ON THE EDGE-HYPER-HAMILTONIAN LACEABILITY OF BALANCED HYPERCUBES

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Abstract

The balanced hypercube $BH_n$, defined by Wu and Huang, is a variant of the hypercube network $Q_n$, and has been proved to have better properties than $Q_n$ with the same number of links and processors. For a bipartite graph $G = (V_0 \cup V_1, E)$, we say $G$ is edge-hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, there is a Hamiltonian path containing $e$ in $G - v$ between any two vertices of $V_{1-i}$. In this paper, we prove that $BH_n$ is edge-hyper-Hamiltonian laceable.

Keywords: balanced hypercubes, hyper-Hamiltonian laceability, edge-hyper-Hamiltonian laceability.

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1. Introduction

An interconnection network is usually represented by an undirected graph, where the vertices represent the processors and the edges represent the communication links between processors. Let $G = (V, E)$ be a simple undirected graph with

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vertex set $V$ and edge set $E$. A graph $G = (V, E)$ is bipartite if $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$ such that the two ends of any edge come from different set. For $e = (u, v)$ (or alternatively $e = uv$), $u$ (resp. $v$) is said to be incident with $e$, and $e$ is said to be incident with $u$ and $v$. A path $P_{[v_0, v_m]} = \langle v_0, v_1, \ldots, v_m \rangle$ is a sequence of distinct vertices from $v_0$ to $v_m$ such that two consecutive vertices are adjacent. A Hamiltonian path (resp. Hamiltonian cycle) of $G$ is a path (resp. cycle) that traverses each vertex of $G$ exactly once. For $x, y \in V$, a Hamiltonian path between $x$ and $y$ in $G$ is called an $(x, y)$-Hamiltonian path. A graph $G$ is Hamiltonian if it has a Hamiltonian cycle. In a Hamiltonian bipartite graph $G$, there exists no Hamiltonian path between two vertices in the same partite set. Simmons [9] introduced the notation of Hamiltonian laceability of a bipartite graph. A bipartite graph $G = (V_0 \cup V_1, E)$ is Hamiltonian laceable if there is a Hamiltonian path between any two vertices $x$ and $y$ with $x \in V_0$ and $y \in V_1$. Hsieh et al. [3] extended this concept to strongly Hamiltonian laceable. A Hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is strongly Hamiltonian laceable if there is a path of length $|V_0 \cup V_1| - 2$ between any two distinct vertices of the same partite set. Lewinter and Widulski [6] further proposed the concept of hyper-Hamiltonian laceability. A graph $G$ is hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in $V_{1-i}$. A graph $G$ is edge-hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, there is a Hamiltonian path containing $e$ in $G - v$ between any pair of vertices in $V_{1-i}$. $G$ is a vertex transitive graph (resp. edge transitive graph), if for any two vertices $x$ and $y$ (resp. edges $e_1$ and $e_2$) of $G$, there is an automorphism $T$ of $G$ such that $T(x) = y$ (resp., $T(e_1) = e_2$). Some other definitions and notations not given in this paper are referred to [1, 10, 12].

Interconnection networks play an important role in parallel and distributed systems. The hypercube network has proved to be one of the most popular interconnection networks. The balanced hypercube $BH_n$, proposed by Huang and Wu [5], is a variant of the hypercube. Like hypercubes, the balanced hypercubes are bipartite, vertex-transitive and edge transitive [4, 11, 15]. The balanced hypercubes are superior to the hypercube in having smaller diameter, supporting an efficient reconfiguration without changing the adjacent relationship among tasks [11]. Plenty of properties of balanced hypercubes have been studied extensively [2, 4, 5, 8, 14]. Xu et al. [13] showed that the balanced hypercube is edge-bipancyclic and Hamiltonian laceable. Lv and Zhang [7] obtained that $BH_n$ is hyper-Hamiltonian laceable. This means that $BH_n$ is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in $V_{1-i}$. So it is natural to propose the following problem.

For any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, does there exist a Hamiltonian path containing $e$ in $G - v$ between any pair of vertices in $V_{1-i}$?
This is the main motivation of this paper, and our answer is yes.

This paper is organized as follows. Section 2 introduces some definitions of balanced hypercubes and their basic properties. The proof of our main result is presented in Section 3. In Section 4, we draw a conclusion of this paper.

2. Preliminaries

In the following ′+′ is an operation with modular 4.

**Definition** [5]. An $n$-dimensional balanced hypercube, denoted by $BH_n$, is defined as follows. For $n \geq 1$, $BH_n$ has $4^n$ vertices with addresses $(a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for each $0 \leq i \leq n - 1$. For $1 \leq i \leq n - 1$, an arbitrary vertex $(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ in $BH_n$ has the following $2^n$ neighbors:

$$(a_0 \pm 1, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}),$$

and

$$(a_0 \pm 1, a_1, \ldots, a_{i-1}, a_i + (-1)^a, a_{i+1}, \ldots, a_{n-1}).$$

$BH_1$ is a cycle of length 4 (see Figure 1), since $BH_n$ is vertex transitive and edge transitive, there are some automorphisms of $BH_n$, Figure 2 shows four graphs that are isomorphic to $BH_2$.

![Figure 1. $BH_1$.](image)

![Figure 2. Four automorphisms of $BH_2$.](image)

The balanced hypercube $BH_n$ can be recursively defined.

