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THE DYNAMICS OF THE FOREST GRAPH OPERATOR

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Abstract

In 1966, Cummins introduced the “tree graph”: the tree graph $\mathbf{T}(G)$ of a graph G (possibly infinite) has all its spanning trees as vertices, and distinct such trees correspond to adjacent vertices if they differ in just one edge, i.e., two spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The tree graph of a connected graph need not be connected. To obviate this difficulty we define the “forest graph”: let G be a labeled graph of order α , finite or infinite, and let $\mathfrak{N}(G)$ be the set of all labeled maximal forests of G . The forest graph of G , denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests F_1, F_2 of G form an edge if and only if they differ exactly by one edge, i.e., $F_2 = F_1 - e + f$ for some edges $e \in F_1$ and $f \notin F_1$.

Using the theory of cardinal numbers, Zorn's lemma, transfinite induction, the axiom of choice and the well-ordering principle, we determine the \mathbf{F} -convergence, \mathbf{F} -divergence, \mathbf{F} -depth and \mathbf{F} -stability of any graph G . In particular it is shown that a graph G (finite or infinite) is \mathbf{F} -convergent if and only if G has at most one cycle of length 3. The \mathbf{F} -stable graphs are precisely K_3 and K_1 . The \mathbf{F} -depth of any graph G different from K_3 and K_1 is finite. We also determine various parameters of $\mathbf{F}(G)$ for an infinite graph G , including the number, order, size, and degree of its components.

Keywords: forest graph operator, graph dynamics.

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1. INTRODUCTION

A *graph dynamical system* is a set X of graphs together with a mapping $\phi : X \rightarrow X$ (see Prisner [12]). We investigate the graph dynamical system on finite and infinite graphs defined by the forest graph operator \mathbf{F} , which transforms G to its graph of maximal forests.

Let G be a labeled graph of order α , finite or infinite. (All our graphs are labeled.) A *spanning tree* of G is a connected, acyclic, spanning subgraph of G ; it exists if and only if G is connected. Any acyclic subgraph of G , connected or not, is called a *forest* of G . A forest F of G is said to be *maximal* if there is no forest F' of G such that F is a proper subgraph of F' . The tree graph $\mathbf{T}(G)$ of G has all the spanning trees of G as vertices, and distinct such trees are adjacent vertices if they differ in just one edge [12, 15]; i.e., two spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The *iterated tree graphs* of G are defined by $\mathbf{T}^0(G) = G$ and $\mathbf{T}^n(G) = \mathbf{T}(\mathbf{T}^{n-1}(G))$ for $n > 0$. There are several results on tree graphs. See [1, 11, 18] for connectivity of the tree graph, [3, 4, 6, 7, 8, 10, 13, 16, 19] for bounds on the order of $\mathbf{T}(G)$ (that is, on the number of spanning trees of G), [2, 14] for Hamilton circuits in a tree graph.

There is one difficulty with iterating the tree graph operator. The tree graph of an infinite connected graph need not be connected [2, 14], so $\mathbf{T}^2(G)$ may be undefined. For example, $\mathbf{T}(K_{\aleph_0})$ is disconnected (see Corollary 2.5 in this paper; \aleph_0 denotes the cardinality of the set \mathbb{N} of natural numbers); therefore $\mathbf{T}^2(K_{\aleph_0})$ is not defined. To obviate this difficulty with iterated tree graphs, and inspired by the tree graph operator \mathbf{T} , we define a forest graph operator. Let $\mathfrak{N}(G)$ be the set of all maximal forests of G . The *forest graph* of G , denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests F_1, F_2 form an edge if and only if they differ by exactly one edge. The *forest graph operator* (or *maximal forest operator*) on graphs, $G \mapsto \mathbf{F}(G)$, is denoted by \mathbf{F} . Zorn's

lemma implies that every connected graph contains a spanning tree (see [5]); similarly, every graph has a maximal forest. Hence, the forest graph always exists. Since, when G is connected, maximal forests are the same as spanning trees, then $\mathbf{F}(G) = \mathbf{T}(G)$; that is, the tree graph is a special case of the forest graph. We write $\mathbf{F}^2(G)$ to denote $\mathbf{F}(\mathbf{F}(G))$, and in general $\mathbf{F}^n(G) = \mathbf{F}(\mathbf{F}^{n-1}(G))$ for $n \geq 1$, with $\mathbf{F}^0(G) = G$.

Definition 1.1. A graph G is said to be **F-convergent** if $\{\mathbf{F}^n(G) : n \in \mathbb{N}\}$ is finite; otherwise it is **F-divergent**.

A graph H is said to be an **F-root** of G if $\mathbf{F}(H)$ is isomorphic to G , $\mathbf{F}(H) \cong G$. The **F-depth** of G is

$$\sup\{n \in \mathbb{N} : G \cong \mathbf{F}^n(H) \text{ for some graph } H\}.$$

The **F-depth** of a graph G that has no **F-root** is said to be zero.

The graph G is said to be **F-periodic** if there exists a positive integer n such that $\mathbf{F}^n(G) = G$. The least such integer is called the **F-periodicity** of G . If $n = 1$, G is called **F-stable**.

