $\kappa_k(K_n - M) \geq n - \frac{k}{2} - 1$;*

(2) *If $|M| = \lfloor \frac{n}{2} \rfloor$, then $\lambda_k(K_n - M) \geq n - \frac{k}{2} - 1$.*

Proof. (1) Set $G = K_n - M$. Since $\Delta(K_n[M]) = 1$, it follows that M is a matching of K_n. By the definition of $\kappa_k(G)$, we need to show that $\kappa(S) \geq n-\frac{k}{2}-1$ for any $S \subseteq V(G)$.

Case 1. There exists no u, w in S such that $uw \in M$. Without loss of generality, let $S = \{u_1, u_2, \dots, u_k\}$ such that $u_1, u_2, \dots, u_r$ are M-saturated but $u_{r+1}, u_{r+2}, \dots, u_k$ are M-unsaturated. Let $M_1 = \{u_iw_i \,|\, 1 \leq i \leq r\} \subseteq M$. Since $|M| = k-1$, it follows that $0 \leq r \leq k-1$. In this case, $u_iu_j \notin M$ $(1 \leq i, j \leq r)$. Clearly, $G[S]$ is a clique of order k. We choose a path $P = u_1u_2 \cdots u_ru_{r+1}$ in $G[S]$. Let $G' = G - E(P)$. Then $G'[S] = K_k - E(P)$. Since $|E(P)| = r \leq k-1$ and $\Delta(K_k[E(P)]) = 2 \leq \frac{k}{2}$, it follows that $G'[S]$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees, which are also $\frac{k-2}{2}$ internally disjoint S-Steiner trees. These trees together with the trees T_i induced by the edges in $\{u_1w_i, u_2w_i, u_{i-1}w_i, u_{i+1}w_i, \dots, u_kw_i, u_iu_{i+1}\}$ $(1 \leq i \leq r)$ (see Figure 1(a)) and the trees T_j induced by the edges in $\{u_1v_j, u_2v_j, \dots, u_kv_j\}$ where $v_j \in \bar{S} - \{w_1, w_2, \dots, w_r\} = \{v_1, v_2, \dots, v_{n-k-r}\}$ form $\frac{k-2}{2} + r + (n-k-r) = n - \frac{k}{2} - 1$ internally disjoint S-Steiner trees. Thus, $\kappa(S) \geq n - \frac{k}{2} - 1$, as desired.

Case 2. There exist u, w in S such that $uw \in M$. Without loss of generality, we let $S = \{u_1, u_2, \dots, u_r, u_{r+1}, u_{r+2}, \dots, u_{r+s}, u_{r+s+1}, \dots, u_{k-r}, w_1, w_2, \dots, w_r\}$ such that the vertices $u_1, u_2, \dots, u_{r+s}, w_1, w_2, \dots, w_r$ are all M-saturated and $u_iw_i \in M$ $(1 \leq i \leq r)$. Set $M_1 = \{u_iw_i \,|\, 1 \leq i \leq r\}$. In this case, $r \geq 1$ and $2r + s \leq k$. Since $|M| = k-1$, it follows that $r + s \leq k-1$ and $s \leq k-2$.

First, we consider $2r+s = k$. Since k is even, it follows that s is even. If $s = 0$, then $r = \frac{k}{2}$. Thus $S = \{u_1, u_2, \dots, u_{\frac{k}{2}}, w_1, w_2, \dots, w_{\frac{k}{2}}\}$. Clearly, $M_1 = \{u_iw_i \,|\, 1 \leq i \leq \frac{k}{2}\}$, $|M_1| = \frac{k}{2} \leq k-1$ and $\Delta(K_n[M_1]) = 1 < \frac{k}{2}$. By Lemma 2.6, $G[S]$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees, which are also $\frac{k-2}{2}$ internally disjoint S-Steiner trees. These trees together with the trees T_j induced by the edges in $\{u_1v_j, u_2v_j, \dots, u_{\frac{k}{2}}v_j\} \cup \{w_1v_j, w_2v_j, \dots, w_{\frac{k}{2}}v_j\}$ form $\frac{k-2}{2} + (n-k)$ internally disjoint S-Steiner trees, where $v_j \in \bar{S} = \{v_1, v_2, \dots, v_{n-k}\}$. So, $\kappa(S) \geq n - \frac{k}{2} - 1$.

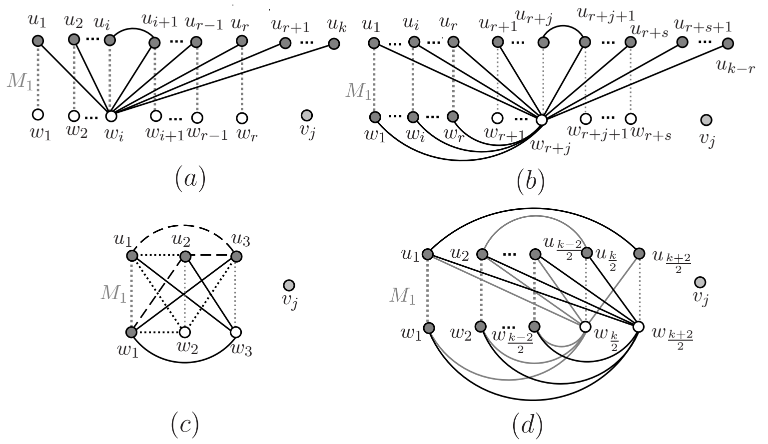

Figure 1. Graphs for (1) of Lemma 2.7.

Consider $s = 2$. Since $2r + s = k$, we have $r = \frac{k-2}{2}$. If $k = 4$, then $r = 1$ and hence $S = \{u_1, u_2, u_3, w_1\}$. Clearly, $M_1 = \{u_1w_1\}$, and the tree T_1 induced by the edges in $\{u_1u_2, u_1w_2, w_1w_2, u_3w_2\}$ and the tree T_2 induced by the edges in $\{u_1u_3, u_2u_3, u_2w_1\}$ and the tree T_3 induced by the edges in $\{u_1w_3, u_2w_3, w_1w_3, u_3w_1\}$ are three spanning trees; see Figure 1(c). These trees together with the trees T_j induced by the edges in $\{u_1v_j, u_2v_j, u_3v_j, w_1v_j\}$ form $3 + (n - 6)$ internally disjoint S-Steiner trees, where $v_j \in \bar{S} - \{w_2, w_3\} = \{v_1, v_2, \ldots, v_{n-6}\}$. Thus, $\kappa(S) \geq n - 3 = n - \frac{k}{2} - 1$. Suppose $k \geq 6$. Then $r \geq 2$, $S = \{u_1, u_2, \ldots, u_{\frac{k+2}{2}}, w_1, w_2, \ldots, w_{\frac{k-2}{2}}\}$ and $M_1 = \{u_iw_i \,|\, 1 \leq i \leq \frac{k-2}{2}\}$. Clearly, the tree T_1 induced by the edges in $\{u_1w_{\frac{k}{2}}, u_2w_{\frac{k}{2}}, \ldots, u_{\frac{k-2}{2}}w_{\frac{k}{2}}, u_{\frac{k+2}{2}}w_{\frac{k}{2}}, u_2u_{\frac{k}{2}}, w_1w_{\frac{k}{2}}, w_2w_{\frac{k}{2}}, \ldots, w_{\frac{k-2}{2}}w_{\frac{k}{2}}\}$ and the tree T_2 induced by the edges in $\{u_1w_{\frac{k+2}{2}}, u_2w_{\frac{k+2}{2}}, \ldots, u_{\frac{k}{2}}w_{\frac{k+2}{2}}\} \cup \{u_1u_{\frac{k+2}{2}}, w_1w_{\frac{k+2}{2}}, w_2w_{\frac{k+2}{2}}, \ldots, w_{\frac{k-2}{2}}w_{\frac{k+2}{2}}\}$ are two internally disjoint S-Steiner trees; see Figure 1(d). Let $M_2 = M_1 \cup \{u_1u_{\frac{k+2}{2}}, u_2u_{\frac{k}{2}}\}$. Then $|M_2| = |M_1| + 2 = \frac{k-2}{2} + 2 = \frac{k+2}{2} < k - 1$ and $\Delta(K_n[M_2]) = 2 \leq \frac{k}{2}$, which implies that $G[S] - \{u_1u_{\frac{k+2}{2}}, u_2u_{\frac{k}{2}}\} = K_k - M_2$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees by Lemma 2.6, which are also $\frac{k-2}{2}$ internally disjoint S-Steiner trees. These trees together with T_1, T_2 and the trees T_j induced by the edges in $\{u_1v_j, u_2v_j, \ldots, u_{\frac{k+2}{2}}v_j, w_1v_j, w_2v_j, \ldots, u_{\frac{k-2}{2}}v_j\}$ are $\frac{k-2}{2}+2+(n-k-2)$ internally disjoint S-Steiner trees, where $v_j \in \bar{S} - \{w_{\frac{k}{2}}, w_{\frac{k+2}{2}}\} = \{v_1, v_2, \ldots, v_{n-k-2}\}$. So, $\kappa(S) \geq n - \frac{k}{2} - 1$.