**Definition.** $BH_n$ is constructed recursively as follows:

1. $BH_1$ is a cycle with 4 vertices labeled as 0, 1, 2, 3, respectively.
(2) For \( n \geq 2 \), \( BH_n \) consists of four copies of \( BH_{n-1} \), denoted by \( BH_i^{n-1} \), for each integer \( i \) with \( 0 \leq i \leq 3 \). The vertex in \( BH_i^{n-1} \) corresponding to a vertex \( (a_0, a_1, \ldots, a_{n-2}, i) \) in \( BH_{n-1} \) is denoted by \( (a_0, a_1, \ldots, a_{n-2}, i) \), where \( a_j \in \{0, 1, 2, 3\} \) for every \( 0 \leq j \leq n - 2 \). Each vertex \( (a_0, a_1, \ldots, a_{n-2}, i) \) of \( BH_i^{n-1} \) has the following two extra neighbors:

\[
(a_0 \pm 1, a_1, \ldots, a_{n-2}, i + 1) \text{ in } BH_i^{n-1+1} \text{ if } a_0 \text{ is even.}
\]

\[
(a_0 \pm 1, a_1, \ldots, a_{n-2}, i - 1) \text{ in } BH_i^{n-1-1} \text{ if } a_0 \text{ is odd.}
\]

The first coordinate \( a_0 \) of vertex \( (a_0, a_1, \ldots, a_{n-2}, a_{n-1}) \) is named inner index, and the other coordinates \( a_j \) (\( 1 \leq j \leq n - 1 \)) are named \( j \)-dimension index. In [11], it is seen that the balanced hypercube is bipartite. The vertex sets

\[
V_0 = \{ x = (a_0, a_1, \ldots, a_{n-1}) | x \in V(BH_n), a_0 \text{ is odd} \}
\]

and

\[
V_1 = \{ x = (a_0, a_1, \ldots, a_{n-1}) | x \in V(BH_n), a_0 \text{ is even} \}
\]

give the desired partition.

In the following, we use black vertices to denote the vertices in \( V_0 \) and white vertices to denote the vertices in \( V_1 \). Now we classify the edges of \( BH_n \). If two adjacent vertices \( u, v \) differ only in the inner index, the edge \((u, v)\) is said to be a 0-dimension edge and \( v \) is a 0-dimension neighbor of \( u \). If two adjacent vertices \( u, v \) not only differ in the inner index, but also differ in some \( i \)-dimension index \((1 \leq i \leq n - 1)\), then the edge \((u, v)\) is said to be an \( i \)-dimension edge and \( v \) is an \( i \)-dimension neighbor of \( u \). Let \( E_i \) denote the set of all edges of \( i \)-dimension edges for \( i \in \{0, 1, 2, \ldots, n - 1\} \). Then \( E(BH_n) = \bigcup_{i=0}^{n-1} E_i \). For \( i \in \{0, 1, 2, 3\} \) and \( 1 \leq j \leq n - 1 \), we use \( BH_{n-1}^{j,i} \) to denote the \((n-1)\)-dimension sub-balanced hypercubes of the \( BH_n \) induced by all vertices labeled by \((a_0, a_1, \ldots, a_{j-1}, i, a_{j+1}, \ldots, a_{n-1})\). Obviously, \( BH_n - E_j = \bigcup_{i=0}^{3} BH_{n-1}^{j,i} \) and \( BH_{n-1}^{j,i} \cong BH_{n-1} \). If \( j = n - 1 \), \( BH_{n-1}^{j,i} \) and \( E_j \) are denoted by \( BH_{n-1}^{j,i} \) and \( E_{c, i} \), respectively, where \( i \in \{0, 1, 2, 3\} \).

One has \( BH_n - E_c = \bigcup_{i=0}^{3} BH_{n-1}^{j,i} \).

In the rest of this paper we often use \( w_{ij} \) and \( b_{ij} \) to denote white and black vertices in \( BH_n \), respectively, where \( i \in N \) and \( j \in \{0, 1, 2, 3\} \), and \( j \) means that the corresponding vertex lies in \( BH_i^{j,i} \). By definition, any white vertex in \( BH_{n-1}^{j,i} \) has two black \((n - 1)\)-dimension neighbors in \( BH_{n-1}^{j+1,1} \).

Some basic properties of \( BH_n \) are given below and will be used in the sequel.

**Lemma 1** [11]. The balanced hypercube \( BH_n \) is bipartite and vertex transitive.

**Lemma 2** [15]. The balanced hypercube \( BH_n \) is edge transitive.

**Lemma 3** [13]. Let \((u, v)\) be an edge of \( BH_n \). Then \((u, v)\) is contained in a cycle \( C \) of length 8 in \( BH_n \) such that \( |E(C) \cap E(BH_{n-1}^{j,i})| = 1 \), where \( i \in \{0, 1, 2, 3\} \).
Lemma 4 [11]. The vertices \((a_0, a_1, \ldots, a_{n-1})\) and \((a_0 + 2, a_1, \ldots, a_{n-1})\) of \(BH_n\) have the same neighborhood.

Lemma 5 [13]. The balanced hypercube \(BH_n\) is Hamiltonian laceable for \(n \geq 1\).

Lemma 6 [7]. The balanced hypercube \(BH_n\) is hyper-Hamiltonian laceable for \(n \geq 1\).

3. Edge-Hyper-Hamiltonian Laceability of Balanced Hypercube

Lemma 7. The balanced hypercube \(BH_2\) is edge-hyper-Hamiltonian laceable.