This paper is organized as follows. In Section 2 we give some basic results. In later sections, using Zorn's lemma, transfinite induction, the well ordering principle and the theory of cardinal numbers, we study the number of **F-roots** and determine the **F-convergence**, **F-divergence**, **F-depth** and **F-stability** of any graph G . In particular we show that:

- (i) A graph G is **F-convergent** if and only if G has at most one cycle of length 3.
- (ii) The **F-depth** of any graph G different from K_3 and K_1 is finite.
- (iii) The **F-stable** graphs are precisely K_3 and K_1 .
- (iv) A graph that has one **F-root** has innumerable many, but only some **F-roots** are important.

2. PRELIMINARIES

For standard notation and terminology in graph theory we follow Diestel [5] and Prisner [12].

Some elementary properties of infinite cardinal numbers that we use are (see, e.g., Kamke [9]):

- (1) $\alpha + \beta = \alpha \cdot \beta = \max(\alpha, \beta)$ if α, β are cardinal numbers and β is infinite. In particular, $2 \cdot \beta = \aleph_0 \cdot \beta = \beta$.
- (2) $\beta^n = \beta$ if β is an infinite cardinal and n is a positive integer.
- (3) $\beta < 2^\beta$ for every cardinal number.
- (4) The number of finite subsets of an infinite set of cardinality β is equal to β .

We consider finite and infinite labeled graphs *without multiple edges or loops*. An *isthmus* of a graph G is an edge e such that deleting e divides one component of G into two of $G - e$. Equivalently, an isthmus is an edge that belongs to no cycle. Each isthmus is in every maximal forest, but no non-isthmus is.

Let $\mathfrak{C}(G)$ and $\mathfrak{N}(G)$ denote the set of all possible cycles and the set of all maximal forests of a graph G , respectively. Note that a maximal forest of G consists of a spanning tree in each component of G . A fundamental fact, whose proof is similar to that of the existence of a maximal forest, is the following forest extension lemma.

Lemma 2.1. *In any graph G , every forest is contained in a maximal forest.*

Lemma 2.2. *If G is a complete graph of infinite order α , then $|\mathfrak{N}(G)| = 2^\alpha$.*

Proof. Let $G = (V, E)$ be a complete graph of order α (α infinite), i.e., $G = K_\alpha$. Let v_1, v_2 be two vertices of G and $V' = V \setminus \{v_1, v_2\}$. Then for every $A \subseteq V'$ there is a spanning tree T_A such that every vertex of A is adjacent only to v_1 and every vertex of $V' \setminus A$ is adjacent only to v_2 . It is easy to see that $T_A \neq T_B$ whenever $A \neq B$. As the cardinality of the power set of V' is 2^α , there are at least 2^α spanning trees of G . Since G is connected, the maximal forests are the spanning trees; therefore $|\mathfrak{N}(G)| \geq 2^\alpha$. Since the degree of each vertex is α and G contains α vertices, the total number of edges in G is $\alpha \cdot \alpha = \alpha$. The edge set of a maximal forest of G is a subset of E and the number of all possible subsets of E is 2^α . Therefore, G has at most 2^α maximal forests, i.e., $|\mathfrak{N}(G)| \leq 2^\alpha$. Hence $|\mathfrak{N}(G)| = 2^\alpha$. ■

For two maximal forests of G , F_1 and F_2 , let $d(F_1, F_2)$ denote the distance between them in $\mathbf{F}(G)$. We connect this distance to the number of edges by which F_1, F_2 differ; the result is elementary but we could not find it anywhere in the literature. We say F_1, F_2 differ by l edges if $|E(F_1) \setminus E(F_2)| = |E(F_2) \setminus E(F_1)| = l$.

Lemma 2.3. *Let l be a natural number. For two maximal forests F_1, F_2 of a graph G , if $|E(F_1) \setminus E(F_2)| = l$, then $|E(F_2) \setminus E(F_1)| = l$. Furthermore, F_1 and F_2 differ by exactly l edges if and only if $d(F_1, F_2) = l$.*

We cannot apply to an infinite graph the simple proof for finite graphs, in which the number of edges in a maximal forest is given by a formula. Therefore, we prove the lemma by edge exchange.

Proof. We prove the first part by induction on l . Let F_1, F_2 be maximal forests of G and let $E(F_1) \setminus E(F_2) = \{e'_1, e'_2, \dots, e'_k\}$, $E(F_2) \setminus E(F_1) = \{e_1, e_2, \dots, e_l\}$. If $l = 0$ then $k = 0 = l$ because $F_2 = F_1$. Suppose $l > 0$; then $k > 0$ also. Deleting e_l from F_2 divides a tree of F_2 into two trees. Since these trees are in the same component of G , there is an edge of F_1 that connects them; this edge

is not e_1 so it is not in F_2 ; therefore, it is an e'_i , say e'_k . Let $F'_2 = F_2 - e_l + e'_k$. Then $E(F_1) \setminus E(F'_2) = \{e'_1, e'_2, \dots, e'_{k-1}\}$, $E(F'_2) \setminus E(F_1) = \{e_1, e_2, \dots, e_{l-1}\}$. By induction, $k - 1 = l - 1$.

