

**$K_3$ -WORM COLORINGS OF GRAPHS:  
LOWER CHROMATIC NUMBER AND GAPS  
IN THE CHROMATIC SPECTRUM**

CSILLA BUJTÁS AND ZSOLT TUZA

*Alfréd Rényi Institute of Mathematics*  
*Hungarian Academy of Sciences, Budapest, Hungary*  
*Department of Computer Science and Systems Technology*  
*University of Pannonia, Veszprém, Hungary*

**e-mail:** bujtas@dcs.uni-pannon.hu  
tuza@dcs.uni-pannon.hu

**Abstract**

A  $K_3$ -WORM coloring of a graph  $G$  is an assignment of colors to the vertices in such a way that the vertices of each  $K_3$ -subgraph of  $G$  get precisely two colors. We study graphs  $G$  which admit at least one such coloring. We disprove a conjecture of Goddard *et al.* [Congr. Numer. 219 (2014) 161–173] by proving that for every integer  $k \geq 3$  there exists a  $K_3$ -WORM-colorable graph in which the minimum number of colors is exactly  $k$ . There also exist  $K_3$ -WORM colorable graphs which have a  $K_3$ -WORM coloring with two colors and also with  $k$  colors but no coloring with any of  $3, \dots, k - 1$  colors. We also prove that it is NP-hard to determine the minimum number of colors, and NP-complete to decide  $k$ -colorability for every  $k \geq 2$  (and remains intractable even for graphs of maximum degree 9 if  $k = 3$ ). On the other hand, we prove positive results for  $d$ -degenerate graphs with small  $d$ , also including planar graphs.

**Keywords:** WORM coloring, lower chromatic number, feasible set, gap in the chromatic spectrum.

**2010 Mathematics Subject Classification:** 05C15.

1. INTRODUCTION

In a vertex-colored graph, a subgraph is *monochromatic* if its vertices have the same color, and it is *rainbow* if its vertices have pairwise different colors. Given two graphs  $F$  and  $G$ , an  $F$ -WORM coloring of  $G$  is an assignment of colors to its vertices such that no subgraph isomorphic to  $F$  is monochromatic or rainbow.

This notion was introduced recently in [9] by Goddard, Wash, and Xu. As noted in [9], however, for some types of  $F$  some earlier results due to Bujtás *et al.* [3, 4] imply upper bounds on the possible number of colors in  $F$ -WORM colorings of graphs  $G$ . The name ‘ $F$ -WORM’ comes as the abbreviation of ‘WithOut a Rainbow or Monochromatic subgraph isomorphic to  $F$ ’.

If  $G$  has at least one  $F$ -WORM coloring, then  $W^-(G, F)$  denotes the minimum number of colors and  $W^+(G, F)$  denotes the maximum number of colors in an  $F$ -WORM coloring of  $G$ ; they are termed the  $F$ -WORM *lower* and *upper chromatic number*, respectively. Moreover, the  $F$ -WORM *feasible set*  $\Phi_w(G, F)$  of  $G$  is the set of those integers  $s$  for which  $G$  admits an  $F$ -WORM coloring with exactly  $s$  colors. In general, we say that  $G$  has a *gap* at  $k$  in its  $F$ -WORM chromatic spectrum, if  $W^-(G, F) < k < W^+(G, F)$  but  $G$  has no  $F$ -WORM coloring with precisely  $k$  colors. Otherwise, if  $\Phi_w(G, F)$  contains all integers between  $W^-(G, F)$  and  $W^+(G, F)$ , we say that the  $F$ -WORM feasible set (or the  $F$ -WORM chromatic spectrum) of  $G$  is *gap-free*.

We shall not mention later in each assertion, but it should be emphasized that the values  $W^-(G, F)$  and  $W^+(G, F)$  are defined only for  $F$ -WORM-colorable graphs. Hence, wherever  $W^-$  or  $W^+$  appears in the text, it is assumed that the graph in question is colorable.

As one can see, four fundamental problems arise in this context: testing whether  $G$  is  $F$ -WORM colorable, computing  $W^-(G, F)$ , computing  $W^+(G, F)$ , and determining  $\Phi_w(G, F)$ .

### 1.1. Results

In this paper we focus on the case of  $F = K_3$ , i.e.,  $K_3$ -WORM colorings of graphs. It is clear that  $K_5$  has no  $K_3$ -WORM coloring. Moreover,  $W^-(G, K_3) = 1$  and  $W^+(G, K_3) = n$  are valid for all triangle-free  $n$ -vertex graphs  $G$  (and only for them), and any number of colors between 1 and  $n$  can occur in this case. Therefore, the interesting examples are the graphs whose clique number equals 3 or 4.

Goddard, Wash, and Xu [8] proved that  $W^-(G, K_3) \leq 2$  holds for outerplanar graphs and also for cubic graphs. They conjectured that every  $K_3$ -WORM-colorable graph admits a  $K_3$ -WORM coloring with two colors ([8, Conjecture 1]). Our Theorem 4 disproves this conjecture in a wide sense, showing that the minimum number of colors in  $K_3$ -WORM-colorable graphs can be arbitrarily large.