Consider the remaining case for s, namely, for $4 \leq s \leq k - 2$. Clearly, there exists a cycle of order s containing $u_{r+1}, u_{r+2}, \ldots, u_{r+s}$ in $K_k - M_1$, say $C_s = u_{r+1}u_{r+2} \cdots u_{r+s}u_{r+1}$. Set $M' = M_1 \cup E(C_s)$. Then $|M'| = r + s \leq k - 1$

and $\Delta(K_n[M']) = 2 \leq \frac{k}{2}$, which implies that $G - E(C_s) = K_k - M'$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees by Lemma 2.6. These trees together with the trees T_{r+j} induced by the edges in $\{u_1w_{r+j}, u_2w_{r+j}, \ldots, u_{r+j-1}w_{r+j}, u_{r+j+1}w_{r+j}, \ldots, u_{r+s}w_{r+j}, u_{r+j}u_{r+j+1}, w_1w_{r+j}, w_2w_{r+j}, \ldots, w_rw_{r+j}\}$ $(1 \leq j \leq s)$ form $\frac{k-2}{2} + s$ internally disjoint trees; see Figure 2(b) (note that $u_{r+s} = u_{k-r}$). These trees together with the trees T'_j induced by the edges in $\{u_1v_j, u_2v_j, \ldots, u_{r+s}v_j, w_1v_j, \ldots, w_rv_j\}$ form $\frac{k-2}{2} + s + (n - 2r - 2s) = n - \frac{k}{2} - 1$ internally disjoint S-Steiner trees where $v_j \in \bar{S} - \{w_{r+1}, w_{r+2}, \ldots, w_{r+s}\} = \{v_1, v_2, \ldots, v_{n-2r-2s}\}$. Thus, $\kappa(S) \geq n - \frac{k}{2} - 1$, as desired.

Next, assume $2r + s < k$. Then $S = \{u_1, u_2, \ldots, u_{r+s}, u_{r+s+1}, \ldots, u_{k-r}, w_1, w_2, \ldots, w_r\}$ and $r+s+1 \leq k-r$. If $s = 0$, then $S = \{u_1, u_2, \ldots, u_{k-r}, w_1, w_2, \ldots, w_r\}$. Clearly, $M_1 = \{u_iw_i \,|\, 1 \leq i \leq r\}$, $|M_1| = r \leq k-1$ and $\Delta(K_n[M_1]) = 1 < \frac{k}{2}$. By Lemma 2.6, $G[S]$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees. These trees together with the trees T_j induced by the edges in $\{u_1v_j, u_2v_j, \ldots, u_{n-r}v_j, w_1v_j, w_2v_j, \ldots, w_rv_j\}$ form $\frac{k-2}{2} + (n - k)$ internally disjoint S-Steiner trees, where $v_j \in \bar{S} = \{v_1, v_2, \ldots, v_{n-k}\}$. Therefore, $\kappa(S) \geq n - \frac{k}{2} - 1$. Assume $s \geq 1$. Clearly, there exists a path of length s containing $u_{r+1}, u_{r+2}, \ldots, u_{r+s}, u_{r+s+1}$ in $G[S]$, say $P_s = u_{r+1}u_{r+2} \cdots u_{r+s}u_{r+s+1}$. Set $M' = M_1 \cup E(P_s)$. Then $|M'| = r + s \leq k - 1$ and $\Delta(K_n[M']) = 2 \leq \frac{k}{2}$, which implies that $G[S] - E(P_s) = K_k - M'$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees by Lemma 2.6, which are also $\frac{k-2}{2}$ internally disjoint S-Steiner trees. These trees together with the trees T_{r+j} induced by the edges in $\{u_1w_{r+j},\ u_2w_{r+j}, \ldots, u_{r+j-1}w_{r+j},\ u_{r+j+1}w_{r+j}, \ldots, u_{k-r}w_{r+j},\ u_{r+j}u_{r+j+1},\ w_1w_{r+j},\ w_2w_{r+j}, \ldots, w_rw_{r+j}\}$ $(1 \leq j \leq s)$ form $\frac{k-2}{2} + s$ internally disjoint S-Steiner trees; see Figure 1(b). These trees together with the trees T'_j induced by the edges in $\{u_1v_j, u_2v_j, \ldots, u_{k-r}v_j,\ w_1v_j, w_2v_j, \ldots, w_rv_j\}$ form $\frac{k-2}{2} + s + (n-k+r) - (r+s) = n - \frac{k}{2} - 1$ internally disjoint S-Steiner trees where $v_j \in \bar{S} - \{w_{r+1}, w_{r+2}, \ldots, w_{r+s}\} = \{v_1, v_2, \ldots, v_{n-k-s}\}$. So, $\kappa(S) \geq n - \frac{k}{2} - 1$, as desired.

We conclude that $\kappa(S) \geq n - \frac{k}{2} - 1$ for any $S \subseteq V(G)$. From the arbitrariness of S, it follows that $\kappa_k(G) \geq n - \frac{k}{2} - 1$.

(2) Set $G = K_n - M$. Assume that n is even. Thus M is a perfect matching of K_n, and all vertices of G are M-saturated. By the definition of $\lambda_k(G)$, we need to show that $\lambda(S) \geq n - \frac{k}{2} - 1$ for any $S \subseteq V(G)$.

Case 3. There exists no u, w in S such that $uw \in M$. Without loss of generality, let $S = \{u_1, u_2, \ldots, u_k\}$. In this case, $u_iu_j \notin M$ $(1 \leq i, j \leq k)$. Let $M_1 = \{u_iw_i \,|\, 1 \leq i \leq k\} \subseteq M = \{u_iw_i \,|\, 1 \leq i \leq \frac{n}{2}\}$. Clearly, $w_i \notin S$ $(1 \leq i \leq \frac{n}{2})$ and $u_j \notin S$ $(k+1 \leq j \leq \frac{n}{2})$. Since $G[S]$ is a clique of order k, it follows that there are $\frac{k}{2}$ edge-disjoint spanning trees in $G[S]$, which are also $\frac{k}{2}$ edge-disjoint S-Steiner trees. These trees together with the trees T_i induced by the edges in $\{u_1w_i, u_2w_i, u_{i-1}w_i, u_{i+1}w_i, \ldots, u_kw_i, u_iw_k, w_iw_k\}$ $(1 \leq i \leq k-1)$

(see Figure 2(a)) and the trees T'_j induced by the edges in $\{u_1u_j, u_2u_j, \ldots, u_ku_j\}$ $(k+1 \leq j \leq \frac{n}{2})$ and the trees T''_j induced by the edges in $\{u_1w_j, u_2w_j, \ldots, u_kw_j\}$ $(k+1 \leq j \leq \frac{n}{2})$ form $\frac{k}{2} + (k-1) + (n-2k) = n - \frac{k}{2} - 1$ edge-disjoint S-Steiner trees. Therefore, $\lambda(S) \geq n - \frac{k}{2} - 1$, as desired.

Case 4. There exist u, w in S such that $uw \in M$. Without loss of generality, let $S = \{u_1, u_2, \ldots, u_{r+s}, w_1, w_2, \ldots, w_r\}$ with $|S| = k = 2r + s$, where $1 \leq r \leq \frac{k}{2}$ and $0 \leq s \leq k-2$. Set $M_1 = \{u_iw_i \,|\, 1 \leq i \leq r\} \subseteq M = \{u_iw_i \,|\, 1 \leq i \leq \frac{n}{2}\}$. We claim that $r + s \leq k - 1$. Otherwise, let $r + s = k$. Combining this with $2r + s = k$, we have $r = 0$, a contradiction. Since $k = 2r + s$ and k is even, it follows that s is even.

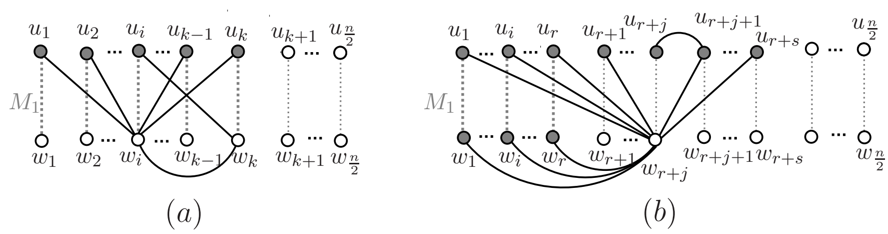

Figure 2. Graphs for (2) of Lemma 2.7.

If $s = 0$, then $r = \frac{k}{2}$. Clearly, $S = \{u_1, u_2, \ldots, u_{\frac{k}{2}}, w_1, w_2, \ldots, w_{\frac{k}{2}}\}$ and $M_1 = M = \{u_iw_i | 1 \leq i \leq \frac{k}{2}\}$. In addition, $|M_1| \leq \frac{k}{2} < k-1$ and $\Delta(M \cap K_n[S]) = 1 < \frac{k}{2}$. Then $G[S]$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees by Lemma 2.6. These trees together with the trees T_i induced by the edges in $\{u_1u_i, u_2u_i, \ldots, u_{\frac{k}{2}}u_i, w_1u_i, w_2u_i, \ldots, w_{\frac{k}{2}}u_i\}$ $(k+1 \leq j \leq \frac{n}{2})$ and the trees T'_i induced by the edges in $\{u_1w_i, u_2w_i, \ldots, u_{\frac{k}{2}}w_i, w_1w_i, w_2w_i, \ldots, w_{\frac{k}{2}}w_i\}$ $(\frac{k}{2}+1 \leq i \leq \frac{n}{2})$ form $n - \frac{k}{2} - 1$ edge-disjoint S-Steiner trees. Thus, $\lambda(S) \geq n - \frac{k}{2} - 1$.