We also prove the second part by induction on l . Assume F_1, F_2 differ by exactly l edges and define F'_2 as above. If $l = 0, 1$, clearly $d(F_1, F_2) = l$. Suppose $l > 1$. In a shortest path from F_1 to F_2 , whose length is $d(F_1, F_2)$, each successive edge of the path can increase the number of edges not in F_1 by at most 1. Therefore, F_1 and F_2 differ by at most $d(F_1, F_2)$ edges. That is, $l \leq d(F_1, F_2)$. Conversely, $d(F_1, F'_2) = l - 1$ by induction and there is a path in $\mathbf{F}(G)$ from F_1 to F'_2 of length $l - 1$, then continuing to F_2 and having total length l . Thus, $d(F_1, F_2) \leq l$. ■

From the above lemma we have two corollaries.

Corollary 2.4. *For any graph G , $\mathbf{F}(G)$ is connected if and only if any two maximal forests of G differ by at most a finite number of edges.*

Corollary 2.5. *If $G = K_\alpha$, α infinite, then $\mathbf{F}(G)$ is disconnected.*

Lemma 2.6. *Let G be a graph with α vertices and β edges and with no isolated vertices. If either α or β is infinite, then $\alpha = \beta$.*

Proof. We know that $|E(G)| \leq |V(G)|^2$, i.e., $\beta \leq \alpha^2$ so if β is infinite, α must also be infinite. We also know, since each edge has two endpoints, that $|V(G)| \leq 2|E(G)|$, i.e., $\alpha \leq 2 \cdot \beta$ so if α is infinite, then β must be infinite. Now assuming both are infinite, $\alpha^2 = \alpha$ and $2 \cdot \beta = \beta$, hence $\alpha = \beta$. ■

The following lemmas are needed in connection with \mathbf{F} -convergence and \mathbf{F} -divergence in Section 5 and \mathbf{F} -depth in Section 6.

Lemma 2.7. *Let G be a graph. If K_n (for finite $n \geq 2$) is a subgraph of G , then $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $\mathbf{F}(G)$.*

Proof. Let G be a graph such that K_n ($n \geq 2$, finite) is a subgraph of G with vertex labels v_1, v_2, \dots, v_n . Then there is a path $L = v_1, v_2, \dots, v_n$ of order n in G . Let F be a maximal forest of G such that F contains the path L . In F if we replace the edge $v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1}$ by any other edge $v_i v_j$ where $i = 1, \dots, \lfloor n/2 \rfloor$ and $j = \lfloor n/2 \rfloor + 1, \dots, n$, we get a maximal forest F_{ij} . Since there are $\lfloor n^2/4 \rfloor$ such edges $v_i v_j$, there are $\lfloor n^2/4 \rfloor$ maximal forests F_{ij} (of which one is F). Any two forests F_{ij} differ by one edge. It follows that they form a complete subgraph in $\mathbf{F}(G)$. Therefore $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $\mathbf{F}(G)$. ■

Lemma 2.8. *If G has a cycle of (finite) length n with $n \geq 3$, then $\mathbf{F}(G)$ contains K_n .*

Proof. Suppose that G has a cycle C_n of length n with edge set $\{e_1, e_2, \dots, e_n\}$. Let $P_i = C_n - e_i$ for $i = 1, 2, \dots, n$ and let F_1 be a maximal forest of G containing the path P_1 . Define $F_i = F_1 \setminus P_1 \cup P_i$ for $i = 2, 3, \dots, n$. These F_i 's are maximal forests of G and any two of them differ by exactly one edge, so they form a complete graph K_n in $\mathbf{F}(G)$. ■

In particular, $\mathbf{F}(C_n) = K_n$.

Lemma 2.9. *Suppose that G contains K_n , where $n \geq 3$. Then $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$.*

Proof. Cayley's formula states that K_n has n^{n-2} spanning trees. Cummins [2] proved that the tree graph of a finite connected graph is Hamiltonian. Therefore, $\mathbf{F}(K_n)$ contains $C_{n^{n-2}}$. Let F_{T_0} be a spanning tree of G that extends one of the spanning trees T_0 of the K_n subgraph. Replacing the edges of T_0 in F_{T_0} by the edges of any other spanning tree T of K_n , we have a spanning tree F_T that contains T . The F_T 's for all spanning trees T of K_n are n^{n-2} spanning trees of G that differ only within K_n ; thus, the graph of the F_T 's is the same as the graph of the T 's, which is Hamiltonian. That is, $\mathbf{F}(G)$ contains $C_{n^{n-2}}$. By Lemma 2.8, $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$. ■

We do not know exactly what graphs $\mathbf{F}(K_n)$ and $\mathbf{F}^2(K_n)$ are.