It was proved in [9] that there exist graphs with gaps in their  $P_3$ -WORM chromatic spectrum. In [8], the authors remark that for trees the  $K_3$ -WORM chromatic spectrum is trivially gap-free (as noted above, it is clearly so for all triangle-free graphs), and they ask whether this is true for every  $K_3$ -WORM colorable graph. Our constructions presented in Section 3 show the existence of graphs  $H_k$  which have  $W^-(H_k, K_3) = 2$  and  $W^+(H_k, K_3) \geq k$ , but the feasible

set  $\Phi_w(G, K_3)$  contains no element from the range  $[3, k - 1]$ . Further types of constructions (applying a different kind of methodology) and a study of the  $K_3$ -WORM upper chromatic number will be presented in our follow-up paper [6].

Goddard, Wash, and Xu proved that the decision problem whether a generic input graph admits a  $K_3$ -WORM coloring is NP-complete ([8, Theorem 3]). We consider complexity issues related to the determination of  $W^-(G, K_3)$ . In Section 4, we show that it is NP-hard to distinguish between graphs which are  $K_3$ -WORM-colorable with three colors and those needing precisely four as minimum. This hardness is true already on the class of graphs with maximum degree 9. Additionally, we prove that for every  $k \geq 4$ , the decision problem whether  $W^-(G, K_3) \leq k$  is NP-complete already when restricted to graphs with a sufficiently large but bounded maximum degree. Deciding  $K_3$ -WORM 2-colorability is hard, too, but so far we do not have a bounded-degree version of this result. We also prove that the algorithmic problem of deciding if the  $K_3$ -WORM chromatic spectrum is gap-free is intractable.

In Section 5 we deal with 4-colorable graphs and some subclasses. It was observed by Ozeki [15] that the property of being  $K_3$ -WORM 2-colorable is valid for planar graphs; and his argument directly extends to any graph of chromatic number at most 4. As a property stronger than 4-colorability, a graph is *3-degenerate* if each of its non-empty subgraphs contains a vertex of degree at most 3. We point out that every 3-degenerate graph has a gap-free  $K_3$ -WORM chromatic spectrum. For graphs of maximum degree 3, a formula for  $W^+(G, K_3)$  can also be given.

We conclude the paper with several open problems and conjectures in Section 6.

## 2. MIXED BI-HYPERGRAPHS

The notion of mixed hypergraph was introduced by Voloshin in the 1990s [16, 17]. A detailed overview of the theory is given in the monograph [18], for up to date information see also [7] and [19]. Many open problems in the area are surveyed in [2] and [5]. In the present context the relevant structures will be what are called ‘mixed bi-hypergraphs’.<sup>1</sup>

A *mixed bi-hypergraph*  $\mathcal{H}$  is a pair  $(X, \mathcal{B})$ , where  $X$  is the vertex set and  $\mathcal{B}$  is a set system over  $X$ . A (feasible) coloring of  $\mathcal{H}$  is a mapping  $\varphi : X \rightarrow \mathbb{N}$  such that each  $B \in \mathcal{B}$  contains two vertices with a common color and also contains two vertices with distinct colors. In other words, no hyperedges are rainbow or monochromatic.

---

<sup>1</sup>In the literature of mixed hypergraphs the term simply is ‘bi-hypergraph’. Since here our main subject is a different structure class, we will emphasize that it is a *mixed* bi-hypergraph.

For a given mixed bi-hypergraph  $\mathcal{H}$ , four fundamental questions arise in a very natural way.

**Colorability.** Does  $\mathcal{H}$  admit any coloring?

**Lower chromatic number.** If  $\mathcal{H}$  is colorable, what is the minimum number  $\chi(\mathcal{H})$  of colors in a coloring?

**Upper chromatic number.** If  $\mathcal{H}$  is colorable, what is the maximum number  $\bar{\chi}(\mathcal{H})$  of colors in a coloring?

**Feasible set.** If  $\mathcal{H}$  is colorable, what is the set  $\Phi(\mathcal{H})$  of integers  $s$  such that  $\mathcal{H}$  admits a coloring with exactly  $s$  colors?

The next observation shows that mixed hypergraph theory provides a proper and very natural general framework for the study of  $F$ -WORM colorings.

**Proposition 1.** *Let  $F$  be a given graph. For any graph  $G$  on a vertex set  $V$ , let  $\mathcal{H} = (X, \mathcal{B})$  be the mixed bi-hypergraph in which  $X = V$ , and  $\mathcal{B}$  consist of those vertex subsets of cardinality  $|V(F)|$  which induce a subgraph containing  $F$  in  $G$ . Then:*

- (i)  $G$  is  $F$ -WORM-colorable if and only if  $\mathcal{H}$  is colorable.
- (ii)  $W^-(G, F) = \chi(\mathcal{H})$ .
- (iii)  $W^+(G, F) = \bar{\chi}(\mathcal{H})$ .
- (iv)  $\Phi_w(G, F) = \Phi(\mathcal{H})$ .