Proof. Let \(u\) be any vertex of \(BH_2\). From Lemma 1, \(BH_2\) is vertex transitive, therefore we may suppose \(u = (0, 0)\) without loss of generality. For any \(e \in BH_2 - u\) and any two black vertices \(x\) and \(y\), we can prove there exists an \((x, y)\)-Hamiltonian path containing \(e\) in \(BH_2 - u\). Here we only prove the result for \(e = (1, 0)(2, 0)\) without loss of generality. By the relative positions of \(e, x\) and \(y\) in \(BH_2\), we distinguish into the following two cases.

Case 1. \(x\) and \(y\) are in the same \(BH_1^1\), \(i \in \{0, 1, 2, 3\}\). Since there are only two black vertices in \(BH_1^1\), \(i \in \{0, 1, 2, 3\}\), so if \(x\) and \(y\) are in the same \(BH_1^1\), then one is \(x\), the other is \(y\). Let \(x, y \in BH_1^1\), \(x = (1, 0), y = (3, 0)\). Then \(\langle(1, 0), (2, 0), (1, 1), (0, 1), (3, 1), (2, 1), (1, 2), (2, 2), (3, 2), (0, 2), (3, 3), (2, 3), (1, 3), (0, 3), (3, 0)\rangle\) is an \((x, y)\)-Hamiltonian path containing \(e\) in \(BH_2 - u\).

Case 2. \(x\) and \(y\) are not in the same \(BH_1^1\), \(i \in \{0, 1, 2, 3\}\). Let \(x \in BH_1^0\), \(y \in BH_1^1\). Suppose \(x = (1, 0), y = (1, 1)\). Then \(\langle(1, 0), (2, 0), (3, 1), (0, 1), (1, 2), (2, 2), (3, 3), (2, 3), (1, 3), (0, 2), (3, 2), (2, 1), (1, 1)\rangle\) is an \((x, y)\)-Hamiltonian path containing \(e\) in \(BH_2 - u\).

For other cases, we can similarly find an \((x, y)\)-Hamiltonian path containing \(e\) in \(BH_2 - u\), their proofs are omitted here.

Theorem 8. The balanced hypercube \(BH_n\) is edge-hyper-Hamiltonian laceable for \(n \geq 2\).

Proof. We prove the result by induction on \(n \geq 2\). The case \(n = 2\) follows from Lemma 7. Next we consider \(n \geq 3\). Suppose that the theorem is true for \(BH_k\), \(3 \leq k \leq n - 1\). We suppose that \(u \in V_0\) is a white vertex. For any edge \(e \in BH_n - u\), then \(e \in E(BH_{n-1}^i), i \in \{0, 1, 2, 3\}\), or \(e\) is an \((n-1)\)-dimension edge. If \(e\) is an \((n-1)\)-dimension edge, since \(BH_n\) is vertex and edge transitive, there is an automorphism of \(BH_n\) so that \(e \in E(BH_{n-1}^i), i \in \{0, 1, 2, 3\}\), therefore in the following we only consider \(e \in BH_{n-1}^i, i \in \{0, 1, 2, 3\}\). For any pair of vertices in \(V_1\), say \(x\) and \(y\), they are black vertices. By the relative positions of \(e, x\) and \(y\) in \(BH_n\), we distinguish into the following cases.
Case 1. $e \in E(BH_0^0)$.

Subcase 1.1. $x$ and $y$ are in the same $BH_{n-1}^i$, $i \in \{0, 1, 2, 3\}$.

Subcase 1.1.1. $x, y \in BH_{n-1}^0$ (see Figure 3). By the induction hypothesis, the graph $BH_{n-1}^0 - e$ contains a Hamiltonian path $P_0$ from $x$ to $y$ containing $e$. Since $n \geq 3$, there exists an edge different from $e$, say $w_{00}b_{00} \in E(P_0)$, such that the removal of the edge $w_{00}b_{00}$ decomposes $P_0$ into two sections $P_{00}[x, w_{00}]$ and $P_{10}[b_{00}, y]$. Let $b_{01} \in BH_{n-1}^1$ (resp. $w_{03} \in BH_{n-1}^3$) be $(n-1)$-dimension neighbors of $w_{00}$ (resp. $b_{00}$). By Lemma 3, $w_{00}b_{00}$ is contained in a cycle $C$ of length 8 in $BH_n$ such that $|E(C) \cap E(BH_{n-1}^i)| = 1$, $i \in \{0, 1, 2, 3\}$. We denote the cycle $C = (w_{00}, b_{01}, w_{01}, b_{02}, w_{02}, b_{03}, w_{03}, b_{00}, w_{00})$. By Lemma 5, $BH_n$ is Hamiltonian laceable for $n \geq 1$, so $BH_{n-1}^i$ contains a $(b_{0i}, w_{0i})$-Hamiltonian path $P_i$, $i \in \{1, 2, 3\}$. Then $(x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_3, w_{03}, b_{00}, P_{10}, y)$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.