Lemma 2.10. *If G has two edge disjoint triangles, then $\mathbf{F}^2(G)$ contains K_9 .*

Proof. Suppose that G has two edge disjoint triangles whose edges are e_1, e_2, e_3 and f_1, f_2, f_3 , respectively. The union of the triangles has exactly 9 maximal forests F'_{ij} , obtained by deleting one e_i and one f_j from the triangles. Extend F'_{11} to a maximal forest F_{11} and let F_{ij} be the maximal forest $F_{11} \setminus E(F'_{11}) \cup F'_{ij}$, for each $i, j = 1, 2, 3$. The nine maximal forests F'_{ij} , and consequently the maximal forests F_{ij} in $\mathbf{F}(G)$, form a Cartesian product graph $C_3 \times C_3$, which contains a cycle of length 9. By Lemma 2.8, $\mathbf{F}^2(G)$ contains K_9 . ■

We now show that repeated application of the forest graph operator to many graphs creates larger and larger complete subgraphs.

Lemma 2.11. *If G has a cycle of (finite) length n with $n \geq 4$ or it has two edge disjoint triangles, then for any finite $m \geq 1$, $\mathbf{F}^m(G)$ contains K_{m^2} .*

Proof. We prove this lemma by induction on m .

Case 1. Suppose that G has a cycle C_n of length n ($n \geq 4$, n finite). By Lemma 2.8, $\mathbf{F}(G)$ contains K_n as a subgraph, which implies that $\mathbf{F}(G)$ contains K_4 . By Lemma 2.9, $\mathbf{F}^3(G)$ contains K_{16} and in particular it contains K_3 .

Case 2. Suppose that G has two edge disjoint triangles. By Lemma 2.10 $\mathbf{F}^2(G)$ contains K_9 as a subgraph. It follows by Lemma 2.7 that $\mathbf{F}^3(G)$ contains $K_{\lfloor 9^2/4 \rfloor} = K_{20}$ as a subgraph. This implies that $\mathbf{F}^3(G)$ contains K_{3^2} as a subgraph.

By Cases 1 and 2 it follows that the result is true for $m = 1, 2, 3$. Let us assume that the result is true for $m = l \geq 3$, i.e., that $\mathbf{F}^l(G)$ contains K_{l^2} as a subgraph. By Lemma 2.7 it follows that $\mathbf{F}(\mathbf{F}^l(G))$ has a subgraph $K_{\lfloor l^4/4 \rfloor}$. Since $\lfloor l^4/4 \rfloor > (l + 1)^2$, it follows that $\mathbf{F}^{l+1}(G)$ contains $K_{(l+1)^2}$. By the induction hypothesis $\mathbf{F}^m(G)$ contains K_{m^2} for any finite $m \geq 1$. ■

With Lemma 2.9 it is clearly possible to prove a much stronger lower bound on complete subgraphs of iterated forest graphs, but Lemma 2.11 is good enough for our purposes.

Lemma 2.12. *A forest graph that is not K_1 has no isolated vertices and no isthmi.*

Proof. Let $G = \mathbf{F}(H)$ for some graph H . Consider a vertex F of G , that is, a maximal forest in H . Let e be an edge of F that belongs to a cycle C in H . Then there is an edge f in C that is not in F and $F' = F - e + f$ is a second maximal forest that is adjacent to F in G . Since C has length at least 3, it has a third edge g . If g is not in F , let $F'' = F - e + g$. If g is in F , let $F'' = F - g + f$. In both cases F'' is a maximal forest that is adjacent to F and F' . Thus, F is not isolated and the edge FF' in G is not an isthmus.

Suppose $F, F' \in \mathfrak{N}(H)$ are adjacent in G . That means there are edges $e \in E(F)$ and $e' \in E(F')$ such that $F' = F - e + e'$. Thus, e belongs to the unique cycle in $F + e'$. As shown above, there is an $F'' \in \mathfrak{N}(H)$ that forms a cycle with F and F' . Therefore the edge FF' of G is not an isthmus.

Let $F \in \mathfrak{N}(H)$ be an isolated vertex in G . If H has an edge e not in F , then $F + e$ contains a cycle so F has a neighboring vertex in G , as shown above. Therefore, no such e can exist; in other words, $H = F$ and G is K_1 . ■

3. BASIC PROPERTIES OF AN INFINITE FOREST GRAPH

We now present a crucial foundation for the proof of the main theorem in Section 5. The *cyclomatic number* $\beta_1(G)$ of a graph G can be defined as the cardinality $|E(G) \setminus E(F)|$ where F is a maximal forest of G .

Proposition 3.1. *Let G be a graph such that $|\mathfrak{C}(G)| = \beta$, an infinite cardinal number. Then*

- (i) $\beta_1(G) = \beta$ and $\beta_1(\mathbf{F}(G)) = 2^\beta$.

- (ii) Both the order of $\mathbf{F}(G)$ and its number of edges equal 2^β . Both the order and the number of edges of G equal β , provided that G has no isolated vertices and no isthmi.
- (iii) $\mathbf{F}(G)$ is β -regular.
- (iv) The order of any connected component of $\mathbf{F}(G)$ is β , and it has exactly β edges.
- (v) $\mathbf{F}(G)$ has exactly 2^β components.
- (vi) Every component of $\mathbf{F}(G)$ has exactly β cycles.
- (vii) $|\mathfrak{C}(\mathbf{F}(G))| = 2^\beta$.