**Proof.** By the definitions, an assignment  $\varphi : V \rightarrow \mathbb{N}$  is an  $F$ -WORM coloring of  $G$  if and only if it is a feasible coloring of the mixed bi-hypergraph  $\mathcal{H}$ . Then, the statements (i)–(iv) immediately follow. ■

A similar bijection between ‘WORM edge colorings’ of  $K_n$  and the colorings of a mixed bi-hypergraph defined in a suitable way on the edge set of  $K_n$  was observed by Voloshin in an e-mail correspondence to us in 2013 [20].

Due to the strong correspondence above, it is meaningful and reasonable to adopt the terminology of mixed hypergraphs to the study of WORM colorings.

### 3. LARGE $W^-$ AND GAP IN THE CHROMATIC SPECTRUM

In several proofs of this paper we will use the following notion and notation. For a graph  $G$  with vertex set  $V(G) = \{v_1, \dots, v_n\}$ , the strong product

$$H = G \boxtimes K_2$$

is obtained from  $G$  by replacing each vertex  $v_i$  with two adjacent vertices  $x_i, y_i$  and each edge  $v_i v_j$  with a copy of  $K_4$  on the vertex set  $\{x_i, y_i, x_j, y_j\}$ .

**Lemma 2.** *Let  $G$  be a connected and triangle-free graph and let  $H = G \boxtimes K_2$ . Then, every  $K_3$ -WORM coloring  $\varphi$  of  $H$  is one of the following two types:*

- (i) *for each  $v_i \in V(G)$ , the vertices  $x_i$  and  $y_i$  receive the same color, and if  $v_i$  and  $v_j$  are adjacent in  $G$  then  $\varphi(x_i) \neq \varphi(x_j)$ ; or*
- (ii)  *$\varphi$  uses two colors and for each  $v_i \in V(G)$ , the vertices  $x_i$  and  $y_i$  receive different colors.*

*Moreover, if  $G$  has at least one edge,  $W^-(H, K_3) = 2$ , and  $H$  has a  $K_3$ -WORM coloring with precisely  $s$  colors for an  $s \geq 3$  if and only if  $\chi(G) \leq s \leq |V(G)|$ .*

**Proof.** Since  $G$  is triangle-free, each triangle of  $H$  is inside a copy of  $K_4$  originating from an edge of  $G$ . Thus, the  $K_3$ -WORM colorings of  $H$  are precisely those vertex colorings in which

- (\*) each copy  $K$  of  $K_4$  gets exactly two colors such that each of them appears on exactly two vertices of this  $K$ .

First, assume that  $x_1$  and  $y_1$  have the same color in the  $K_3$ -WORM coloring  $\varphi$ . If a vertex  $v_j$  is adjacent to  $v_1$  then, by (\*), the only way in a  $K_3$ -WORM coloring is to assign  $x_j$  and  $y_j$  to the same color which is different from the color of  $\{x_1, y_1\}$ . This property of monochromatic pairs propagates along paths, therefore each pair  $\{x_i, y_i\}$  ( $1 \leq i \leq n$ ) is monochromatic whenever  $G$  is connected, and for every edge  $v_i v_j \in E(G)$  the colors  $\varphi(x_i)$  and  $\varphi(x_j)$  are different.

On the other hand, if  $x_1$  and  $y_1$  have distinct colors, and a vertex  $v_j$  is adjacent to  $v_1$ , then again by (\*), the only way in a  $K_3$ -WORM coloring is to assign  $\{x_j, y_j\}$  to the same pair of colors. Then, if  $G$  is connected, precisely two colors are used in the entire graph.

A  $K_3$ -WORM coloring of  $H$  is easily obtained by assigning color 1 to all vertices  $x_i$  and color 2 to all vertices  $y_i$ . Hence,  $W^-(H, K_3) = 2$ . Further, if  $\chi(G) \leq s \leq |V(G)|$  then  $G$  has a proper coloring  $\phi$  which uses precisely  $s$  colors. Assigning the color  $\phi(v_i)$  to the vertices  $x_i$  and  $y_i$ , yields a  $K_3$ -WORM coloring of  $H$  with exactly  $s$  colors. This completes the proof of the lemma. ■

**Theorem 3.** *The feasible sets of  $K_3$ -WORM-colorable graphs may contain arbitrarily large gaps.*

**Proof.** For an integer  $k \geq 4$ , consider a connected triangle-free graph  $G_k$  whose chromatic number equals  $k$ . It is well known<sup>2</sup> that such a graph exists for each positive  $k$ . By Lemma 2, the  $K_3$ -WORM feasible set of the graph  $H_k = G_k \boxtimes K_2$  is

$$\{2\} \cup \left\{ s \mid k \leq s \leq \frac{|V(H_k)|}{2} \right\},$$

which contains a gap of size  $k - 3$ . ■

<sup>2</sup>The existence is known for over a half century, by explicit constructions and also by applying the probabilistic method; see e.g. [10, Section 1.5] for references.