Subcase 1.1.2. $x, y \in BH_{n-1}^1$ (see Figure 4). Choose an arbitrary white vertex, say $w_{01}$, in $BH_{n-1}^1$. By Lemma 5, there exists an $(x, w_{01})$-Hamiltonian path $P_1$ in $BH_{n-1}^1$. Then there exists an edge, say $w_{11}y \in E(P_1)$, whose removal divides $P_1$ into two sections $P_{11}[x, w_{11}]$ and $P_{01}[y, w_{01}]$. Let $b_{02}$ and $b_{12}$ be $(n-1)$-dimension neighbors of $w_{01}$ and $w_{11}$ in $BH_{n-1}^2$ respectively. Since every white vertex of $BH_{n-1}^1$ has two $(n-1)$-dimension neighbors in $BH_{n-1}^2$, so we can always choose two vertices $b_{02}$ and $b_{12}$ such that $b_{02} \neq b_{12}$. Similarly, choose an arbitrary white vertex, say $w_{02}$, in $BH_{n-1}^2$. By Lemma 5, there exists a $(b_{12}, w_{02})$-Hamiltonian path $P_2$ in $BH_{n-1}^2$ via the edge $b_{12}w_{12}$. We delete $b_{02}w_{12}$, then $P_2$ is divided to two sections $P_{02}[w_{02}, b_{02}]$ and $P_{12}[w_{12}, b_{12}]$. Let $b_{03}$ and $b_{13}$ be $(n-1)$-dimension neighbors of $w_{02}$ and $w_{12}$ in $BH_{n-1}^3$ respectively, and $b_{03} \neq b_{13}$. Choose any white vertex $w_{03}$ in $BH_{n-1}^3$; by Lemma 5, there is a $(b_{13}, w_{03})$-Hamiltonian path $P_3$ in $BH_{n-1}^3$. Therefore there exists an edge, say $b_{03}w_{13}$, in $P_3$ whose deletion divides $P_3$ into two parts $P_{03}[w_{03}, b_{03}]$ and $P_{13}[w_{13}, b_{13}]$. Let $b_{00}$ and $b_{10}$ be $(n-1)$-dimension neighbors of $w_{03}$ and $w_{13}$ in $BH_{n-1}^4$ respectively, and $b_{00} \neq b_{10}$. By the induction hypothesis, there exists a
On the Edge-Hyper-Hamiltonian Laceability of ... 811

(b₀₀, b₁₀)-Hamiltonian path \( P₀ \) containing \( e \) in \( BH^{0}_{n-1} - u \). Then \( \langle x, P₁₁, w₁₁, b₁₂, P₁₂, w₁₂, b₁₃, P₁₃, w₁₃, b₁₀, P₀₀, w₀₀, b₀₀, w₀₁, P₀₁, b₀₁, P₀₃, b₀₃, w₀₂, P₀₂, b₀₂, w₀₁, P₀₁, y \rangle \) is an \( (x, y) \)-Hamiltonian path containing \( e \) in \( BH_n - u \).

Subcase 1.1.3. \( x, y \in V(BH^{2}_{n-1}) \) (see Figure 5). Choose an arbitrary white vertex, say \( w₀₃ \), in \( BH^{3}_{n-1} \). Let \( b₀₀ \) and \( b₀₂ \) be two \((n - 1)\)-dimension neighbors (different from \( u \)) of \( w₀₃ \) in \( BH^{0}_{n-1} \). By the induction hypothesis, for any \( e \in BH^{0}_{n-1} \), the graph \( BH^{0}_{n-1} - u \) contains a \((b₀₀, b₀₂)\)-Hamiltonian path \( P₀ \) containing \( e \). Now we choose two arbitrary white vertices in \( BH^{2}_{n-1} \) with different \((n - 1)\)-dimension neighbors in \( BH^{3}_{n-1} \), say \( w₀₂ \) and \( w₁₂ \). By Lemma 6, there is a \((w₀₂, w₁₂)\)-Hamiltonian path \( P₁ \) in \( BH^{2}_{n-1} - x \). There exists an edge, say \( yw₂₂ \in E(P₂) \), whose deletion divides \( P₂ \) into two sections \( P₁₀[w₁₂, y] \) and \( P₀₂[w₂₂, w₀₂] \).

Let \( b₀₃, b₁₃ \) and \( b₂₃ \) (they are different from each other) be \((n - 1)\)-dimension neighbors of \( w₀₂, w₁₂ \) and \( w₂₂ \) in \( BH^{3}_{n-1} \), respectively. By Lemma 6, there exists a \((b₀₃, b₂₃)\)-Hamiltonian path \( P₃ \) in \( BH^{3}_{n-1} - w₀₃ \). Then there exists an edge \( w₁₃b₁₃ \in E(P₃) \), whose deletion divides \( P₃ \) into two sections \( P₀₃[w₀₃, w₁₃] \) and \( P₁₃[b₁₃, b₂₃] \). Let \( b₁₀ \) be an \((n - 1)\)-dimension neighbor of \( w₁₃ \) in \( BH^{0}_{n-1} \) such that \( b₁₀ \) is not incident with \( e \). Then there exists an edge \( b₁₀w₀₀ \in E(P₀) \), whose deletion divides \( P₀ \) into two sections \( P₀₀[b₀₀, w₀₀] \) and \( P₁₀[b₁₀, b₂₀] \). Let \( b₀₁ \) and \( w₀₁ \) be \((n - 1)\)-dimension neighbors of \( w₀₀ \) and \( x \) in \( BH^{3}_{n-1} \), respectively. By Lemma 5, there exists a \((w₀₁, b₀₁)\)-Hamiltonian path \( P₁ \) in \( BH^{3}_{n-1} - x \). Then \( \langle x, w₀₁, P₁, b₀₁, w₀₀, P₀₀, b₀₀, w₀₃, b₂₀, P₁₀, b₁₀, w₁₃, P₀₃, b₀₃, w₀₂, P₀₂, w₂₂, b₂₃, P₁₃, b₁₃, w₁₂, P₁₂, y \rangle \) is an \((x, y)\)-Hamiltonian path containing \( e \) in \( BH_n - u \).