Proof. Let G be a graph with $|\mathfrak{C}(G)| = \beta$ (β infinite).

(i) Let F be a maximal forest of G . The number of cycles in G is not more than the number of finite subsets of $E(G) \setminus E(F)$. This number is finite if $E(G) \setminus E(F)$ is finite, but it cannot be finite because $|\mathfrak{C}(G)|$ is infinite. Therefore $E(G) \setminus E(F)$ is infinite and the number of its finite subsets equals $|E(G) \setminus E(F)| = \beta_1(G)$. Thus, $\beta_1(G) \geq |\mathfrak{C}(G)|$. The number of cycles is at least as large as the number of edges not in F , because every such edge makes a different cycle with F . Thus, $|\mathfrak{C}(G)| \geq \beta_1(G)$. It follows that $\beta_1(G) = |\mathfrak{C}(G)| = \beta$. Note that this proves $\beta_1(G)$ does not depend on the choice of F .

The value of $\beta_1(\mathbf{F}(G))$ follows from this and part (vii).

(ii) For the first part, let F be a maximal forest of G and let F_0 be a maximal forest of $G \setminus E(F)$. As $G \setminus E(F)$ has $\beta_1(G) = \beta$ edges by part (i), it has β non-isolated vertices by Lemma 2.6. F_0 has the same non-isolated vertices, so it too has β edges.

Any edge set $A \subseteq F_0$ extends to a maximal forest F_A in $F \cup A$. Since $F_A \setminus F = A$, the F_A 's are distinct. Therefore, there are at least 2^β maximal forests in $F_0 \cup F$. The maximal forest F consists of a spanning tree in each component of G ; therefore, the vertex sets of components of F are the same as those of G , and so are those of $F_0 \cup F$. Therefore, a maximal forest in $F_0 \cup F$, which consists of a spanning tree in each component of $F_0 \cup F$, contains a spanning tree of each component of G .

We conclude that a maximal forest in $F_0 \cup F$ is a maximal forest of G and hence that there are at least 2^β maximal forests in G , i.e., $|\mathfrak{N}(G)| \geq 2^\beta$. Since G is a subgraph of K_β , and since $|\mathfrak{N}(K_\beta)| = 2^\beta$ by Lemma 2.2, we have $|\mathfrak{N}(G)| \leq 2^\beta$. Therefore $|\mathfrak{N}(G)| = 2^\beta$. That is, the order of $\mathbf{F}(G)$ is 2^β . By Lemmas 2.12 and 2.6, that is also the number of edges of $\mathbf{F}(G)$.

For the second part, note that G has infinite order or else $\beta_1(G)$ would be finite. If G has no isolated vertices and no isthmi, then $|V(G)| = |E(G)|$ by Lemma 2.6. By part (i) there are β edges of G outside a maximal forest; hence $\beta \leq |E(G)|$.

Since every edge of G is in a cycle, by the axiom of choice we can choose a

cycle $C(e)$ containing e for each edge e of G . Let $\mathfrak{C} = \{C(e) : e \in E(G)\}$. The total number of pairs (f, C) such that $f \in C \in \mathfrak{C}$ is no more than $\aleph_0 \cdot |\mathfrak{C}| \leq \aleph_0 \cdot |\mathfrak{C}(G)| = \aleph_0 \cdot \beta = \beta$. This number of pairs is not less than the number of edges, so $|E(G)| \leq \beta$. It follows that G has exactly β edges.

(iii) Let F be a maximal forest of G . By part (i), $|E(G) \setminus E(F)| = \beta$. By adding any edge e from $E(G) \setminus E(F)$ to F we get a cycle C . Removing any edge other than e from the cycle C gives a new maximal forest which differs by exactly one edge with F . The number of maximal forests we get in this way is $\beta_1(G)$ because there are $\beta_1(G)$ ways to choose e and a finite number of edges of C to choose to remove, and $\beta_1(G)$ is infinite. Thus we get β maximal forests of G , each of which differs by exactly one edge with F . Every such maximal forest is generated by this construction. Therefore, the degree of any vertex in $\mathbf{F}(G)$ is β .

(iv) Let A be a connected component of $\mathbf{F}(G)$. As $\mathbf{F}(G)$ is β -regular by part (iii), it follows that $|V(A)| \geq \beta$. Fix a vertex v in A and define the n^{th} neighborhood $D_n = \{v' : d(v, v') = n\}$ for each n in \mathbb{N} . Since every vertex has degree β , $|D_0| = 1$, $|D_1| = \beta$ and $|D_k| \leq \beta|D_{k-1}|$. Thus, by induction on n , $|D_n| \leq \beta$ for $n > 0$.

Since A is connected, it follows that $V(A) = \bigcup_{i \in \mathbb{N} \cup \{0\}} D_i$, i.e., $V(A)$ is the countable union of sets of order β . Therefore $|V(A)| = \beta$, as $|\mathbb{N}| \cdot \beta = \beta$. Hence any connected component of $\mathbf{F}(G)$ has β vertices. By Lemma 2.6 it has β edges.