**Theorem 4.** *For every  $k \geq 3$  there exists a graph  $F_k$  such that  $W^-(F_k, K_3) = k$ .*

**Proof.** We start with a connected triangle-free graph  $G_k$  whose chromatic number is equal to  $k$ . Let  $H_k$  be again  $G_k \boxtimes K_2$ , as above. We define  $F_k$  as the graph obtained from three vertex-disjoint copies  $H_k^i$  of  $H_k$  ( $i = 1, 2, 3$ ) by the following three identifications of vertices:

$$x_1^1 = y_1^2, \quad x_1^2 = y_1^3, \quad x_1^3 = y_1^1.$$

This graph is  $K_3$ -WORM-colored if and only if so is each  $H_k^i$  and moreover the triangle  $\{x_1^1, x_1^2, x_1^3\}$  gets precisely two colors.

Suppose, without loss of generality, that  $x_1^1$  and  $x_1^2$  get color 1, and  $x_1^3$  gets color 2. Then, according to Lemma 2, both  $H_k^1$  and  $H_k^3$  are colored entirely with  $\{1, 2\}$ . On the other hand, we have  $\{x_1^2, y_1^2\} = \{x_1^1, x_1^2\}$ , hence this vertex pair is monochromatic in color 1, therefore  $H_k^2$  is colored according to a proper vertex coloring of  $H_k$ . Thus, the smallest possible number of colors equals the chromatic number  $k$  of  $G_k$ . ■

We close this section with an example which shows that  $W^-(G, K_3)$  can exceed 2 even when  $K_4$  is not a subgraph of  $G$ . Note first that in every  $K_3$ -WORM 2-coloring of  $W_5$  the 5-cycle contains a monochromatic edge and the center of the wheel gets the opposite color. Thus, making complete adjacencies between consecutive members of the sequence of two vertex-disjoint 5-cycles  $C^1, C^2$  and further three independent vertices  $x, y, z$  in the order

$$x, C^1, y, C^2, z$$

the vertices  $x$  and  $z$  get the same color in every  $K_3$ -WORM coloring with two colors. Let us take two copies of this graph with ends  $x', z'$  and  $x'', z''$ , respectively; moreover, take a triangle  $w_1 w_2 w_3$  and make the following identifications:

$$w_1 = x', \quad w_2 = z' = z'', \quad w_3 = x''.$$

Should this  $K_4$ -free 25-vertex graph  $G$  have a  $K_3$ -WORM 2-coloring  $\phi$ , we should have  $\phi(w_1) = \phi(w_2) = \phi(w_3)$  due to the construction of the copies, and

$$|\{\phi(w_1), \phi(w_2), \phi(w_3)\}| = 2$$

due to the triangle  $w_1 w_2 w_3$ . This contradiction implies  $W^-(G, K_3) \geq 3$ .

#### 4. ALGORITHMIC COMPLEXITY

In this section we consider two algorithmic problems: to determine the minimum number of colors, and to decide whether no gaps occur in the chromatic spectrum.

**4.1. Lower chromatic number**

Here we prove that the determination of  $W^-(G, K_3)$  is NP-hard, and it remains hard even when the input is restricted to graphs with maximum degree 9. We give degree-restricted versions of such results for every number  $k \geq 3$  of colors. At the end of the subsection we prove a theorem on 2-colorings, but without upper bound on vertex degrees.

More formally, we will consider the case  $F = K_3$  of the following decision problem for every positive integer  $k$ .

$F$ -WORM  $k$ -COLORABILITY

**Input:** An  $F$ -WORM-colorable graph  $G = (V, E)$ .

**Question:** Is  $W^-(G, F) \leq k$ ?

The membership of this problem in NP is obvious for every  $F$  and every  $k$ . To prove NP-completeness for  $F = K_3$  and any  $k \geq 3$ , we will refer to our constructions from Section 3 and the following result of Maffray and Preissmann concerning the complexity of deciding whether a graph has a proper vertex coloring with a given number  $k$  of colors, which we shall refer to as GRAPH  $k$ -COLORABILITY.

**Theorem 5** [14]. (i) *The GRAPH 3-COLORABILITY problem remains NP-complete when the input is restricted to the class of triangle-free graphs with maximum degree four.*

(ii) *For each  $k \geq 4$ , the GRAPH  $k$ -COLORABILITY problem is NP-complete on the restricted class of triangle-free graphs with maximum degree  $3 \cdot 2^{k-1} + 2k - 2$ .*

By a closer look into the proof in [14] we see that this theorem is also valid if one restricts to *connected non-regular graphs* with that maximum degree.