If $s = 2$, then $r = \frac{k-2}{2}$. Then $S = \{u_1, u_2, \ldots, u_{\frac{k+2}{2}}, w_1, w_2, \ldots, w_{\frac{k-2}{2}}\}$ and $M_1 = \{u_iw_i \,|\, 1 \leq i \leq \frac{k-2}{2}\} \subseteq M$. If $k = 4$, then $r = 1$ and hence $S = \{u_1, u_2, u_3, w_1\}$. Clearly, $M_1 = \{u_1w_1\}$, and the tree T_1 induced by the edges in $\{u_1u_2, u_1w_2, w_1w_2, u_3w_2\}$ and the tree T_2 induced by the edges in $\{u_1u_3, u_2u_3, u_2w_1\}$ and the tree T_3 induced by the edges in $\{u_1w_3, u_2w_3, w_1w_3, u_3w_1\}$ are three edge-disjoint spanning trees; see Figure 1(c). These trees together with the trees T_j induced by the edges in $\{u_1u_j, u_2u_j, u_3u_j, w_1u_j\}$ $(4 \leq k \leq \frac{n}{2})$ and the trees T'_j induced by the edges in $\{u_1w_j, u_2w_j, u_3w_j, w_1u_j\}$ $(4 \leq k \leq \frac{n}{2})$ form $3 + (n-6)$ edge-disjoint S-Steiner trees. So, $\lambda(S) \geq n - 3 = n - \frac{k}{2} - 1$, as desired. Suppose $k \geq 6$. Then $r \geq 2$, $S = \{u_1, u_2, \ldots, u_{\frac{k+2}{2}}, w_1, w_2, \ldots, w_{\frac{k-2}{2}}\}$ and $M_1 = \{u_iw_i \,|\, 1 \leq i \leq \frac{k-2}{2}\}$. Clearly, the tree T_1 induced by the edges in $\{u_1w_{\frac{k}{2}}, u_2w_{\frac{k}{2}}, \ldots, u_{\frac{k-2}{2}}w_{\frac{k}{2}},$

$u_{\frac{k+2}{2}}w_{\frac{k}{2}}, u_2u_{\frac{k}{2}}, w_1w_{\frac{k}{2}}, w_2w_{\frac{k}{2}}, \ldots, w_{\frac{k-2}{2}}w_{\frac{k}{2}}\}$ and the tree T_2 induced by the edges in $\{u_1w_{\frac{k+2}{2}}, u_2w_{\frac{k+2}{2}}, \ldots, u_{\frac{k}{2}}w_{\frac{k+2}{2}}, u_1u_{\frac{k+2}{2}}, w_1w_{\frac{k+2}{2}}, w_2w_{\frac{k+2}{2}}, \ldots, w_{\frac{k-2}{2}}w_{\frac{k+2}{2}}\}$ are two edge-disjoint S-Steiner trees; see Figure 1(d). Let $M_2 = M_1 \cup \{u_1u_{\frac{k+2}{2}}, u_2u_{\frac{k}{2}}\}$. Then $|M_2| = |M_1|+2 = \frac{k-2}{2}+2 = \frac{k+2}{2} < k-1$ and $\Delta(K_n[M_2]) = 2 \leq \frac{k}{2}$, which implies that $G[S] - \{u_1u_{\frac{k+2}{2}}, u_2u_{\frac{k}{2}}\} = K_k - M_2$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees by Lemma 2.6. These trees together with T_1, T_2 and the trees T_j induced by the edges in $\{u_1u_j, u_2u_j, \ldots, u_{\frac{k+2}{2}}u_j, w_1u_j, w_2u_j, \ldots, u_{\frac{k-2}{2}}u_j\}$ $(\frac{k}{2}+2 \leq j \leq \frac{n}{2})$ and the trees T'_j induced by the edges in $\{u_1w_j, u_2w_j, \ldots, u_{\frac{k+2}{2}}w_j, w_1w_j, w_2w_j, \ldots, u_{\frac{k-2}{2}}w_j\}$ $(\frac{k}{2}+2 \leq j \leq \frac{n}{2})$ are $\frac{k-2}{2}+2+(n-k-2)$ edge-disjoint S-Steiner trees. Therefore, $\lambda(S) \geq n - \frac{k}{2} - 1$, as desired.

Consider the remaining case s with $4 \leq s \leq k-2$. Clearly, there exists a cycle of order s containing $u_{r+1}, u_{r+2}, \ldots, u_{r+s}$ in $K_k - M_1$, say $C_s = u_{r+1}u_{r+2}\cdots u_{r+s}u_{r+1}$. Set $M' = M_1 \cup E(C_s)$. Then $|M'| = r+s \leq k-1$ and $\Delta(K_n[M']) = 2 \leq \frac{k}{2}$, which implies that $G - E(C_s)$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees by Lemma 2.6. These trees together with the trees T_{r+j} induced by the edges in $\{u_1w_{r+j}, u_2w_{r+j}, \ldots, u_{r+j-1}w_{r+j}, u_{r+j+1}w_{r+j}, \ldots, u_{r+s}w_{r+j}, u_{r+j}u_{r+j+1}, w_1w_{r+j}, w_2w_{r+j}, \ldots, w_rw_{r+j}\}$ $(1 \leq j \leq s)$ form $\frac{k-2}{2}+s$ edge-disjoint S-Steiner trees; see Figure 2(b). These trees together with the trees T'_i induced by the edges in $\{u_1u_i, u_2u_i, \ldots, u_{r+s}u_i, w_1u_i, \ldots, w_ru_i\}$ $(r+s+1 \leq i \leq \frac{n}{2})$ and the trees T''_i induced by the edges in $\{u_1w_i, u_2w_i, \ldots, u_{r+s}w_i, w_1w_i, \ldots, w_rw_i\}$ $(r+s+1 \leq i \leq \frac{n}{2})$ form $(n-2r-2s)+(\frac{k-2}{2}+s) = n - \frac{k}{2} - 1$ edge-disjoint S-Steiner trees since $2r+s = k$. Thus, $\lambda(S) \geq n - \frac{k}{2} - 1$, as desired.

We conclude that $\lambda(S) \geq n - \frac{k}{2} - 1$ for any $S \subseteq V(G)$. From the arbitrariness of S, it follows that $\lambda_k(G) \geq n - \frac{k}{2} - 1$. For n odd, M is a maximum matching and we can also check that $\lambda_k(G) \geq n - \frac{k}{2} - 1$ similarly. ■

Lemma 2.8. *Let n and k be two integers such that k is even and $4 \leq k \leq n$. If M is a set of edges in the complete graph K_n such that $|M| = k-1$, and $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$, then $\kappa_k(K_n - M) \geq n - \frac{k}{2} - 1$.*

Proof. Set $G = K_n - M$. For $n = k$, there are $\frac{n-2}{2}$ edge-disjoint spanning trees by Lemma 2.6, and hence $\kappa_n(G) = \lambda_n(G) \geq \frac{n-2}{2}$. So from now on, we assume $n \geq k+1$. Let $S = \{u_1, u_2, \ldots, u_k\} \subseteq V(G)$ and $\bar{S} = V(G) - S = \{w_1, w_2, \ldots, w_{n-k}\}$. We have the following two cases to consider.

Case 1. $M \subseteq E(K_n[S]) \cup E(K_n[\bar{S}])$. Let $M' = M \cap E(K_n[S])$ and $M'' = M \cap E(K_n[\bar{S}])$. Then $|M'| + |M''| = |M| = k-1$ and $0 \leq |M'|, |M''| \leq k-1$. We can regard $G[S]$ as a complete graph K_k by deleting $|M'|$ edges. Since $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $M' \subseteq M$, it follows that $\Delta(K_n[M']) \leq \Delta(K_n[M]) \leq \frac{k}{2}$.

From Lemma 2.6, there exist $\frac{k-2}{2}$ edge-disjoint spanning trees in $G[S]$. Actually, these $\frac{k-2}{2}$ edge-disjoint spanning trees are all internally disjoint S-Steiner trees in $G[S]$. All these trees together with the trees T_i induced by the edges in $\{w_iu_1, w_iu_2, \ldots, w_iu_k\}$ $(1 \leq i \leq n-k)$ form $\frac{k-2}{2} + (n-k) = n - \frac{k}{2} - 1$ internally disjoint S-Steiner trees, and hence $\kappa(S) \geq n - \frac{k}{2} - 1$. From the arbitrariness of S, we have $\kappa_k(G) \geq n - \frac{k}{2} - 1$, as desired.

Case 2. $M \nsubseteq E(K_n[S]) \cup E(K_n[\bar{S}])$. In this case, there exist some edges of M in $E_{K_n}[S, \bar{S}]$. Let $M' = M \cap E(K_n[S])$, $M'' = M \cap E(K_n[\bar{S}])$, and $|M'| = m_1$ and $|M''| = m_2$. Clearly, $0 \leq m_i \leq k-2$ $(i = 1, 2)$. For $w_i \in \bar{S}$, let $|E_{K_n[M]}[w_i, S]| = x_i$, where $1 \leq i \leq n-k$. Without loss of generality, let $x_1 \geq x_2 \geq \cdots \geq x_{n-k}$. Because there exist some edges of M in $E_{K_n}[S, \bar{S}]$, we have $x_1 \geq 1$. Since $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$, it follows that $x_i = |E_{K_n[M]}[w_i, S]| \leq d_{K_n[M]}(w_i) \leq \Delta(K_n[M]) \leq \frac{k}{2}$ for $1 \leq i \leq n-k$. We claim that there exists at most one vertex in $K_n[M]$ such that its degree is $\frac{k}{2}$. Assume, to the contrary, that there are two vertices, say w and w', such that $d_{K_n[M]}(w) = d_{K_n[M]}(w') = \frac{k}{2}$. Then $|M| \geq d_{K_n[M]}(w) + d_{K_n[M]}(w') = \frac{k}{2} + \frac{k}{2} = k$, contradicting $|M| = k-1$. We conclude that there exists at most one vertex in $K_n[M]$ such that its degree is $\frac{k}{2}$. Recall that $x_{n-k} \leq x_{n-k-1} \leq \cdots \leq x_2 \leq x_1 \leq \frac{k}{2}$. So $x_1 = \frac{k}{2}$ and $x_i \leq \frac{k-2}{2}$ $(2 \leq i \leq n-k)$, or $x_i \leq \frac{k-2}{2}$ $(1 \leq i \leq n-k)$. Since $|E_{K_n[M]}[w_i, S]| = x_i$, we have $|E_G[w_i, S]| = k - x_i$. Since $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$, it follows that $\delta(G[S]) \geq k - 1 - \frac{k}{2} = \frac{k-2}{2}$.