Subcase 1.1.4. \( x, y \in V(BH^{2}_{n-1}) \) (see Figure 6). Let \( w₀₃ \) and \( w₁₃ \) be any two white vertices in \( BH^{3}_{n-1} \), with different \((n - 1)\)-dimension neighbors in \( BH^{0}_{n-1} \). By the induction hypothesis, for any edge, say \( yw₂₂ \in E(BH^{3}_{n-1} - x) \), there exists a \((w₀₃, w₁₃)\)-Hamiltonian path \( P₃ \) in \( BH^{3}_{n-1} - x \). We choose such an edge \( yw₂₂ \) so that \( b₂₀ \) is not incident with \( e \) and \( b₂₀ \neq b₀₀ \), where \( b₀₀ \) and \( b₂₀ \) are \((n - 1)\)-dimension neighbors of \( w₀₃ \) and \( w₂₂ \) in \( BH^{0}_{n-1} \), respectively. Now \( P₃ = \langle w₀₃, P₀₃[w₀₃, w₂₃], w₂₃, y, P₁₃[y, w₁₃], w₁₃ \rangle \). Let \( b₁₀ \) be an \((n - 1)\)-dimension neigh-
bor of \(w_{13}\) in \(BH_{n-1}^0\). By the induction hypothesis, there exists a \((b_{00}, b_{10})\)-Hamiltonian path \(P_0\) containing \(e\) in \(BH_{n-1}^0 - u\). Then there exists an edge, say \(w_{00}b_{20} \in E(P_3)\), whose deletion divides \(P_3\) into two sections \(P_{00}[b_{00}, w_{00}]\) and \(P_{10}[b_{20}, b_{10}]\). Since \(w_{03}\) and \(w_{13}\) are arbitrary vertices, so we can always choose two such vertices so that \(w_{00}b_{20} \neq e\). Let \(b_{01}\) and \(w_{02}\) be \((n-1)\)-dimension neighbors of \(w_{00}\) and \(x\) respectively. Furthermore, let \(b_{02}\) be any black vertex in \(BH_{n-1}^2\) and \(w_{03}\) be an \((n-1)\)-dimension neighbor of \(b_{02}\) in \(BH_{n-1}^1\). By Lemma 5, there exists an \((b_{03}, w_{03})\)-Hamiltonian path \(P_i\) in \(BH_{n-1}^3\), \(i \in \{1, 2\}\). Then \((x, w_{02}, P_2, b_{02}, w_{03}, P_{03}, b_{03}, b_{00}, w_{00}, P_{00}, b_{00}, b_{10}, w_{13}, P_{13}, y)\) is an \((x, y)\)-Hamiltonian path containing \(e\) in \(BH_{n-1}^0 - u\).

**Subcase 1.2.** \(x \in V(BH_{n-1}^1), y \in V(BH_{n-1}^2), 0 \leq i \leq j \leq 3\).

**Subcase 1.2.1.** \(x \in V(BH_{n-1}^0), y \in V(BH_{n-1}^1)\). We choose any black vertex in \(BH_{n-1}^0\), say \(b_{00}\). By the induction hypothesis, there exists an \((x, b_{00})\)-Hamiltonian path \(P_0\) containing \(e\) in \(BH_{n-1}^0 - u\). Let \(w_{03}\) be an \((n-1)\)-dimension neighbor of \(b_{00}\) in \(BH_{n-1}^3\). By Lemma 5, for any black vertex \(b_{03} \in BH_{n-1}^3\), there exists a \((w_{03}, b_{03})\)-Hamiltonian path \(P_3\) in \(BH_{n-1}^3\). Similarly, for any white vertex \(w_{01} \in BH_{n-1}^1\), there exists a \((y, w_{01})\)-Hamiltonian path \(P_1\) in \(BH_{n-1}^1\). Let \(b_{02}\) and \(w_{02}\) be \((n-1)\)-dimension neighbors of \(w_{01}\) and \(b_{02}\) in \(BH_{n-1}^2\), respectively. By Lemma 5, there exists a \((b_{02}, w_{02})\)-Hamiltonian path \(P_2\) in \(BH_{n-1}^2\). Then \((x, P_0, b_{00}, w_{03}, P_3, b_{03}, w_{02}, P_2, b_{02}, w_{01}, P_1, y)\) is an \((x, y)\)-Hamiltonian path containing \(e\) in \(BH_{n-1}^0 - u\).