**Theorem 6.** (i) *The decision problem of  $K_3$ -WORM 3-COLORABILITY is NP-complete already on the class of graphs with maximum degree 9.*

(ii) *The decision problem of  $K_3$ -WORM  $k$ -COLORABILITY is NP-complete for each  $k \geq 4$  already on the class of graphs with maximum degree  $3 \cdot 2^k + 4k - 3$ .*

**Proof.** As noted above, the problems are clearly in NP. To prove (i), we reduce the GRAPH 3-COLORABILITY problem on the class of triangle-free graphs to the problem of  $K_3$ -WORM 3-COLORABILITY. Consider a generic input graph  $G'$  of the former problem with  $\Delta(G') = 4$ . Without loss of generality we can assume that  $G'$  is connected and non-regular. Hence attaching a pendant edge to a vertex of minimum degree we get a graph  $G$  without increasing the maximum degree, such that it has a degree-1 vertex  $v_0$ . Then, we define  $H$  to be the graph  $G \boxtimes K_2$ , as in Section 3. Observe that  $\Delta(H) = 9$ . In the next step, we take three vertex-disjoint copies  $H^1, H^2$ , and  $H^3$  of  $H$ , and make the following three identifications of vertices, each of which originates from the vertex  $v_0$  of  $G$ :

$$x_0^1 = y_0^2, \quad x_0^2 = y_0^3, \quad x_0^3 = y_0^1.$$

The maximum degree of the obtained graph  $F$  remains 9, as the vertices  $x_0^i$  and  $y_0^i$  had only degree 3 in  $H^i$ . By Lemma 2, and similarly to the proof of Theorem 4, we obtain that  $\chi(G) = 3$  if and only if  $W^-(F, K_3) = 3$ . Thus, part (i) of Theorem 5 implies the NP-completeness of  $K_3$ -WORM 3-COLORABILITY for graphs of maximum degree 9.

Part (ii) of our theorem follows from Theorem 5 (ii) by similar steps of reductions as discussed above. ■

The following result states that the case of two colors is already hard.

**Theorem 7.** *The decision problem of  $K_3$ -WORM 2-COLORABILITY is NP-complete on  $K_3$ -WORM-colorable graphs.*

**Proof.** We apply reduction from the 2-colorability of 3-uniform hypergraphs; we denote by  $\mathcal{H} = (X, \mathcal{F})$  a generic input of this problem. Hence,  $X$  is the vertex set of  $\mathcal{H}$ , and  $\mathcal{F}$  is a family of 3-element subsets of  $X$ . It is NP-complete to decide whether there exists a proper 2-coloring of  $\mathcal{H}$ , that is a partition  $(X_1, X_2)$  of  $X$  such that each  $F \in \mathcal{F}$  meets both  $X_1$  and  $X_2$  [12].

>From  $\mathcal{H} = (X, \mathcal{F})$  we construct a graph  $G = (V, E)$  such that  $\mathcal{H}$  has a proper 2-coloring if and only if  $G$  has a  $K_3$ -WORM coloring with two colors. This correspondence between  $\mathcal{H}$  and  $G$  will imply the validity of the theorem.

For each hyperedge  $F \in \mathcal{F}$  of  $\mathcal{H}$  and each vertex  $x \in F$ , we create a vertex  $(x, F) \in V$  of  $G$ . If  $F = \{x, x', x''\}$ , then the vertices  $(x, F), (x', F), (x'', F)$  will be mutually adjacent in  $G$ . Moreover, small gadgets will ensure that any two vertices  $(x, F'), (x, F'') \in V$  with the same  $x$  get the same color whenever  $G$  is  $K_3$ -WORM-colored.

To ensure this, suppose that an  $x$  is incident with the hyperedges  $F_1, \dots, F_d$ . Then, for any two edges  $F_i, F_{i+1}$  having consecutive indices in this set (where  $1 \leq i < d$ ), we take a graph  $H(x, i)$  which is isomorphic to  $K_5 - e$ , and identify its two non-adjacent vertices — say  $y$  and  $z$  — with  $(x, F_i)$  and  $(x, F_{i+1})$ , respectively. We make this kind of extension for each pair  $(x, i)$  in such a way that the triangles  $H(x, i) - y - z$  are mutually vertex-disjoint. Let  $G$  denote the graph obtained in this way.

Consider any of the gadgets  $H = H(x, i)$ ; we shall abbreviate it as  $H$ . Every  $K_3$ -WORM coloring of  $H$  uses a color twice on  $H - y - z$ , therefore the second color of  $H - y - z$  (which occurs just once there) must be repeated on  $y$  and on  $z$  as well, for otherwise  $H - y$  or  $H - z$  would violate the conditions of  $K_3$ -WORM coloring. Thus, all of  $(x, F_1), \dots, (x, F_d)$  sharing any  $x$  must have the same color. Consequently, every  $K_3$ -WORM coloring of the obtained graph  $G$  defines a proper vertex coloring of  $\mathcal{H}$  in a natural way.

Conversely, if  $\mathcal{H}$  is properly colored, we can assign the color of each  $x \in X$  to all vertices of type  $(x, F)$  with the same  $x$ . Then, in each  $H(x, i)$ , the non-adjacent vertices  $y$  and  $z$  have the same color. Repeating this color on one vertex

of  $H(x, i) - y - z$  and assigning one different color to its remaining vertex pair we eventually obtain a  $K_3$ -WORM coloring of  $G$ . Moreover, if  $\mathcal{H}$  is 2-colored, we do not need to introduce any further colors for  $G$ .

The two-way correspondence between the 2-colorings of  $\mathcal{H}$  (if they exist) and the  $K_3$ -WORM colorings of  $G$  with two colors verifies the validity of the theorem. ■

#### 4.2. The CHROMATIC GAP decision problem

The problem considered in this subsection is as follows.