Our basic idea is to seek for some edges in $G[S]$, and combine them with the edges of $E_G[S, \bar{S}]$ to form $n-k$ internally disjoint trees, say $T_1, T_2, \ldots, T_{n-k}$, with their roots $w_1, w_2, \ldots, w_{n-k}$, respectively. Let $G' = G - (\bigcup_{j=1}^{n-k} E(T_j))$. We will prove that $G'[S]$ satisfies the conditions of Lemma 2.6 so that $G'[S]$ contains $\frac{k-2}{2}$ edge-disjoint spanning trees, which are also $\frac{k-2}{2}$ internally disjoint S-Steiner trees. These trees together with $T_1, T_2, \ldots, T_{n-k}$ are our $n - \frac{k}{2} - 1$ desired trees. Thus, $\kappa(S) \geq n - \frac{k}{2} - 1$. So we can complete our proof by the arbitrariness of S.

For $w_1 \in \bar{S}$, without loss of generality, let $S = S_1^1 \cup S_2^1$ and $S_1^1 = \{u_1, u_2, \ldots, u_{x_1}\}$ such that $u_jw_1 \in M$ for $1 \leq j \leq x_1$. Set $S_2^1 = S - S_1^1 = \{u_{x_1+1}, u_{x_1+2}, \ldots, u_k\}$. Then $u_jw_1 \in E(G)$ for $x_1 + 1 \leq j \leq k$. One can see that the tree T_1' induced by the edges in $\{w_1u_{x_1+1}, w_1u_{x_1+2}, \ldots, w_1u_k\}$ is a Steiner tree connecting S_2^1. Our current idea is to seek for x_1 edges in $E_G[S_1^1, S_2^1]$ and add them to T_1' to form a Steiner tree connecting S. For each $u_j \in S_1^1$ $(1 \leq j \leq x_1)$, we claim that $|E_G[u_j, S_2^1]| \geq 1$. Otherwise, let $|E_G[u_j, S_2^1]| = 0$. Then $|E_{K_n[M]}[u_j, S_2^1]| = k - x_1$ and hence $|M| \geq |E_{K_n[M]}[u_j, S_2^1]| + d_{K_n[M]}(w_1) \geq (k - x_1) + x_1 = k$, which contradicts $|M| = k-1$. We conclude that for each $u_j \in S_1^1$ $(1 \leq j \leq x_1)$ there is at least one edge in G connecting it to a vertex of S_2^1. Choose the vertex with the smallest subscript among all the vertices of S_1^1 having maximum degree in $G[S]$, say u_1'. Then we select the vertex adjacent to u_1' with the smallest subscript among all the vertices of S_2^1 having maximum degree in $G[S]$, say u_1''. Let $e_{11} = u_1'u_1''$. Consider the graph $G_{11} = G - e_{11}$, and choose the vertex with the smallest subscript among all the vertices of $S_1^1 - u_1'$ having maximum degree in $G_{11}[S]$, say u_2'. Then we select the vertex adjacent to u_2' with the smallest subscript among all the vertices of S_2^1 having maximum degree in $G_{11}[S]$, say u_2''. Set $e_{12} = u_2'u_2''$. Consider the graph $G_{12} = G_{11} - e_{12} = G - \{e_{11}, e_{12}\}$. Choose the one with the smallest subscript among all the vertices of $S_1^1 - \{u_1', u_2'\}$ having maximum degree in $G_{12}[S]$, say u_3', and select the vertex adjacent to u_3' with the smallest subscript among all the vertices of S_2^1 having maximum degree in $G_{12}[S]$, say u_3''. Put $e_{13} = u_3'u_3''$. Consider the graph $G_{13} = G_{12} - e_{11} = G - \{e_{11}, e_{12}, e_{13}\}$. For each $u_j \in S_1^1$ $(1 \leq j \leq x_1)$, we proceed to find $e_{14}, e_{15}, \ldots, e_{1x_1}$ in the same way, and obtain graphs $G_{1j} = G - \{e_{11}, e_{12}, \ldots, e_{1(j-1)}\}$ $(1 \leq j \leq x_1)$. Let $M_1 = \{e_{11}, e_{12}, \ldots, e_{1x_1}\}$ and $G_1 = G - M_1$. Thus the tree T_1 induced by the edges in $\{w_1u_{x_2+1}, w_1u_{x_2+2}, \ldots, w_1u_k\} \cup \{e_{11}, e_{12}, \ldots, e_{1x_1}\}$ is our desired tree.

Let us now prove the following claim.

Claim 1. $\delta(G_1[S]) \geq \frac{k-2}{2}$.

Proof. Assume, to the contrary, that $\delta(G_1[S]) \leq \frac{k-4}{2}$. Then there exists a vertex $u_p \in S$ such that $d_{G_1[S]}(u_p) \leq \frac{k-4}{2}$. If $u_p \in S_2^1$, then by our procedure $d_{G[S]}(u_p) = d_{G_1[S]}(u_p) + 1 \leq \frac{k-2}{2}$, which implies that $d_{M \cap K_n[S]}(u_p) \geq k - 1 - \frac{k-2}{2} = \frac{k}{2}$. Since $w_1u_p \in M$, it follows that $d_{K_n[M]}(u_p) \geq d_{M \cap K_n[S]}(u_p) + 1 \geq \frac{k+2}{2}$, which contradicts $\Delta(K_n[M]) \leq \frac{k}{2}$. Let us now assume $u_p \in S_2^1$. By the above procedure, there exists a vertex $u_q \in S_1^1$ such that when we select the edge $e_{1j} = u_pu_q$ $(1 \leq j \leq x_1)$ from $G_{1(j-1)}[S]$, then the degree of u_p in $G_{1j}[S]$ is equal to $\frac{k-4}{2}$. Thus, $d_{G_{1j}[S]}(u_p) = \frac{k-4}{2}$ and $d_{G_{1(j-1)}[S]}(u_p) = \frac{k-2}{2}$. From our procedure, $|E_G[u_q, S_2^1]| = |E_{G_{1(j-1)}}[u_q, S_2^1]|$. Without loss of generality, let $|E_G[u_q, S_2^1]| = t$ and $u_qu_j \in E(G)$ for $x_1 + 1 \leq j \leq x_1 + t$; see Figure 3$(a)$. Thus $u_p \in \{u_{x_1+1}, u_{x_1+2}, \ldots, u_{x_1+t}\}$, and $u_qu_j \in M$ for $x_1 + t + 1 \leq j \leq k$. Because $|E_G[u_j, S_2^1]| \geq 1$ for each $u_j \in S_1^1$ $(1 \leq j \leq x_1)$, we have $t \geq 1$. Since $|M| = k - 1$ and $u_jw_1 \in M$ for $1 \leq j \leq x_1$, it follows that $1 \leq t \leq k - 2$. Since $d_{G_{1(j-1)}[S]}(u_p) = \frac{k-2}{2}$, by our procedure $d_{G_{1(j-1)}[S]}(u_j) \leq \frac{k-2}{2}$ for each $u_j \in S_2^1$ $(x_1 + 1 \leq j \leq x_1 + t)$. Assume, to the contrary, that there is a vertex u_s $(x_1+1 \leq s \leq x_1+t)$ such that $d_{G_{1(j-1)}[S]}(u_s) \geq \frac{k-2}{2}$. Then we should have selected the edge u_qu_s instead of $e_{1j} = u_qu_p$ by our procedure, a contradiction. We conclude that $d_{G_{1(j-1)}[S]}(u_r) \leq \frac{k-2}{2}$ for each $u_r \in S_1^1$ $(x_1 + 1 \leq r \leq x_1 + t)$. Clearly, there are at least $k - 1 - \frac{k-2}{2} = \frac{k}{2}$ edges incident to each u_r $(x_1 + 1 \leq r \leq x_1 + t)$ belonging to $M \cup \{e_{11}, e_{12}, \ldots, e_{1(j-1)}\}$. Since $j \leq x_1$ and $u_qu_j \in M$ for $x_i + t + 1 \leq j \leq k$, we have

$$
\begin{aligned}
|E_{K_n[M]}[u_q, S_2^1]| + \sum_{j=1}^{t} d_{K_n[M]}(u_j) &\geq k - x_1 - t + \frac{k}{2}t - (j-1) - \binom{t}{2} \\
&= k + \frac{(k-2)}{2}t - x_1 - j + 1 - \binom{t}{2}
\end{aligned}
$$

and hence

$$\begin{aligned}|M| &\geq |M\cap (E_{K_n}[w_1,S])| + \sum_{j=1}^{t} d_{K_n[M]}(u_j) + |E_{K_n[M]}[u_q,S_1^1]|\\
&\geq x_1 + \left(k+\frac{(k-2)}{2}t - x_1 - j + 1\right) - \binom{t}{2}\\
&= -\frac{t^2}{2}+\frac{t}{2}+\frac{(k-2)}{2}t+k-j+1 = -\frac{t^2}{2}+\frac{(k-1)}{2}t+k-j+1\\
&= -\frac{1}{2}\left(t-\frac{k-1}{2}\right)^2+\frac{(k-1)^2}{8}+k-j+1\\
&\geq \frac{k}{2}-1+k-j+1 \qquad\qquad (\text{since } 1\leq t\leq k-2)\\
&= \frac{k}{2}+k-j\geq k, \qquad\qquad \left(\text{since } j\leq x_1 \text{ and } x_1\leq \frac{k}{2}\right)\end{aligned}$$

contradicting $|M| = k-1$. □

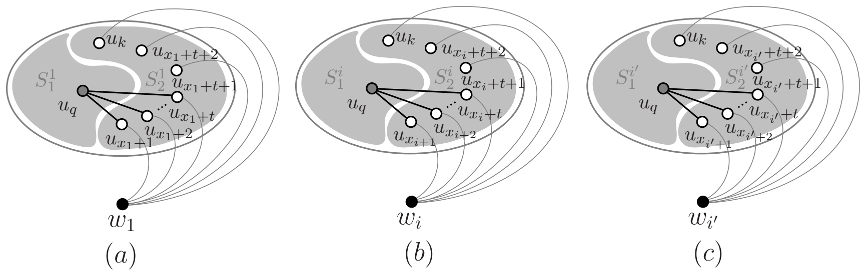

Figure 3. Graphs for Lemma 2.8.