(v) By parts (ii), (iv) the order of $\mathbf{F}(G)$ is 2^β and the order of each component of $\mathbf{F}(G)$ is β . Since $|\mathbf{F}(G)| = 2^\beta$, $\mathbf{F}(G)$ has at most 2^β components. Suppose that $\mathbf{F}(G)$ has β' components where $\beta' < 2^\beta$. As each component has β vertices, it follows that $\mathbf{F}(G)$ has order at most $\beta' \cdot \beta = \max\{\beta', \beta\}$. This is a contradiction to part (ii). Therefore $\mathbf{F}(G)$ has exactly 2^β components.

(vi) Let A be a component of $\mathbf{F}(G)$. Since it is infinite, by part (iv) it has exactly β edges. Suppose that $|\mathfrak{C}(A)| = \beta'$. Then β' is at most the number of finite subsets of $E(A)$, which is β since $|E(A)| = \beta$ is infinite; that is, $\beta' \leq \beta$. By the argument in part (iii) every edge of $\mathbf{F}(G)$ lies on a cycle. The length of each cycle is finite. Thus A has at most $\aleph_0 \cdot \beta' = \max\{\beta', \aleph_0\} = \beta'$ edges if β' is infinite and it has a finite number of edges if β' is finite. Since $|E(A)| = \beta$, which is infinite, $\beta' \geq \beta$. We conclude that $\beta' = \beta$.

(vii) By parts (v), (vi) $\mathbf{F}(G)$ has 2^β components and each component has β cycles. Since every cycle is contained in a component, $|\mathfrak{C}(\mathbf{F}(G))| = \beta \cdot 2^\beta = 2^\beta$. ■

From the above proposition it follows that an infinite graph cannot be a forest graph unless every component has the same infinite order β and there are 2^β components. A consequence is that the infinite graph itself must have order 2^β . Hence,

Corollary 3.2. *Any infinite graph whose order is not a power of 2, including \aleph_0 and all other limit cardinals, is not a forest graph.*

Corollary 3.3. *For a graph G the following statements are equivalent.*

- (i) $\mathbf{F}(G)$ is connected.
- (ii) $\mathbf{F}(G)$ is finite.
- (iii) The union of all cycles in G is a finite graph.

Proof. (i) \implies (iii). Suppose that $\mathbf{F}(G)$ is connected. If G has infinitely many cycles then by Proposition 3.1(v) $\mathbf{F}(G)$ is disconnected. Therefore G has finitely many cycles. Let $A = \{e \in E(G) : \text{edge } e \text{ lies on a cycle in } G\}$. Then $|A|$ is finite because the length of each cycle is finite. That proves (iii).

(iii) \implies (ii). As every maximal forest of G consists of a maximal forest of A and all the edges of G which are not in A , G has at most 2^n maximal forests where $n = |A|$. Hence $\mathbf{F}(G)$ has a finite number of vertices and consequently is finite.

(ii) \implies (i). By identifying vertices in different components (Whitney vertex identification; see Section 4) we can assume G is connected so $\mathbf{F}(G) = \mathbf{T}(G)$. Cummins [2] proved that the tree graph of a finite graph is Hamiltonian; therefore it is connected. ■

4. \mathbf{F} -ROOTS

In this section we establish properties of \mathbf{F} -roots of graphs. We begin with the question of what an \mathbf{F} -root should be.

Since any graph H' that is isomorphic to an \mathbf{F} -root H of G is immediately also an \mathbf{F} -root, the number of non-isomorphic \mathbf{F} -roots is a better question than the number of labeled \mathbf{F} -roots. We now show in some detail that a still better question is the number of non-isomorphic \mathbf{F} -roots without isthmi.

Let t_β be the number of non-isomorphic rooted trees of order β . We note that $t_{\aleph_0} \geq 2^{\aleph_0}$, by a construction of Reinhard Diestel (personal communication, July 10, 2015). (We do not know a corresponding lower bound on t_β for $\beta > \aleph_0$.) Let P be a one-way infinite path whose vertices are labelled by natural numbers, with root 1; choose any subset S of \mathbb{N} and attach two edges at every vertex in S , forming a rooted tree T_S (rooted at 1). Then S is determined by T_S because the vertices in S are those of degree at least 3 in T_S . (If $2 \in S$ but $1 \notin S$, then vertex 1 is determined only up to isomorphism by T_S , but S itself is determined uniquely.) The number of sets S is 2^{\aleph_0} , hence $t_{\aleph_0} \geq 2^{\aleph_0}$.

Proposition 4.1. *Let G be a graph with an \mathbf{F} -root of order α . If α is finite, then G has infinitely many non-isomorphic finite \mathbf{F} -roots. If α is finite or infinite, then G has at least t_β non-isomorphic \mathbf{F} -roots of order β for every infinite $\beta \geq \alpha$.*

Proof. Let G be a graph which has an \mathbf{F} -root H , i.e., $\mathbf{F}(H) \cong G$, and let α be the order of H . We may assume H has no isthmi and no isolated vertices unless it is K_1 .