*F*-WORM CHROMATIC GAP

**Input:** An *F*-WORM-colorable graph  $G$ .

**Question:** Does the *F*-WORM chromatic spectrum of  $G$  have a gap?

Here we prove

**Theorem 8.** *The  $K_3$ -WORM CHROMATIC GAP problem is NP-hard.*

**Proof.** Part (ii) of Lemma 2 yields that the  $K_3$ -WORM chromatic spectrum of the graph  $G_k \boxtimes K_2$  is gap-free if and only if  $G_k$  has a proper vertex coloring with at most three colors. This property is NP-hard (actually NP-complete) to decide. ■

### 5. 4-COLORABLE AND 3-DEGENERATE GRAPHS

Here we show that two of the four basic problems listed in Section 2 have a simple solution on 4-colorable graphs. Moreover, we prove further results on 3-degenerate graphs and on graphs of maximum degree 3.

The complete graph  $K_5$  shows that not every 5-colorable graph is  $K_3$ -WORM-colorable. On the other hand, as we shall see, every 4-colorable graph is  $K_3$ -WORM-colorable, and this can be done by using only two colors. This was commented to us after our talk at the AGTAC 2015 conference by Kenta Ozeki; hence, the next proposition should be attributed to him [15].

**Proposition 9.** *Every 4-colorable graph  $G$  is  $K_3$ -WORM-colorable, and the lower chromatic number  $W^-(G, K_3)$  is at most 2.*

**Proof.** If  $(V_1, V_2, V_3, V_4)$  is a vertex partition of  $G$  into four independent sets, then each of  $V_1 \cup V_2$  and  $V_3 \cup V_4$  meets all triangles of  $G$ . Hence, the two color classes  $V_1 \cup V_2$  and  $V_3 \cup V_4$  determine a  $K_3$ -WORM-coloring of  $G$ . ■

**Remark 10.** It follows from the Four Color Theorem and Proposition 9 that every planar graph is  $K_3$ -WORM colorable with (at most) two colors. We note that this can also be derived by a modification of the proof of [11, Theorem 2.1], without using the 4CT. In the quoted result, Kündgen and Ramamurthi prove

WORM 2-colorability of triangular faces of planar graphs; i.e., the condition is not required there for separating triangles. Planar graphs containing separating triangles can be handled recursively, using a local condition stronger than the global one given in [11]. We omit the details.

Next we prove that for 3-degenerate graphs not only  $W^-(G, K_3) \leq 2$  holds but there are no gaps in their  $K_3$ -WORM chromatic spectrum.

**Proposition 11.** *If  $G$  is a 3-degenerate graph, then  $G$  has a gap-free  $K_3$ -WORM chromatic spectrum.*

**Proof.** The proof proceeds by induction on the order of the graph. Consider a 3-degenerate graph  $G$ , and a vertex  $v \in V(G)$  which has three neighbors, say  $a$ ,  $b$ , and  $c$ . By the induction hypothesis, the graph  $G^-$  obtained by removing  $v$  and its incident edges has a gap-free chromatic spectrum. We show that  $G$  has a  $K_3$ -WORM coloring with exactly  $t$  colors for each  $t \geq 2$  in the range  $W^-(G^-, K_3) \leq t \leq W^+(G^-, K_3)$ . To do this, we start with a  $t$ -coloring  $\varphi$  of  $G^-$ . First, assume that  $\varphi(a)$ ,  $\varphi(b)$ , and  $\varphi(c)$  are pairwise distinct. Then  $abc$  is not a triangle. If  $a, b, c$  induce a  $P_3$ , the color of its central vertex can be repeated on  $v$ . If  $a, b, c$  induce only one edge, say  $ab$ , then  $\varphi(a)$  can be assigned to  $v$ . If  $a, b, c$  are pairwise non-adjacent, then  $v$  can get any of the  $t$  colors of  $G^-$ . Next, consider the case of  $\varphi(a) = \varphi(b)$ . If this color is different from  $\varphi(c)$ , then it is appropriate to define  $\varphi(v) = \varphi(c)$ . In the last case,  $\{a, b, c\}$  is monochromatic and  $v$  can be assigned to any color which is different from  $\varphi(a)$ . This proves that  $G$  is  $K_3$ -WORM colorable with exactly  $t$  colors for each  $t$  with  $t \geq 2$  and  $W^-(G^-, K_3) \leq t \leq W^+(G^-, K_3)$ .

Note that  $W^-(G, K_3) = 1$  if and only if  $G$  is triangle-free, and this implies gap-free spectrum; moreover observe that  $W^+(G, K_3) \leq W^+(G^-, K_3) + 1$ . By induction, we obtain that the statement holds for every 3-degenerate graph. ■

Suppose now that  $G$  has maximum degree 3. By Proposition 9 and 11 we know that  $G$  is  $K_3$ -WORM-colorable, has  $W^-(G, K_3) = 2$ , and its chromatic spectrum is gap-free. Next, we show that  $W^+(G, K_3)$  can be computed efficiently.