By Claim 1, we have $\delta(G_1[S]) \geq \frac{k-2}{2}$. Recall that there exists at most one vertex in $K_n[M]$ such that its degree is $\frac{k}{2}$, and $x_{n-k} \leq x_{n-k-1} \leq \cdots \leq x_2 \leq x_1 \leq \frac{k}{2}$. Then $x_i \leq \frac{k-2}{2}$ for $2 \leq i \leq n-k$. Now we continue to introduce our procedure.

For $w_2 \in \bar{S}$, without loss of generality, let $S = S_1^2 \cup S_2^2$ and $S_1^2 = \{u_1, u_2, \ldots, u_{x_2}\}$ such that $u_j w_2 \in M$ for $1 \leq j \leq x_2$. Let $S_2^2 = S - S_1^2 = \{u_{x_2+1}, u_{x_2+2}, \ldots, u_k\}$. Then $u_j w_2 \in E(G)$ for $x_2+1 \leq j \leq k$. Clearly, the tree T_2' induced by the edges in $\{w_2u_{x_2+1}, w_2u_{x_2+2}, \ldots, w_2u_k\}$ is a Steiner tree connecting S_2^2. Our idea is to seek for x_2 edges in $E_{G_1}[S_1^2, S_2^2]$ and add them to T_2' to form a Steiner tree connecting S. For each $u_j \in S_1^2$ $(1 \leq j \leq x_2)$, we claim that $|E_{G_1}[u_j, S_2^2]| \geq 1$. Otherwise, let $|E_{G_1}[u_j, S_2^2]| = 0$. Recall that $|M_1| = x_1$. Then there exist $k-x_2$ edges between u_j and S_2^2 belonging to $M \cup M_1$, and hence $|E_{K_n[M]}[u_j, S_2^2]| \geq k - x_2 - x_1$. Therefore,

$|M| \geq |E_{K_n[M]}[u_j, S_2^2]| + d_{K_n[M]}(w_1) + d_{K_n[M]}(w_2) \geq (k - x_2 - x_1) + x_1 + x_2 = k$, which contradicts $|M| = k-1$. Choose the vertex with the smallest subscript among all the vertices of S_1^2 having maximum degree in $G_1[S]$, say u_1'. Then we select the vertex adjacent to u_1' with the smallest subscript among all the vertices of S_2^2 having maximum degree in $G_1[S]$, say u_1''. Let $e_{21} = u_1'u_1''$. Consider the graph $G_{21} = G_1 - e_{21}$, and choose the one with the smallest subscript among all the vertices of $S_1^2 - u_1'$ having maximum degree in $G_{21}[S]$, say u_2'. Then we select the vertex adjacent to u_2' with the smallest subscript among all the vertices of S_2^2 having maximum degree in $G_{21}[S]$, say u_2''. Set $e_{22} = u_2'u_2''$. Consider the graph $G_{22} = G_{21} - e_{22} = G_1 - \{e_{21}, e_{22}\}$. For each $u_j \in S_1^2$ $(1 \leq j \leq x_2)$, we proceed to find $e_{23}, e_{24}, \ldots, e_{2x_2}$ in the same way, and get graphs $G_{2j} = G_1 - \{e_{21}, e_{22}, \ldots, e_{2(j-1)}\}$ $(1 \leq j \leq x_2)$. Let $M_2 = \{e_{21}, e_{22}, \ldots, e_{2x_2}\}$ and $G_2 = G_1 - M_1$. Thus the tree T_2 induced by the edges in $\{w_2u_{x_2+1}, w_2u_{x_2+2}, \ldots, w_2u_k\} \cup \{e_{21}, e_{22}, \ldots, e_{2x_2}\}$ is our desired tree. Furthermore, T_2 and T_1 are two internally disjoint S-Steiner trees.

For $w_i \in \bar{S}$, without loss of generality, let $S = S_1^i \cup S_2^i$ and $S_1^i = \{u_1, u_2, \ldots, u_{x_i}\}$ such that $u_jw_i \in M$ for $1 \leq j \leq x_i$. Set $S_2^i = S - S_1^i = \{u_{x_i+1}, u_{x_i+2}, \ldots, u_k\}$. Then $u_jw_i \in E(G)$ for $x_i+1 \leq j \leq k$. One can see that the tree T_i' induced by the edges in $\{w_iu_{x_i+1}, w_iu_{x_i+2}, \ldots, w_iu_k\}$ is a Steiner tree connecting S_2^i. Our idea is to seek for x_i edges in $E_{G_{i-1}}[S_1^2, S_2^2]$ and add them to T_i' to form a Steiner tree connecting S. For each $u_j \in S_1^i$ $(1 \leq j \leq x_i)$, we claim that $|E_{G_{i-1}}[u_j, S_2^i]| \geq 1$. Otherwise, let $|E_{G_{i-1}}[u_j, S_2^i]| = 0$. Recall that $|M_j| = x_j$ $(1 \leq j \leq i)$. Then there are $k - x_i$ edges between u_j and S_2^i belonging to $M \cup (\bigcup_{j=1}^{i-1} M_j)$, and hence $|E_{K_n[M]}[u_j, S_2^i]| \geq k - x_i - \sum_{j=1}^{i-1} x_j$. Therefore, $|M| \geq |E_{K_n[M]}[u_j, S_2^i]| + \sum_{j=1}^{i} |M \cap (K_n[w_j, S])| \geq k - x_i - \sum_{j=1}^{i-1} x_j + \sum_{j=1}^{i} x_j = k$, contradicting $|M| = k-1$. Choose the vertex with the smallest subscript among all the vertices of S_1^i having maximum degree in $G_{i-1}[S]$, say u_1'. Then we select the vertex adjacent to u_1' with the smallest subscript among all the vertices of S_2^i having maximum degree in $G_{i-1}[S]$, say u_1''. Let $e_{i1} = u_1'u_1''$. Consider the graph $G_{i1} = G_{i-1} - e_{i1}$; choose the vertex with the smallest subscript among all the vertices of $S_1^i - u_1'$ having maximum degree in $G_{i1}[S]$, say u_2'. Then we select the vertex adjacent to u_2' with the smallest subscript among all the vertices of S_2^i having maximum degree in $G_{i1}[S]$, say u_2''. Set $e_{i2} = u_2'u_2''$. Consider the graph $G_{i2} = G_{i1} - e_{i2} = G_{i-1} - \{e_{i1}, e_{i2}\}$. For each $u_j \in S_1^i$ $(1 \leq j \leq x_i)$, we proceed to find $e_{i3}, e_{i4}, \ldots, e_{ix_i}$ in the same way, and get graphs $G_{ij} = G_{i-1} - \{e_{i1}, e_{i2}, \ldots, e_{i(j-1)}\}$ $(1 \leq j \leq x_i)$. Let $M_i = \{e_{i1}, e_{i2}, \ldots, e_{ix_2}\}$ and $G_i = G_{i-1} - M_i$. Thus the tree T_i induced by the edges in $\{w_iu_{x_2+1}, w_iu_{x_2+2}, \ldots, w_iu_k\} \cup \{e_{i1}, e_{i2}, \ldots, e_{ix_i}\}$ is our desired tree. Furthermore, $T_1, T_2, \ldots, T_i$ are pairwise internally disjoint S-Steiner trees.

We continue this procedure until we obtain $n-k$ pairwise internally disjoint trees $T_1, T_2, \ldots, T_{n-k}$. Note that if there exists some x_j such that $x_j = 0$ then $x_{j+1} = x_{j+2} = \cdots = x_{n-k} = 0$ since $x_1 \geq x_2 \geq \cdots \geq x_{n-k}$. Then the tree T_i

induced by the edges in $\{w_iu_1, w_iu_2, \ldots, w_iu_k\}$ $(j \leq i \leq n-k)$ is our desired tree. From the above procedure, the resulting graph must be $G_{n-k} = G - \bigcup_{i=1}^{n-k} M_i$. Let us show the following claim.

Claim 2. $\delta(G_{n-k}[S]) \geq \frac{k-2}{2}$.

Proof. Assume, to the contrary, that $\delta(G_{n-k}[S]) \leq \frac{k-4}{2}$, namely, there exists a vertex $u_p \in S$ such that $d_{G_{n-k}[S]}(u_p) \leq \frac{k-4}{2}$. Since $\delta(G[S]) \geq \frac{k-2}{2}$, by our procedure there exists an edge e_{ij} in $G_{i(j-1)}$ incident to the vertex u_p such that when we pick up this edge, $d_{G_{ij}[S]}(u_p) = \frac{k-4}{2}$ but $d_{G_{i(j-1)}[S]}(u_p) = \frac{k-2}{2}$.