![Figure 7. Subcase 1.2.2.](image1)

**Subcase 1.2.2.** \(x \in V(BH_{n-1}^0), y \in V(BH_{n-1}^2)\) (see Figure 7). Choose an arbitrary black vertex, say \(b_{00}\) in \(BH_{n-1}^0\). By the induction hypothesis, the graph \(BH_{n-1}^0 - u\) contains an \((x, b_{00})\)-Hamiltonian path \(P_0\) containing \(e\). Then there exists an edge, say \(w_{00}b_{10} \in E(P_0)\) so that \(w_{00}b_{10} \neq e\) and \(P_0 = (x, P_{00}, w_{00}, P_{01}, b_{10}, P_{10}, b_{00}, P_{00})\). Let \(b_{01} \in BH_{n-1}^1\), \(w_{03} \in BH_{n-1}^3\) and \(w_{13} \in BH_{n-1}^3\) be \((n-1)\)-dimension neighbors of \(w_{00}, b_{00}\) and \(b_{10}\), respectively. Let \(b_{03}\) be an arbitrary black vertex in \(BH_{n-1}^3\). By Lemma 5, there exists a \((b_{03}, w_{13})\)-Hamiltonian path containing \(e\) in \(BH_{n-1}^0 - u\).
path $P_3$. Hence there exists an edge in $P_3$, say $w_{03}b_{13}$, whose deletion divides $P_3$ into two sections $P_{03}[b_{03}, w_{03}]$ and $P_{13}[b_{13}, w_{13}]$. Let $w_{02}$ and $w_{12}$ be $(n - 1)$-dimension neighbors of $b_{03}$ and $b_{13}$ in $BH_{n-1}^2$, respectively, and $w_{02} \neq w_{12}$. Since $y$ is black vertex in $BH_{n-1}^2$, in view of Lemma 5, there exists a $(w_{02}, y)$-Hamiltonian path $P_2$ in $BH_{n-1}^2$. Then there exists an edge in $P_2$, say $b_{02}w_{12}$, whose deletion divides $P_2$ into two sections $P_{02}[w_{02}, b_{02}]$ and $P_{12}[w_{12}, y]$. Let $w_{01}$ be an $(n - 1)$-dimension neighbor of $b_{02}$ in $BH_{n-1}^1$. By Lemma 5, there exists a $(b_{01}, w_{01})$-Hamiltonian path $P_1$ in $BH_{n-1}^1$. Then $(x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y)$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.

Subcase 1.2.3. $x \in V(BH_{n-1}^0)$, $y \in V(BH_{n-1}^0)$ (see Figure 8). Assume $b_{10}$ is an arbitrary black vertex in $BH_{n-1}^0$. By the induction hypothesis, $BH_{n-1}^0 - u$ contains an $(x, b_{10})$-Hamiltonian path $P_0$ containing $e$. Then there exists an edge different from $e$, say $w_{00}b_{00} \in E(P_0)$, whose deletion divides $P_0$ into two sections $P_{00}[x, w_{00}]$ and $P_{00}[b_{00}, b_{01}]$. Let $b_{01}$ be an $(n - 1)$-dimension neighbor of $w_{00}$ in $BH_{n-1}^1$, $w_{03}$ and $w_{13}$ be $(n - 1)$-dimension neighbors of $b_{00}$ and $b_{10}$ in $BH_{n-1}^1$, respectively. By Lemma 5, there exists a $(w_{03}, y)$-Hamiltonian path $P_3$ in $BH_{n-1}^1$. Then there exists an edge, say $w_{13}b_{03} \in E(P_3)$, whose deletion divides $P_3$ into two sections $P_{03}[w_{03}, b_{03}]$ and $P_{13}[w_{13}, y]$. Let $w_{02}$ be an $(n - 1)$-dimension neighbor of $b_{03}$ in $BH_{n-1}^2$. For any black vertex in $BH_{n-1}^2$, say $b_{02}$, we assume $w_{01}$ is an $(n - 1)$-dimension neighbor of $b_{02}$ in $BH_{n-1}^1$. By Lemma 5, there is a $(b_{01}, w_{01})$-Hamiltonian path $P_1$ in $BH_{n-1}^1, i \in \{1, 2\}$. Then $(x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y)$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.

![Figure 9. Subcase 1.2.4.](image1)

![Figure 10. Subcase 1.2.6.](image2)

Subcase 1.2.4. $x \in V(BH_{n-1}^1)$, $y \in V(BH_{n-1}^2)$ (see Figure 9). Assume $w_{01} \in BH_{n-1}^1$ and $w_{02} \in BH_{n-1}^2$ are arbitrary white vertices. By Lemma 5, there exists an $(x, w_{01})$-Hamiltonian path $P_1$ in $BH_{n-1}^1$ and a $(y, w_{02})$-Hamiltonian path $P_2$ in $BH_{n-1}^2$. Let $b_{02} \neq y$ be an $(n - 1)$-dimension neighbor of $w_{01}$ in $BH_{n-1}^2$. Then there exists an edge, say $b_{02}w_{12} \in E(P_2)$, whose deletion divides $P_2$ into two sec-
tions $P_{02}[b_{02}, w_{02}]$ and $P_{12}[y, w_{12}]$. Let $b_{03}$ and $b_{13}$ be $(n-1)$-dimension neighbors of $w_{02}$ and $w_{12}$, respectively. $b_{03} \neq b_{13}$ since every white vertex in $BH_{n-1}^3$ has two black $(n-1)$-dimension neighbors in $BH_{n-1}^3$. Let $w_{13}$ be any white vertex in $BH_{n-1}^3$. By Lemma 5, there exists a $(b_{03}, w_{13})$-Hamiltonian path $P_3$ in $BH_{n-1}^3$. Then there exists an edge, say $w_{03}b_{13} \in E(P_3)$, whose deletion divides $P_3$ into two sections $P_{03}[b_{03}, w_{03}]$ and $P_{13}[b_{13}, w_{13}]$. Let $b_{00}$ and $b_{10}$ be an $(n-1)$-dimension neighbors of $w_{03}$ and $w_{13}$ in $BH_{n-1}^0$, respectively and $b_{00} \neq b_{10}$. By the induction hypothesis, there is a $(b_{00}, b_{10})$-Hamiltonian path $P_0$ containing $e$ in $BH_{n-1}^0$. Then $\langle x, P_1, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_0, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y \rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.