Let  $G^\Delta$  be the graph obtained from  $G$  by removing all edges which are not contained in any triangles. This  $G^\Delta$  can have the following types of connected components:

$$K_1, \quad K_3, \quad K_4 - e, \quad K_4.$$

For these four types of  $F$ , let us denote by  $n_G(F)$  the number of components isomorphic to  $F$  in  $G^\Delta$ .

**Proposition 12.** *If  $G$  has  $n$  vertices, and has maximum degree at most 3, then*

$$W^+(G, K_3) = n - n_G(K_3) - n_G(K_4 - e) - 2n_G(K_4).$$

Moreover,  $W^+(G, K_3)$  can be determined in  $O(n)$  time.

**Proof.** A vertex coloring is a  $K_3$ -WORM coloring of  $G$  if and only if it is a  $K_3$ -WORM coloring of each connected component in  $G^\Delta$ . Starting from the rainbow coloring of the vertex set, a  $K_3$ -WORM coloring with maximum number of colors needs:

- to decrease the number of colors from 3 to exactly 2 in a  $K_3$  component,
- to make the pair of the two degree-3 vertices monochromatic in a  $K_4 - e$  component,
- to reduce the number of colors from 4 to 2 in a  $K_4$  component.

This proves the correctness of the formula on  $W^+(G, K_3)$ . Linear time bound follows from the fact that one can construct  $G^\Delta$  and enumerate its components of the three relevant types in  $O(n)$  steps in any graph of maximum degree at most 3. ■

>From the formula above, the following tight lower bounds can be derived; part (ii) was proved for cubic graphs by Goddard *et al.* in [8].

**Corollary 13.** *If  $G$  is a graph of order  $n$  and maximum degree 3, then*

- (i)  $W^+(G, K_3) \geq n/2$ , with equality if and only if  $G \cong \frac{n}{4}K_4$ ;
- (ii) if  $G$  does not have any  $K_4$  components, then  $W^+(G, K_3) \geq 2n/3$ , with equality if and only if  $G$  contains  $\frac{n}{3}K_3$  as a subgraph;
- (iii) if  $G$  does not have any  $K_4$  components, and each of its triangles shares an edge with another triangle, then  $W^+(G, K_3) \geq 3n/4$ , with equality if and only if  $G$  contains  $\frac{n}{4}(K_4 - e)$  as a subgraph.

**Proof.** The formula in Theorem 12 shows that the number of colors lost, when compared to the number of vertices, is 2 from 4 in  $K_4$ , 1 from 3 in  $K_3$ , and 1 from 4 in  $K_4 - e$ . ■

A notable particular case of (ii) is where  $n \geq 5$  and  $G$  is connected. Moreover, since  $K_4 - e$  has just two vertices of degree 2, contracting each copy of  $K_4 - e$  in the extremal structure described in (iii) we obtain a collection of vertex-disjoint paths and cycles (where cycles of length 2 are also possible).

## 6. CONCLUDING REMARKS

We have solved several problems — some of them raised in [8] — concerning the  $K_3$ -WORM colorability and the corresponding lower chromatic number of graphs. Further properties of  $K_3$ -WORM feasible sets and the complexity of determining the upper chromatic number will be studied in the successor of this paper, [6].

Below we mention several problems which remain open. The first one proposes a strengthening of Theorem 4. Recall that at the end of Section 3 we gave an example of  $K_4$ -free graph with  $W^-(G, K_3) = 3$ .

**Conjecture 14.** *For every integer  $k \geq 4$  there exists a  $K_3$ -WORM-colorable  $K_4$ -free graph  $G$  such that  $W^-(G, K_3) = k$ .*

The other problems deal with algorithmic complexity. We have proved that it is NP-hard to test whether  $\Phi_W(G, K_3)$  is gap-free. On the other hand, after checking that  $G$  is not triangle-free,  $n - 2$  questions to an NP-oracle in parallel (asking in a non-adaptive manner whether the input graph  $G$  of order  $n$  admits a  $K_3$ -WORM coloring with exactly  $k$  colors, for  $k = 2, 3, \dots, n - 1$ ) solves the problem, hence it is in the class  $\Theta_2^P$  (see [13] for a nice introduction to  $\Theta_2^P$ , or the last part of [1] for short comments on its properties). However, the exact status of the problem is unknown so far.

**Problem 15.** Is the decision problem  $K_3$ -WORM CHROMATIC GAP  $\Theta_2^P$ -complete?

On several natural classes of graphs we do not even have a lower bound on the complexity of this problem.

**Problem 16.** What is the time complexity of deciding whether the  $K_3$ -WORM chromatic spectrum is gap-free, if the input is restricted to  $K_3$ -WORM-colorable

- (i)  $K_4$ -free graphs, or
- (ii) 4-colorable graphs, or
- (iii) planar graphs?

It is not even known at the time of writing this paper whether the feasible sets of graphs from the above classes can contain any gaps.

Also, the classes of  $d$ -degenerate graphs for various values of  $d$  offer interesting questions.