First, we consider the case $u_p \in S_2^i$. Then there exists a vertex $u_q \in S_1^i$ such that when we select the edge $e_{ij} = u_pu_q$ from $G_{i(j-1)}[S]$, then the degree of u_p in $G_{ij}[S]$ is equal to $\frac{k-4}{2}$. Thus, $d_{G_{ij}[S]}(u_p) = \frac{k-4}{2}$ and $d_{G_{i(j-1)}[S]}(u_p) = \frac{k-2}{2}$. From our procedure, $|E_{G_{i-1}}[u_q, S_2^i]| = |E_{G_{i(j-1)}}[u_q, S_2^i]|$. Without loss of generality, let $|E_{G_{i-1}}[u_q, S_2^i]| = t$ and $u_qu_j \in E(G_{i-1})$ for $x_i + 1 \leq j \leq x_i + t$; see Figure 3$(b)$. Thus $u_p \in \{u_{x_i+1}, u_{x_i+2}, \ldots, u_{x_i+t}\}$, and $u_qu_j \in M \cup (\bigcup_{r=1}^{i-1} M_r)$ for $x_i + t + 1 \leq j \leq k$. Since $x_i \leq \frac{k-2}{2}$ $(2 \leq i \leq n-k)$, it follows that $|S_1^i| \leq \frac{k-2}{2}$. From this together with $\delta(G_{i-1}[S]) \geq \frac{k-2}{2}$, we have $|E_{G_{i-1}}[u_q, S_1^i]| \geq 1$, that is, $t \geq 1$. Since $d_{G_{i(j-1)}[S]}(u_p) = \frac{k-2}{2}$, by our procedure $d_{G_{i(j-1)}[S]}(u_j) \leq \frac{k-2}{2}$ for each $u_j \in S_2^i$ $(x_i + 1 \leq j \leq x_i + t)$. Assume, to the contrary, that there exists a vertex u_s $(x_i + 1 \leq s \leq x_i + t)$ such that $d_{G_{i(j-1)}[S]}(u_s) \geq \frac{k-2}{2}$. Then we should have selected the edge u_qu_s instead of $e_{ij} = u_qu_p$ by our procedure, a contradiction. We conclude that $d_{G_{i(j-1)}[S]}(u_r) \leq \frac{k-2}{2}$ for each $u_r \in S_2^i$ $(x_i + 1 \leq r \leq x_i + t)$. Clearly, there are at least $k - 1 - \frac{k-2}{2} = \frac{k}{2}$ edges incident to each u_r $(x_i + 1 \leq r \leq x_i + t)$ belonging to $M \cup (\bigcup_{j=1}^{i-1} M_j) \cup \{e_{i1}, e_{i2}, \ldots, e_{i(j-1)}\}$. Since $j \leq x_i$ and $u_qu_j \in M \cup (\bigcup_{r=1}^{i-1} M_r)$ for $x_i + t + 1 \leq j \leq k$, we have

$$
\begin{aligned}
&|E_{K_n[M]}[u_q, S_2^i]| + \sum_{j=1}^{t} d_{K_n[M]}(u_j) \\
&\geq k - x_i - t + \frac{k}{2}t - \sum_{j=1}^{i-1} x_j - (j-1) - \binom{t}{2} \\
&\geq k + \frac{(k-2)}{2}t - \sum_{j=1}^{i} x_j - x_i + 1 - \binom{t}{2} \qquad \text{(since } j \leq x_i\text{)} \\
&= -\frac{t^2}{2} + \frac{(k-1)}{2}t + k - \sum_{j=1}^{i} x_j - x_i + 1 \\
&= -\frac{1}{2}\left(t - \frac{k-1}{2}\right)^2 + \frac{(k-1)^2}{8} + k - \sum_{j=1}^{i} x_j - x_i + 1
\end{aligned}
$$

and hence

$$\begin{aligned}|M| &\geq \sum_{j=1}^{i}|M\cap(E_{K_n}[w_j,S])| + \sum_{j=1}^{t} d_{K_n[M]}(u_j) + |E_{K_n[M]}[u_q,S_2^i]|\\ &\geq \sum_{j=1}^{i}x_j - \frac{1}{2}\left(t-\frac{k-1}{2}\right)^2 + \frac{(k-1)^2}{8} + k - \sum_{j=1}^{i}x_j - x_i + 1\\ &= -\frac{1}{2}\left(t-\frac{k-1}{2}\right)^2 + \frac{(k-1)^2}{8} + k - x_i + 1\\ &\geq \frac{k}{2} - 1 + k - x_i + 1 \qquad \text{(since } 1\leq t\leq k-2\text{)}\\ &\geq \frac{k}{2} + k - x_i \geq k+1, \qquad \left(\text{since } x_i \leq \frac{k-2}{2}\right)\end{aligned}$$

which contradicts $|M| = k-1$.

Next, assume $u_p \in S_1^i$. Recall that $d_{G_{ij}[S]}(u_p) = \frac{k-4}{2}$. Since $u_p \in S_1^i$, it follows that $d_{G_{i-1}[S]}(u_p) = \frac{k-2}{2}$. If $u_p \in \bigcap_{j=1}^{i} S_1^j$, namely, $u_pw_j \in M$ $(1\leq j\leq i)$, then by our procedure $d_{G[S]}(u_p) = \frac{k-2}{2} + i - 1$ and hence $d_{K_n[S]\cap M}(u_p) = k-1-(\frac{k-2}{2}+i-1) = \frac{k}{2} - i + 1$. Since $u_pw_j \in M$ for each $w_j \in \bar{S}$ $(1\leq j\leq i)$, we have $d_{K_n[M]}(u_p) = d_{K_n[S]\cap M}(u_p) + d_{K_n[S,\bar{S}]\cap M}(u_p) \geq (\frac{k}{2}-i+1)+i = \frac{k+2}{2}$, contradicting $\Delta(K_n[M]) \leq \frac{k}{2}$. Combining this with $u_p \in S_1^i$, we have $u_p \notin \bigcap_{j=1}^{i-1} S_1^i$ and we can assume that there exists an integer i' $(i' \leq i-1)$ satisfying the following conditions:

- $u_p \in S_2^{i'}$ and $d_{G_{i'}[S]}(u_p) < d_{G_{i'-1}[S]}(u_p)$;
- if u_p belongs to some S_2^j $(i'+1 \leq j \leq i)$, then $d_{G_j[S]}(u_p) = d_{G_{j-1}[S]}(u_p)$.

The above two conditions can be restated as follows:

- $u_pw_{i'} \in E(G)$ and $d_{G_{i'}[S]}(u_p) < d_{G_{i'-1}[S]}(u_p)$;
- if $u_pw_j \in E(G)$ $(i'+1 \leq j \leq i)$, then $d_{G_j[S]}(u_p) = d_{G_{j-1}[S]}(u_p)$.

In fact, we can find the integer i' such that $u_pw_{i'} \in E(G)$ and $d_{G_{i'}[S]}(u_p) < d_{G_{i'-1}[S]}(u_p)$. Assume, to the contrary, that for each w_j $(1\leq j\leq i)$, $u_pw_j \in M$, or $u_pw_j \in E(G)$ but $d_{G_j[S]}(u_p) = d_{G_{j-1}[S]}(u_p)$. Let i_1 $(i_1 \leq i)$ be the number of vertices nonadjacent to $u_p \in S$ in $\{w_1, w_2, \ldots, w_{i-1}\} \subseteq \bar{S}$. Without loss of generality, let $w_ju_p \in M$ $(1\leq j\leq i_1)$. Recall that $d_{G_{ij}[S]}(u_p) = \frac{k-4}{2}$. Thus $d_{G[S]}(u_p) = \frac{k-4}{2} + i_1$ and hence $d_{K_n[S]\cap M}(u_p) \geq k-1-(\frac{k-4}{2}+i_1) = \frac{k+2}{2} - i_1$. Since $w_ju_p \in M$ $(1\leq j\leq i_1)$, it follows that $d_{K_n[S,\bar{S}]\cap M}(u_p) \geq i_1$, which results in $d_{K_n[M]}(u_p) = d_{K_n[S]\cap M}(u_p) + d_{K_n[S,\bar{S}]\cap M}(u_p) \geq (\frac{k+2}{2} - i_1) + i_1 = \frac{k+2}{2}$, contradicting $\Delta(K_n[M]) \leq \frac{k}{2}$.

Now we turn our attention to $u_p \in S_2^{i'}$. Without loss of generality, let $u_pw_j \in M$ ($j \in \{j_1, j_2, \ldots, j_{i_1}\}$), namely, $u_p \in S_1^{j_1} \cap S_1^{j_2} \cap \cdots \cap S_1^{j_{i_1}}$, where $j_1, j_2, \ldots, j_{i_1} \in \{i'+1, i'+2, \ldots, i\}$. Then $u_pw_j \in E(G)$ ($j \in \{i'+1, i'+2, \ldots, i\} - \{j_1, j_2, \ldots, j_{i_1}\}$). Clearly, $i_1 \leq i - i'$. Recall that $u_p \in S_1^i$ and $d_{G_{ij}[S]}(u_p) = \frac{k-4}{2}$. Thus $d_{G_{i'}[S]}(u_p) = \frac{k-4}{2} + i_1$. By our procedure, there exists a vertex $u_q \in S_1^{i'}$ such that when we select the edge $e_{i'j} = u_pu_q$ from $G_{i'(j-1)}[S]$, then the degree of u_p in $G_{i'j}[S]$ is equal to $\frac{k-4}{2} + i_1$, that is, $d_{G_{i'j}[S]}(u_p) = \frac{k-4}{2} + i_1$ and $d_{G_{i'(j-1)}[S]}(u_p) = \frac{k-2}{2} + i_1$. From our procedure, $|E_{G_{i'-1}}[u_q, S_2^{i'}]| = |E_{G_{i'(j-1)}}[u_q, S_2^{i'}]|$. Without loss of generality, let $|E_{G_{i'-1}}[u_q, S_2^{i'}]| = t$ and $u_qu_j \in E(G_{i'-1})$ for $x_{i'} + 1 \leq j \leq x_{i'} + t$; see Figure 3($c$). Thus $u_p \in \{u_{x_{i'}+1}, u_{x_{i'}+2}, \ldots, u_{x_{i'}+t}\}$, and $u_qu_j \in M \cup (\bigcup_{r=1}^{i'-1} M_r)$ for $x_{i'} + t + 1 \leq j \leq k$. Since $x_j \leq \frac{k-2}{2}$ ($2 \leq j \leq n-k$), it follows that $|S_1^{i'}| \leq \frac{k-2}{2}$. From this together with $\delta(G_{i'-1}[S]) \geq \frac{k-2}{2}$, we have $|E_{G_{i'-1}}[u_q, S_1^{i'}]| \geq 1$, that is, $t \geq 1$. Since $d_{G_{i'(j-1)}[S]}(u_p) = \frac{k-2}{2} + i_1$, by our procedure $d_{G_{i'(j-1)}[S]}(u_j) \leq \frac{k-2}{2} + i_1$ for each $u_j \in S_2^{i'}$ ($x_{i'} + 1 \leq j \leq x_{i'} + t$). Assume, to the contrary, that there is a vertex u_s ($x_{i'} + 1 \leq s \leq x_{i'} + t$) such that $d_{G_{i'(j-1)}[S]}(u_s) \geq \frac{k-2}{2} + i_1 + 1$. Then we should have selected the edge u_qu_s instead of $e_{i'j} = u_qu_p$ by our procedure, a contradiction. We conclude that $d_{G_{i'(j-1)}[S]}(u_r) \leq \frac{k-2}{2} + i_1$ for each $u_r \in S_2^{i'}$ ($x_{i'} + 1 \leq r \leq x_{i'} + t$). Clearly, there are at least $k - 1 - (\frac{k-2}{2} + i_1) = \frac{k}{2} - i_1$ edges incident to each u_r ($x_{i'} + 1 \leq r \leq x_{i'} + t$) belonging to $M \cup (\bigcup_{j=1}^{i'-1} M_j) \cup \{e_{i'1}, e_{i'2}, \ldots, e_{i'(j-1)}\}$. Since $j \leq x_{i'}$ and $u_qu_j \in M \cup (\bigcup_{r=1}^{i'-1} M_r)$ for $x_{i'} + t + 1 \leq j \leq k$, we have