Suppose α is finite; then let T be a tree, disjoint from H , of any finite order n . Identify any vertex v of H with any vertex w of T . The resulting graph H_T also has G as its forest graph since T is contained in every maximal forest of H_T . As the order of H_T is $\alpha + n - 1$ and n can be any natural number, the graphs H_T are an infinite number of non-isomorphic finite graphs with the same forest graph up to isomorphism.

Suppose α is finite or infinite and $\beta \geq \alpha$ is infinite. Let T be a rooted tree of order β with root vertex w ; for instance, T can be a star rooted at the star center. Attach T to a vertex v of H by identifying v with the root vertex w . Denote the resulting graph by H_T ; it is an \mathbf{F} -root of G and it has order β because it has order $\alpha + \beta$, which equals β because β is infinite and $\beta \geq \alpha$. As H has no isthmi, T and w are determined by H_T ; therefore, if we have a non-isomorphic rooted tree T' with root w' (that means there is no isomorphism of T with T' in which w corresponds to w'), $H_{T'}$ is not isomorphic to H_T . (The one exception is when $H = K_1$, which is easy to treat separately.) The number of non-isomorphic \mathbf{F} -roots of G of order β is therefore at least the number of non-isomorphic rooted trees of order β , i.e., t_β . ■

Proposition 4.1 still does not capture the essence of the number of \mathbf{F} -roots. Whitney's 2-operations on a graph G are the following [17]:

- (1) *Whitney vertex identification.* Identify a vertex in one component of G with a vertex in another component of G , thereby reducing the number of components by 1. For an infinite graph we modify this by allowing an infinite number of vertex identifications; specifically, let W be a set of vertices with at most one from each component of G , and let $\{W_i : i \in I\}$ be a partition of W into $|I|$ sets (where I is any index set); then for each $i \in I$ we identify all the vertices in W_i with each other.
- (2) *Whitney vertex splitting.* The reverse of vertex identification.
- (3) *Whitney twist.* If u, v are two vertices that separate G (that is, $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{u, v\}$ and $|V(G_1)|, |V(G_2)| > 2$), then reverse the names u and v in G_2 and then take the union $G_1 \cup G_2$ (so vertex u in G_1 is identified with the former vertex v in G_2 and v with the former vertex u). Call the new graph G' . For an infinite graph we allow an infinite number of Whitney twists.

It is easy to see that the edge sets of maximal forests in G and G' are identical, hence $\mathbf{F}(G)$ and $\mathbf{F}(G')$ are naturally isomorphic. It follows by Whitney vertex identification that every graph with an \mathbf{F} -root has a connected \mathbf{F} -root, and it follows from Whitney vertex splitting that every graph with an F -root has an \mathbf{F} -root without cut vertices.

We may conclude from Proposition 4.1 that the most interesting question about the number of \mathbf{F} -roots of a graph G that has an \mathbf{F} -root is not the total number of non-isomorphic \mathbf{F} -roots (which by Proposition 4.1 cannot be assigned

any cardinality); it is not the number of a given order; it is not even the number that have no isthmi; it is the number of non-2-isomorphic, connected \mathbf{F} -roots with no isthmi and (except when $G = K_1$) no isolated vertices.

We do not know which graphs have \mathbf{F} -roots, but we do know two large classes that cannot have \mathbf{F} -roots.

Theorem 4.2. *No infinite connected graph has an \mathbf{F} -root.*

Proof. This follows by Corollary 3.3. ■

Theorem 4.3. *No bipartite graph G has an \mathbf{F} -root.*

Proof. Let G be a bipartite graph of order p ($p \geq 2$) and let H be a root of G , i.e., $\mathbf{F}(H) \cong G$. Suppose H has no cycle; then $\mathbf{F}(H)$ is K_1 , which is a contradiction. Therefore H has a cycle of length ≥ 3 . It follows by Lemma 2.8 that $\mathbf{F}(H)$ contains K_3 , a contradiction. Hence no bipartite graph G has a root. ■

5. \mathbf{F} -CONVERGENCE AND \mathbf{F} -DIVERGENCE

In this section we establish the necessary and sufficient conditions for \mathbf{F} -convergence of a graph.

Lemma 5.1. *Let G be a finite graph that contains a C_n (for $n \geq 4$) or at least two edge disjoint triangles; then G is \mathbf{F} -divergent.*

Proof. Let G be a finite graph. By Lemma 2.11, $\mathbf{F}^m(G)$ contains K_{m^2} as a subgraph. Therefore, as m increases the clique size of $\mathbf{F}^m(G)$ increases. Hence G is \mathbf{F} -divergent. ■

Lemma 5.2. *If $|\mathfrak{C}(G)| = \beta$ where β is infinite, then G is \mathbf{F} -divergent.*

Proof. Assume $|\mathfrak{C}(G)| = \beta$ (β infinite). By Proposition 3.1(vii), as $2^\beta < 2^{2^\beta} < 2^{2^{2^\beta}} < \dots$, it follows that $|\mathfrak{C}(\mathbf{F}(G))| < |\mathfrak{C}(\mathbf{F}^2(G))| < |\mathfrak{C}(\mathbf{F}^3(G))| < \dots$. Therefore, as n increases $|\mathfrak{C}(\mathbf{F}^n(G))|$ increases. Hence G is \mathbf{F} -divergent. ■

Theorem 5.3. *Let G be a graph. Then,*

- (i) *G is \mathbf{F} -convergent if and only if either G is acyclic or G has only one cycle, which is of length 3.*
- (ii) *If G is \mathbf{F} -convergent, then it converges in at most two steps.*

Proof. (i) If G has no cycle, then it is a forest and $\mathbf{F}(G)$ is K_1 . If G has only one cycle and that cycle has length 3, then $\mathbf{F}(G)$ is K_3 . Therefore in each case G is \mathbf{F} -convergent.