Case 2. $e \in BH_{n-1}^1$. The proof of this case is similar to that of Case 1, and is omitted.

Subcase 1.2.5. $x \in V(BH_{n-1}^1), y \in V(BH_{n-1}^3)$. Assume that $b_{00}$ and $b_{10}$ are any two black vertices in $BH_{n-1}^0$. By the induction hypothesis, there exists a $(b_{00}, b_{10})$-Hamiltonian path $P_0$ containing $e$ in $BH_{n-1}^0$. Let $w_{03}$ and $w_{13}$ be an $(n-1)$-dimension neighbors of $b_{00}$ and $b_{10}$ in $BH_{n-1}^0$, respectively, and $w_{03} \neq w_{13}$. By Lemma 5, there is a $(y, w_{03})$-Hamiltonian path $P_3$ in $BH_{n-1}^3$. Hence there exists an edge, say $w_{13}b_{03} \in E(P_3)$, whose deletion divides $P_3$ into two sections $P_{13}[y, w_{13}]$ and $P_{03}[b_{03}, w_{03}]$. Let $w_{02}$ be an $(n-1)$-dimension neighbor of $b_{03}$ in $BH_{n-1}^2$, $b_{02}$ be any black vertex in $BH_{n-1}^2$, $w_{01}$ be an $(n-1)$-dimension neighbor of $b_{02}$ in $BH_{n-1}^1$. By Lemma 5, there exists an $(x, w_{02})$-Hamiltonian path $P_1$ in $BH_{n-1}^3$ and $(b_{02}, w_{01})$-Hamiltonian path $P_2$ in $BH_{n-1}^1$. Then $\langle x, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_0, b_{10}, w_{13}, P_{13}, y \rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.

Subcase 1.2.6. $x \in V(BH_{n-1}^2), y \in V(BH_{n-1}^3)$ (see Figure 10). Let $w_{03}$ be an arbitrary white vertex in $BH_{n-1}^2$, $b_{00}$ and $b_{10}$ be two $(n-1)$-dimension neighbors of $w_{03}$ in $BH_{n-1}^0$. By the induction hypothesis, for any $e$, there exists a $(b_{00}, b_{10})$-Hamiltonian path $P_0$ containing $e$ in $BH_{n-1}^0$. Then there exists an edge in $P_0$, say $w_{00}b_{20}$ and $w_{00}b_{20} \neq e$, whose deletion divides $P_0$ into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{20}, b_{10}]$. Let $b_{01} \in BH_{n-1}^1$ (resp. $w_{13} \in BH_{n-1}^3$) be an $(n-1)$-dimension neighbor of $w_{00}$ (resp. $b_{20}$). Hence we choose $w_{13} \neq w_{03}$ since every black vertex in $BH_{n-1}^0$ has two white $(n-1)$-dimension neighbors in $BH_{n-1}^3$. By Lemma 6, there exists a $(w_{03}, w_{13})$-Hamiltonian path $P_3$ in $BH_{n-1}^3 - y$. Hence there exists an edge, say $w_{03}b_{03} \in E(P_3)$, so that $P_3 = \langle w_{03}, b_{03}, P_{03}[b_{03}, w_{13}], w_{13} \rangle$. Let $w_{02}$ and $w_{12}$ be $(n-1)$-dimension neighbors of $b_{03}$ and $y$ in $BH_{n-1}^2$ respectively, and $w_{02} \neq w_{12}$. By Lemma 6, there exists a $(w_{02}, w_{12})$-Hamiltonian path $P_2$ in $BH_{n-1}^2 - x$. Let $w_{01} \in BH_{n-1}^1$ be an $(n-1)$-dimension neighbor of $x$. By Lemma 5, there exists a $(b_{01}, w_{01})$-Hamiltonian path $P_1$ in $BH_{n-1}^1$. Thus it follows that $\langle x, w_{01}, P_1, b_{01}, w_{00}, P_{00}, w_{03}, b_{10}, w_{13}, P_{03}, b_{03}, w_{02}, P_2, w_{12}, y \rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.
Case 3. $e \in BH_{n-1}^2$. If $x, y \in V(BH_{n-1}^2)$, then since $xy \neq e$, at most one of $x$ and $y$ can be incident with $e$. Hence the proof is similar to that of Subcase 1.1.3 since we can exchange $x$ and $y$ in this case when $y$ is incident with $e$.