$$
\begin{aligned}
&|E_{K_n[M]}[u_q, S_2^{i'}]| + \sum_{j=1}^{t} d_{K_n[M]}(u_j) \\
&\geq k - x_{i'} - t + \left(\frac{k}{2} - i_1\right)t - \sum_{j=1}^{i'-1} x_j - (j-1) - \binom{t}{2} \\
&\geq k - \sum_{j=1}^{i'} x_j + \left(\frac{k-2}{2} - i_1\right)t - x_{i'} + 1 - \frac{t(t-1)}{2} \qquad (\text{since } j \leq x_{i'}) \\
&= -\frac{t^2}{2} + \frac{t}{2} + k - \sum_{j=1}^{i'} x_j + \left(\frac{k-2}{2} - i + i'\right)t - x_{i'} + 1 \qquad (\text{since } i_1 \leq i - i') \\
&= -\frac{t^2}{2} + \left(\frac{k-1}{2} - i + i'\right)t + k - \sum_{j=1}^{i'} x_j - x_{i'} + 1 \\
&= -\frac{1}{2}\Big(t^2 - (k-1-2i+2i')t\Big) + k - \sum_{j=1}^{i'} x_j - x_{i'} + 1 \\
&= -\frac{1}{2}\left(t - \frac{k-1-2i+2i'}{2}\right)^2 + \frac{(k-1-2i+2i')^2}{8} + k - \sum_{j=1}^{i'} x_j - x_{i'} + 1
\end{aligned}
$$

and hence

$$\begin{aligned}
|M| &\geq \sum_{j=1}^{i} |M \cap (E_{K_n}[w_j, S])| + \sum_{j=1}^{p} d_{K_n[M]}(u_j) + |E_{K_n[M]}[u_q, S_2^i]| \\
&\geq \sum_{j=1}^{i} x_j - \frac{1}{2}\left(t - \frac{k-1-2i+2i'}{2}\right)^2 + \frac{(k-1-2i+2i')^2}{8} + k - \sum_{j=1}^{i'} x_j - x_{i'} + 1 \\
&= -\frac{1}{2}\left(t - \frac{k-1-2i+2i'}{2}\right)^2 + \frac{(k-1-2i+2i')^2}{8} + k + \sum_{j=i'+1}^{i} x_j - x_{i'} + 1 \\
&\geq \frac{k}{2} - 1 - i + i' + k + \sum_{j=i'+1}^{i} x_j - x_{i'} + 1 \qquad (\text{since } 1 \leq t \leq k-2 \text{ and } k-1-2i+2i' \leq k-2) \\
&\geq k, \qquad \left(\text{since } x_{i'} \leq \frac{k-2}{2} \text{ and } x_j \geq 1 \text{ for } i'+1 \leq j \leq i\right)
\end{aligned}$$

contradicting $|M| = k-1$. This completes the proof of Claim 2. □

From our procedure, we get $n-k$ internally disjoint Steiner trees connecting S in G, say $T_1, T_2, \ldots, T_{n-k}$. Recall that $G_{n-k} = G - (\bigcup_{i=1}^{n-k} M_i)$. We can regard $G_{n-k}[S] = G[S] - (\bigcup_{i=1}^{n-k} M_i)$ as a graph obtained from the complete graph K_k by deleting $|M'| + \sum_{i=1}^{n-k} |M_i|$ edges. Since $|M'| + \sum_{i=1}^{n-k} |M_i| + |M''| = m_1 + \sum_{i=1}^{n-k} x_i + m_2 = k-1$, we have $1 \leq \sum_{i=1}^{n-k} |M_i| + m_1 \leq k-1$. By Claim 2, $\delta(G_{n-k}[S]) \geq \frac{k-2}{2}$ and hence $2 \leq \Delta(\overline{G_{n-k}[S]}) \leq \frac{k}{2}$. From Lemma 2.6, there exist $\frac{k-2}{2}$ edge-disjoint spanning trees connecting S in $G_{n-k}[S]$. These trees together with $T_1, T_2, \ldots, T_{n-k}$ are $n - \frac{k}{2} - 1$ internally disjoint Steiner trees connecting S in G. Thus, $\kappa(S) \geq n - \frac{k}{2} - 1$. From the arbitrariness of S, we have $\kappa_k(G) \geq n - \frac{k}{2} - 1$, as desired. ■

We are now in a position to prove our main results.

***Proof of Theorem* 1.8.** Assume that $\kappa_k(G) = n - \frac{k}{2} - 1$. Since G of order n is connected, we can regard G as a graph obtained from the complete graph K_n by deleting some edges. From Lemma 1.7, it follows that $|M| \geq 1$ and hence $\Delta(K_n[M]) \geq 1$. If $G = K_n - M$ where $M \subseteq E(K_n)$ such that $\Delta(K_n[M]) \geq \frac{k}{2} + 1$, then $\kappa_k(G) \leq \lambda_k(G) < n - \frac{k}{2} - 1$ by Observation 1.2 and Corollary 2.2, a contradiction. So $1 \leq \Delta(K_n[M]) \leq \frac{k}{2}$. If $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $|M| \geq k$, then $\kappa_k(G) \leq \lambda_k(G) < n - \frac{k}{2} - 1$ by Observation 1.2 and Lemma 2.4, a contradiction. Therefore, $1 \leq |M| \leq k-1$. If $\Delta(K_n[M]) = 1$, then $1 \leq |M| \leq k-1$ by Lemma 2.5. We conclude that $1 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $1 \leq |M| \leq k-1$, as desired.

Conversely, let $G = K_n - M$ where $M \subseteq E(K_n)$ such that $1 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $1 \leq |M| \leq k-1$. In fact, we only need to show that $\kappa_k(G) \geq n - \frac{k}{2} - 1$ for

$\Delta(K_n[M]) = 1$ and $|M| = k-1$, or $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$ and $|M| = k-1$. The results follow by (1) of Lemma 2.7 and Lemma 2.8. ■

***Proof of Theorem* 1.9.** If G is a connected graph satisfying condition (2), then $\kappa_k(G) = n - \frac{k}{2} - 1$ by Theorem 1.8. From Observation 1.2, $\lambda_k(G) \geq \kappa_k(G) = n - \frac{k}{2} - 1$. From this together with Lemma 1.7, we have $\lambda_k(G) = n - \frac{k}{2} - 1$. Assume that G is a connected graph satisfying condition (1). We only need to show that $\lambda_k(G) = n - \frac{k}{2} - 1$ for $|M| = \lfloor \frac{n}{2} \rfloor$. The result follows by (2) of Lemma 2.7 and Lemma 1.7.

Conversely, assume that $\lambda_k(G) = n - \frac{k}{2} - 1$. Since G of order n is connected, we can consider G as a graph obtained from a complete graph K_n by deleting some edges. From Corollary 2.2, $G = K_n - M$ such that $\Delta(K_n[M]) \leq \frac{k}{2}$, where $M \subseteq E(K_n)$. Combining this with Lemma 1.7, we have $|M| \geq 1$ and $\Delta(K_n[M]) \geq 1$. So $1 \leq \Delta(K_n[M]) \leq \frac{k}{2}$. It is clear that if $\Delta(K_n[M]) = 1$ then $1 \leq |M| \leq \lfloor \frac{n}{2} \rfloor$. If $2 \leq \Delta(K_n[M]) \leq \frac{k}{2}$, then $1 \leq |M| \leq k-1$ by Lemma 2.4. So (1) or (2) holds. ■

Remark 2.3. As we know, $\lambda(G) = n-2$ if and only if $G = K_n - M$ such that $\Delta(K_n[M]) = 1$ and $1 \leq |M| \leq \lfloor \frac{n}{2} \rfloor$, where $M \subseteq E(K_n)$. So we can restate the above conclusion as follows: $\lambda_2(G) = n-2$ if and only if $G = K_n - M$ such that $\Delta(K_n[M]) = 1$ and $1 \leq |M| \leq \lfloor \frac{n}{2} \rfloor$, where $M \subseteq E(K_n)$. This means that $4 \leq k \leq n$ in Theorem 1.9 can be replaced by $2 \leq k \leq n$.