Conversely, suppose that G has a cycle of length greater than 3 or has at least two triangles. If G has infinitely many cycles, then it follows by Lemma 5.2 that G is \mathbf{F} -divergent. Therefore we may assume that G has a finite number of cycles. If G has a finite number of vertices, then it is finite and by Lemma 5.1 it is \mathbf{F} -divergent. Therefore G has an infinite number of vertices. However, it can have only a finite number of edges that are not isthmi, because each cycle is finite. Thus G consists of a finite graph G_0 and any number of isthmi and isolated vertices. Since $\mathbf{F}(G)$ depends only on the edges that are not isthmi and the vertices that are not isolated, $\mathbf{F}(G) = \mathbf{F}(G_0)$ (under the natural identification of maximal forests in G_0 with their extensions in G by adding all isthmi of G). Therefore, G is \mathbf{F} -divergent.

(ii) If G has no cycle, then G is a forest and $\mathbf{F}(G) \cong \mathbf{F}^2(G) \cong K_1$. If G has only one cycle, which is of length 3, then $\mathbf{F}(G) \cong \mathbf{F}^2(G) \cong K_3$. Therefore G converges in at most 2 steps. ■

Corollary 5.4. *A graph G is \mathbf{F} -stable if and only if $G = K_1$ or K_3 .*

6. \mathbf{F} -DEPTH

In this section we establish results about the \mathbf{F} -depth of a graph.

Theorem 6.1. *Let G be a finite graph. The \mathbf{F} -depth of G is infinite if and only if G is K_1 or K_3 .*

Proof. Let G be a finite graph. Suppose that G is K_1 or K_3 . Then by Corollary 5.4, it follows that G is \mathbf{F} -stable. Therefore, the \mathbf{F} -depth of G is infinite.

Conversely, suppose that G is different from K_1 and K_3 .

Case 1. Let $|V| < 4$. Then G has no \mathbf{F} -root so its \mathbf{F} -depth is zero.

Case 2. Let $|V| = 4$. Suppose G has an \mathbf{F} -root H (i.e., $\mathbf{F}(H) \cong G$). Then H should have exactly 4 maximal forests. That is possible only when H has only one cycle, which is of length 4. By Lemma 2.8 it follows that $\mathbf{F}(H)$ contains K_4 , hence it is K_4 . Therefore G has an \mathbf{F} -root if and only if it is K_4 . Hence the \mathbf{F} -depth of G is zero, except that the depth of K_4 is 1.

Case 3. Let $|V| = n$ where $n > 4$. Suppose that G has infinite \mathbf{F} -depth. Then for every m there is a graph H_m such that $\mathbf{F}^m(H_m) = G$. If H_m does not have two triangles or a cycle of length greater than 3, then H_m has only one cycle which is of length 3, or no cycle and H_m converges to K_1 or K_3 in at most two steps, a contradiction. Therefore H_m has two triangles or a cycle of length greater than 3. By Lemma 2.11 it follows that $\mathbf{F}^m(H_m)$ contains K_{m^2} for each $m \geq 2$, so that in particular $\mathbf{F}^n(H_n)$ contains K_{n^2} . That is, G contains K_{n^2} . This is impossible as G has order n . Hence the \mathbf{F} -depth of G is finite. ■

Theorem 6.2. *The \mathbf{F} -depth of any infinite graph is finite.*

Proof. Let G be a graph of infinite order α . If G has an \mathbf{F} -root, then G is without isthmi or isolated vertices.

If G is connected, Theorem 4.2 implies that G has no root. Therefore its \mathbf{F} -depth is zero.

If G is disconnected, assume it has infinite depth. Then for each natural number n there exists a graph H_n such that $G \cong \mathbf{F}^n(H_n)$. Let β_n denote the order of H_n . Since $\mathbf{F}(H_1) \cong G$, by Proposition 3.1(ii) $\alpha = 2^{\beta_1}$, from which we infer that $\beta_1 < \alpha$. This is independent of which root H_1 is, so in particular we can take $H_1 = \mathbf{F}(H_2)$ and conclude that $\beta_1 = 2^{\beta_2}$, hence that $\beta_2 < \beta_1$. Continuing in like manner we get an infinite decreasing sequence of cardinal numbers starting with α . The cardinal numbers are well ordered [9], so they cannot contain such an infinite sequence. It follows that the \mathbf{F} -depth of G must be finite. ■

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