For $x \in V(BH_{n-1}^2), y \in V(BH_{n-1}^3)$ (see Figure 11), if $x$ is not incident with $e$, then the proof is similar to Subcase 1.2.6. If $x$ is incident with $e$, we may assume $w_{02}$ is an $(n-1)$-dimension neighbor of $y$ in $BH_{n-1}^2$ and $b_{03}$ is another $(n-1)$-dimension neighbor of $w_{02}$ in $BH_{n-1}^3$. Choose any white vertex, say $w_{13}$ in $BH_{n-1}^3$ and let $b_{10}$ and $b_{20}$ be two $(n-1)$-dimension neighbors of $w_{13}$ in $BH_{n-1}^0$. By Lemma 6, there exists a $(b_{03}, y)$-Hamiltonian path $P_3$ in $BH_{n-1}^3 - w_{13}$. Then there exists an edge, say $y_{w_{03}} \in E(P_3)$, and $P_3 = \langle y, w_{03}, P_{03}, w_{03}, b_{03} \rangle$. Let $b_{00}$ be the $(n-1)$-dimension neighbor of $w_{03}$ in $BH_{n-1}^0$ and $b_{00} \neq b_{20}$ as $w_{03}$ has two $(n-1)$-dimension neighbors in $BH_{n-1}^0$. Similarly, by Lemma 6, there exists a $(b_{10}, b_{20})$-Hamiltonian path $P_0$ in $BH_{n-1}^0 - u$. Then there exists an edge, say $b_{00}w_{00} \in E(P_0)$, whose deletion divides $P_0$ into two sections $P_{00}[b_{10}, b_{00}]$ and $P_{10}[w_{00}, b_{20}]$. Let $b_{01}$ be one $(n-1)$-dimension neighbor of $w_{00}$ in $BH_{n-1}^1$. By Lemma 5, for any white vertex $w_{01}$ in $BH_{n-1}^1$, there exists a $(b_{01}, w_{01})$-Hamiltonian path $P_1$. Let $b_{02} \in BH_{n-1}^3$ be one $(n-1)$-dimension neighbor of $w_{01}$. By the induction hypothesis, for any $e$, there exists an $(x, b_{02})$-Hamiltonian path $P_2$ containing $e$ in $BH_{n-1}^2 - w_{02}$. Then $(x, P_2, b_{02}, w_{01}, P_1, b_{01}, w_{00}, P_{10}, b_{20}, w_{13}, b_{10}, P_0, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, y)$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_{n-1} - u$.

Since the other subcases are similar to the corresponding subcases of Case 1, we omit their proofs.

Case 4. $e \in BH_{n-1}^3$. We just prove the subcase that $x \in V(BH_{n-1}^2), y \in V(BH_{n-1}^3)$, the other cases are similar to the corresponding subcases of Case 1.

Subcase 4.1. $x \in V(BH_{n-1}^2), y \in V(BH_{n-1}^3)$ (see Figure 12). For any $e \in E(BH_{n-1}^2)$, if $y$ is not incident with $e$, then in the proof of Subcase 1.2.6, we choose any white vertex $w_{03}$ which is not incident with $e$, then the remainder of the proof is similar to that of Subcase 1.2.6. If $y$ is incident with $e$, as $n \geq 3$,
there exist two white vertices in $BH^3_{n-1}$, say $w_{03}$ and $w_{13}$, who are not adjacent to $y$ and have a common $(n-1)$-dimension neighbor, say $b_{00} \in BH^0_{n-1}$. For any black vertex in $BH^3_{n-1}$, say $b_{13}$, by induction hypothesis, the graph $BH^3_{n-1} - w_{13}$ contains a $(y, b_{13})$-Hamiltonian path $P_3$ containing $e$. Thus there exists an edge, say $w_{03}b_{03} \in E(P_3)$ and $w_{03}b_{03} \neq e$ since $w_{03}$ is not adjacent to $y$ and $y$ is incident with $e$. Now $P_3 = \langle y, P_{03}[y, b_{03}], b_{03}, w_{03}, P_{13}[w_{03}, b_{13}], b_{13}\rangle$. Let $b_{10}$ be another $(n-1)$-dimension neighbor of $w_{13}$ in $BH^0_{n-1}$, $w_{02}$ and $w_{12}$ be $(n-1)$-dimension neighbors of $b_{03}$ and $b_{13}$ in $BH^2_{n-1}$ respectively, and $w_{02} \neq w_{12}$ since every black vertex in $BH^3_{n-1}$ has two $(n-1)$-dimension neighbors in $BH^2_{n-1}$. By Lemma 6, there exists one $(w_{02}, w_{12})$-Hamiltonian path $P_2$ in $BH^2_{n-1} - x$, and one $(b_{00}, b_{10})$-Hamiltonian path $P_0$ in $BH^0_{n-1} - u$. Then there exists an edge $w_{00}b_{00} \in E(P_0)$ such that $P_0 = \langle b_{10}, P_{00}[b_{10}, w_{00}], w_{00}, b_{00}\rangle$. Let $w_{04}$ and $b_{01}$ be $(n-1)$-dimension neighbors of $x$ and $w_{04}$ in $BH^1_{n-1}$ respectively. By Lemma 5, there exists one $(b_{01}, w_{01})$-Hamiltonian path $P_1$ in $BH^1_{n-1}$. Then $\langle x, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{10}, w_{13}, b_{00}, w_{03}, P_{13}, b_{13}, w_{12}, P_2, w_{02}, b_{03}, P_{03}, y\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $BH_n - u$.

Combining the above cases, the proof of this theorem is completed. 

4. Conclusion

The balance hypercube $BH_n$, proposed by Huang and Wu [5], is a variant of the hypercube that gives better performance with the same number of edges and vertices. It has been shown that the balanced hypercube $BH_n$ is Hamiltonian laceable and hyper-Hamiltonian laceable for $n \geq 1$. In this paper, we show that, for any vertex $v \in V_i$, $i \in \{0, 1\}$, and any $e \in E(BH_n - v)$, there exists a Hamiltonian path containing $e$ in $G - v$ between any pair of vertices in $V_i$.

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