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Appendix: An example for Case 2 of Lemma 2.8

Let $k = 8$ and let $G = K_n - M$, where $M \subseteq E(K_n)$, be a connected graph of order n such that $|M| = k - 1 = 7$ and $\Delta(K_n[M]) \leq \frac{k}{2} = 4$. Let $S = \{u_1, u_2, \ldots, u_8\}$, $\bar{S} = V(G) - S = \{w_1, w_2, \ldots, w_{n-8}\}$ and $M = \{w_1u_1, w_1u_2, w_1u_3, w_2u_2, w_2u_4, u_5u_6, u_6u_8\}$; see Figure 4$(a)$. Clearly, $x_1 = |E_{K_n[M]}[w_1, S]| = 3 \geq x_2 = |E_{K_n[M]}[w_2, S]| = 2 > x_i = |E_{K_n[M]}[w_i, S]| = 0$ $(3 \leq i \leq n - 8)$.

For w_1, we let $S_1^1 = \{u_1, u_2, u_3\}$ since $w_1u_1, w_1u_2, w_1u_3 \in M$. Set $S_2^1 = S - S_1^1 = \{u_4, u_5, u_6, u_7, u_8\}$. Clearly, $d_{G[S]}(u_1) = d_{G[S]}(u_2) = d_{G[S]}(u_3) = 7 = k - 1$ and hence u_1, u_2, u_3 are all the vertices of S_1^1 having maximum degree in $G[S]$. But u_1 is the one with the smallest subscript, so we choose $u_1' = u_1$ in S_1^1 and select the vertex adjacent to u_1' in S_2^1 and obtain $u_4, u_5, u_6, u_7, u_8 \in S_2^1$ since $u_1'u_j \in E(G)$ $(j = 4, \ldots, 8)$. Obviously, $d_{G[S]}(u_4) = d_{G[S]}(u_7) = 7 > d_{G[S]}(u_5) = d_{G[S]}(u_8) = 6 > d_{G[S]}(u_6) = 5$ and hence u_4, u_7 are two vertices of S_2^1 having maximum degree in $G[S]$. Since u_4 is the one with the smallest subscript, we choose $u_1'' = u_4 \in S_2^1$ and put $e_{11} = u_1'u_1''(= u_1u_4)$. Consider the graph $G_{11} = G - e_{11}$. Since $d_{G_{11}[S]}(u_2) = d_{G_{11}[S]}(u_3) = 7$ and the subscript of u_2 is smaller than u_3, we let $u_2' = u_2$ in $S_1^1 - u_1'$ and select the vertices adjacent to u_2' in S_2^1 and obtain $u_4, u_5, u_6, u_7, u_8 \in S_2^1$ since $u_2'u_j \in E(G_{11})$ $(j = 4, \ldots, 8)$. Since $d_{G_{11}[S]}(u_7) = 7 > d_{G_{11}[S]}(u_j) = 6 > d_{G_{11}[S]}(u_6) = 5$ $(j = 4, 5, 8)$, we select $u_2'' = u_7 \in S_2^1$ and get $e_{12} = u_2'u_2''$ $(= u_2u_7)$. Consider the graph $G_{12} = G_{11} - e_{12} = G - \{e_{11}, e_{12}\}$. There is only one vertex u_3 in $S_1 - \{u_1', u_2'\} = S_1 - \{u_1, u_2\}$. Therefore, let $u_3' = u_3$ and select the vertices adjacent to u_3' in S_2^1 and obtain $u_j \in S_2^1$ since $u_3'u_j \in E(G_{12})$ $(j = 4, \ldots, 8)$.

Since $d_{G_{12}[S]}(u_j) = 6 > d_{G_{12}[S]}(u_6) = 5$ $(i = 4, 5, 7, 8)$, it follows that u_4, u_5, u_7, u_8 are all the vertices of S_2^1 having maximum degree in $G_{12}[S]$. But u_4 is the one with the smallest subscript, so we choose $u_3'' = u_4 \in S_2^1$ and get $e_{13} = u_3'u_3''$ $(= u_3u_4)$. Since $x_1 = |E_{K_n[M]}[w_1, S]| = 3$, we terminate this procedure. Set $M_1 = \{e_{11}, e_{12}, e_{13}\}$ and $G_1 = G - M_1$. Thus the tree T_1 induced by the edges in $\{w_1u_4, w_1u_5, w_1u_6, w_1u_7, w_1u_8, u_1u_4, u_2u_7, u_3u_4\}$ is our desired tree; see Figure 4(b).

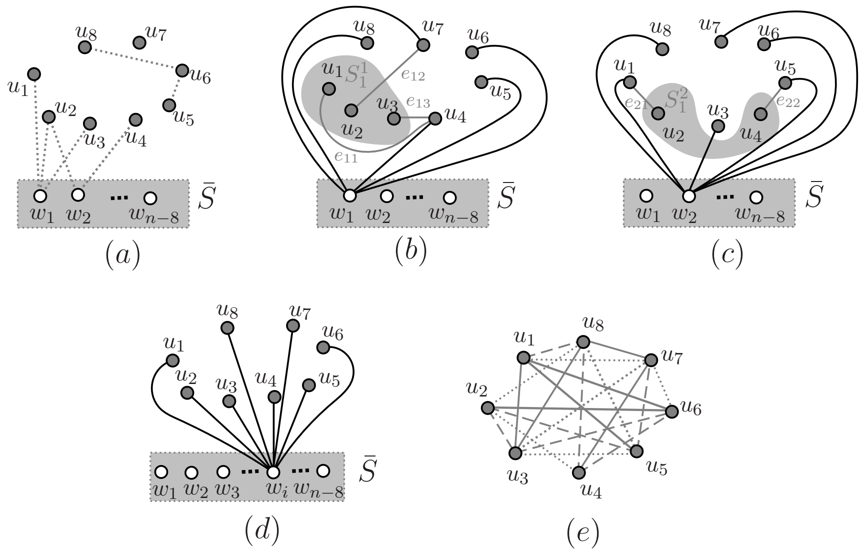

Figure 4. Graphs for the appendix.

For w_2, we let $S_1^2 = \{u_2, u_4\}$ since $w_2u_2, w_2u_4 \in M$. Let $S_2^2 = S - S_1^2 = \{u_1, u_3, u_5, u_6, u_7, u_8\}$. Since $d_{G_1[S]}(u_2) = 6 > d_{G_1[S]}(u_4) = 5$, it follows that u_2 is the vertex of S_1^2 having maximum degree in $G_1[S]$. So we choose $u_1' = u_2$ in S_1^2 and find the vertices adjacent to u_1' $(= u_2)$ in S_2^2 and obtain $u_1, u_3, u_5, u_6, u_8 \in S_2^2$ since $u_1'u_j \in E(G_{21})$ $(j = 1, 3, 5, 6, 8)$. Since $d_{G_1[S]}(u_j) = 6 > d_{G_1[S]}(u_6) = 5$ $(j = 1, 3, 5, 8)$ and u_1 is the vertex having maximum degree with the smallest subscript, we choose $u_1'' = u_1 \in S_2^2$. Put $e_{21} = u_1'u_1''$ $(= u_2u_1)$. Consider the graph $G_{21} = G_1 - e_{21}$. Clearly, $S_1 - \{u_1'\} = S_1 - \{u_2\} = \{u_4\}$, so we let $u_2' = u_4$ and select the vertices adjacent to u_2' $(= u_4)$ in S_2^2 and obtain u_5, u_6, u_7, u_8 since $u_2u_j \in E(G)$ $(j = 5, 6, 7, 8)$. Since $d_{G_{21}[S]}(u_j) = 6 > d_{G_{21}[S]}(u_6) = 5$ $(j = 5, 7, 8)$ and u_5 is the vertex with the smallest subscript, we let $u_2'' = u_5 \in S_2^2$ and get $e_{22} = u_2'u_2''$ $(= u_4u_5)$. Since $x_2 = |E_{K_n[M]}[w_2, S]| = 2$, we terminate this procedure. Let $M_2 = \{e_{21}, e_{22}\}$ and $G_2 = G_1 - M_2$. Then the tree T_2 induced by the edges in $\{w_2u_1, w_2u_3, w_2u_5, w_2u_6, w_2u_7, w_2u_8, u_2u_1, u_4u_5\}$ is our desired tree; see Figure 4(c). Obviously, T_2 and T_1 are two internally disjoint Steiner trees connecting S.

Since $x_i = |E_{K_n[M]}[w_i, S]| = 0$ for $3 \leq i \leq n-8$, we terminate this procedure. For $w_3, \ldots, w_{n-8}$, the trees T_i induced by the edges $\{w_iu_1, w_iu_2, \ldots, w_iu_8\}$ $(3 \leq i \leq n-8)$ (see Figure $4(d)$) are our desired trees.

We can consider $G_2[S] = G[S] - \{M_1, M_2\}$ as a graph obtained from complete graph K_k by deleting $|M \cap K_n[S]| + |M_1| + |M_2|$ edges. Since $|M \cap K_n[S]| + |M_1| + |M_2| = 2 + 3 + 2 = 7 = k - 1$, it follows from Lemma 2.6 that there exist three edge-disjoint spanning trees in $G[S]$. (Actually, we can give three edge-disjoint spanning trees; see Figure $4(e)$. For example, the trees T_1' induced by the edges in $\{u_1u_8 \cup u_8u_4 \cup u_4u_6 \cup u_6u_3 \cup u_3u_2 \cup u_2u_5 \cup u_5u_7\}$, T_2' induced by the edges in $\{u_4u_7 \cup u_7u_8 \cup u_8u_3 \cup u_3u_1 \cup u_1u_5 \cup u_1u_6 \cup u_6u_2\}$ and T_3' induced by the edges in $\{u_2u_4 \cup u_2u_8 \cup u_8u_5 \cup u_5u_3 \cup u_3u_7 \cup u_1u_7 \cup u_7u_6\}$ can be our desired trees.) These three trees together with $T_1, T_2, \ldots, T_{n-8}$ are $n - 5 = n - \frac{k}{2} - 1$ internally disjoint Steiner trees connecting S. Thus, $\kappa(S) \geq n - 5$.