- Problem 17.**
- (i) Can the value of  $W^+(G, K_3)$  be determined in polynomial time on 3-degenerate graphs?
  - (ii) If the answer is yes, what is the smallest  $d$  such that the computation of  $W^+(G, K_3)$  is NP-hard on the class of  $d$ -degenerate graphs?
  - (iii) Prove that a finite threshold value  $d$  with the property described in part (ii) exists.

**Problem 18.** Consider the class of graphs with maximum degree at most  $d$ .

- (i) Is it NP-complete to decide whether  $W^-(G, K_3) = 2$  if  $d$  is large enough?
- (ii) What is the smallest  $d_k$  as a function of  $k$  such that the decision problem of  $W^-(G, K_3) \leq k$  is NP-hard on the class of graphs with maximum degree  $d_k$ ?
- (iii) What is the smallest  $d$  for which it is NP-complete to decide whether a generic input graph of maximum degree at most  $d$  is  $K_3$ -WORM-colorable?

### Acknowledgements

This research was supported in part by the European Union and Hungary co-financed by the European Social Fund through the project TÁMOP-4.2.2.C-11/1/KONV-2012-0004 — National Research Center for Development and Market Introduction of Advanced Information and Communication Technologies. We thank Kenta Ozeki for his comment on 4-colorable graphs and for drawing our attention to the paper [11]. We also thank the referee for constructive suggestions.

### REFERENCES

- [1] S. Arumugam, B.K. Jose, Cs. Bujtás and Zs. Tuza, *Equality of domination and transversal numbers in hypergraphs*, Discrete Appl. Math. **161** (2013) 1859–1867. doi:10.1016/j.dam.2013.02.009
- [2] G. Bacsó, Cs. Bujtás, Zs. Tuza and V. Voloshin, *New challenges in the theory of hypergraph coloring*, in: Advances in Discrete Mathematics and Applications, Ramanujan Mathematical Society Lecture Notes Series **13** (2010) 45–57.
- [3] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, Ch. Dominic and L. Pushpalatha, *Vertex coloring without large polychromatic stars*, Discrete Math. **312** (2012) 2102–2108. doi:10.1016/j.disc.2011.04.013
- [4] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, M.S. Subramanya and Ch. Dominic, *3-consecutive  $C$ -colorings of graphs*, Discuss. Math. Graph Theory **30** (2010) 393–405. doi:10.7151/dmgt.1502
- [5] Cs. Bujtás and Zs. Tuza, *Maximum number of colors:  $C$ -coloring and related problems*, J. Geom. **101** (2011) 83–97. doi:10.1007/s00022-011-0082-2
- [6] Cs. Bujtás and Zs. Tuza,  *$K_n$ -WORM colorings of graphs: Feasible sets and upper chromatic number*, manuscript in preparation (2015).
- [7] Cs. Bujtás, Zs. Tuza and V.I. Voloshin, *Hypergraph colouring*, Chapter 11 in: L.W. Beineke and R.J. Wilson, (Eds.), Topics in Chromatic Graph Theory, Encyclopedia of Mathematics and Its Applications **156** (Cambridge University Press, 2014), 230–254.
- [8] W. Goddard, K. Wash and H. Xu, *WORM colorings forbidding cycles or cliques*, Congr. Numer. **219** (2014) 161–173.
- [9] W. Goddard, K. Wash and H. Xu, *WORM colorings*, Discuss. Math. Graph Theory **35** (2015) 571–584. doi:10.7151/dmgt.1814
- [10] T.R. Jensen and B. Toft, Graph Coloring Problems (Wiley-Interscience, 1995).
- [11] A. Kündgen and R. Ramamurthi, *Coloring face-hypergraphs of graphs on surfaces*, J. Combin. Theory Ser. B **85** (2002) 307–337. doi:10.1006/jctb.2001.2107

- [12] L. Lovász, *Coverings and colorings of hypergraphs*, Congr. Numer. **8** (1973) 3–12.
- [13] D. Marx, *The complexity of chromatic strength and chromatic edge strength*, Comput. Complexity **14** (2006) 308–340.  
doi:10.1007/s00037-005-0201-2
- [14] F. Maffray and M. Preissmann, *On the NP-completeness of the  $k$ -colorability problem for triangle-free graphs*, Discrete Math. **162** (1996) 313–317.  
doi:10.1016/S0012-365X(97)89267-9
- [15] K. Ozeki, private communication, June 2015.
- [16] V.I. Voloshin, *The mixed hypergraphs*, Comput. Sci. J. Moldova **1** (1993) 45–52.
- [17] V.I. Voloshin, *On the upper chromatic number of a hypergraph*, Australas. J. Combin. **11** (1995) 25–45.
- [18] V.I. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications* (Fields Institute Monographs 17, Amer. Math. Soc., 2002).
- [19] V.I. Voloshin, *Mixed hypergraph coloring web site*.  
<http://spectrum.troy.edu/voloshin/mh.html>
- [20] V.I. Voloshin, private communication, November 2013.

Received 8 September 2015

Revised 27 November 2015

Accepted 27 November 2015