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Note

SPANNING TREES WHOSE STEMS HAVE A BOUNDED NUMBER OF BRANCH VERTICES

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Abstract

Let T be a tree, a vertex of degree one and a vertex of degree at least three is called a leaf and a branch vertex, respectively. The set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the stem of T and denoted by $Stem(T)$. In this paper, we give two sufficient conditions for a connected graph to have a spanning tree whose stem has a bounded number of branch vertices, and these conditions are best possible.

Keywords: spanning tree, stem, branch vertex.

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1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For two vertices u and v of G, the distance between u and v in G is denoted by $d_G(u, v)$. For an integer $l \geq 2$, let $\alpha^l(G)$ denote the number defined by

$$\alpha^l(G) = \max\{|S| : S \subset V(G), d_G(x, y) \geq l \text{ for all distinct } x, y \in S\}.$$

For an integer $k \geq 2$, we define

$$\sigma_k^l(G) = \min\left\{ \sum_{x\in S} \deg_G(x) : S \subset V(G),\ |S| = k,\ d_G(x, y) \geq l \text{ for all distinct } x, y \in S \right\}.$$

For convenience, we define $\sigma_k^l(G) = \infty$ if $\alpha^l(G) < k$. Note that $\alpha^2(G)$ is often written $\alpha(G)$, which is the *independence number* of G, and $\sigma_k^2(G)$ is often written $\sigma_k(G)$, which is the minimum degree sum of k independent vertices.

For a tree T, a vertex of degree at least three is called a *branch vertex*, and a tree having at most one branch vertex is called a *spider*. Many researchers have investigated the independence number conditions and the degree sum conditions for the existence of a spanning tree with bounded number of branch vertices [1, 2, 3, 4, 7, 8]. A vertex of T, which has degree one, is often called a *leaf* of T, and the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the *stem* of T and is denote by $Stem(T)$. A spanning tree with specified stem was first considered in [5], and the following theorem was obtained.

Theorem 1 (Kano, Tsugaki and Yan [5]). *Let $k \geq 2$ be an integer, and G be a connected graph. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has maximum degree at most k.*

The following theorems give two sufficient conditions for a connected graph to have a spanning tree whose stem has a few number of leaves.

Theorem 2 (Tsugaki and Zhang [9]). *Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree whose stem has at most k leaves.*

Theorem 3 (Kano and Yan [6]). *Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has at most k leaves.*

In this paper, we give two sufficient conditions for a connected graph to have a spanning tree whose stem has a bounded number of branch vertices, and these conditions are best possible.

Theorem 4. *Let G be a connected graph and k be a non-negative integer. If one of the following conditions holds, then G has a spanning tree whose stem has at most k branch vertices.*

(i) $\alpha^4(G) \leq k + 2$.
(ii) $\sigma_{k+3}^4 \geq |G| - 2k - 3$.

Before proving Theorem 4, we first show that the conditions of Theorem 4 are best possible. Let $m, k \geq 1$ be integers, and let $D_0, D_1, \ldots, D_{k+2}$ be disjoint copies of K_m. Let $P = z_1 z_2, \ldots, z_{k+1}$ be a path. Let $v_0, v_1, \ldots, v_{k+2}$ be vertices not contained in $D_0 \cup D_1 \cdots \cup D_{k+2}$. Join z_i, v_i to all the vertices of D_i $(1 \leq i \leq k+1)$ by edges, and join z_1, v_0 (z_{k+1}, v_{k+2}) to all vertices of D_0 (D_{k+1}) by edges, respectively. Let G denote the resulting graph. Then G satisfies $\alpha^4(G) = k + 3$ and $\sigma_{k+3}^4(G) = |G| - 2k - 4$. Since for any spanning tree T of G, $z_1, z_2, \ldots, z_{k+1}$ have to be the branch vertices of $Stem(T)$, G has no spanning tree whose stem has at most k branch vertices.

2. PROOF OF THEOREM 4

In order to prove Theorem 4, we need the following lemma.

Lemma 5. *Let T be a tree, and let X be the set of vertices of degree at least* 3. *Then the number of leaves in T is counted as follows:*

$$|Leaf(T)| = \sum_{x \in X} (\deg_T(x) - 2) + 2.$$

***Proof of Theorem* 4.** Assume that G satisfies the conditions in Theorem 4 and does not have a spanning tree whose stem has at most k branch vertices. We choose a tree T whose stem has k branch vertices in G so that

(T1) $|T|$ is as large as possible.
(T2) $|Leaf(Stem(T))|$ is as small as possible subject to (T1).
(T3) $|Stem(T)|$ is as small as possible subject to (T1) and (T2).

For the remaining of the proof v is a vertex of G not in T. By the choice (T1), we have the following claim.

Claim 1. For every $v \in V(G) - V(T)$, $N_G(v) \subseteq Leaf(T) \cup (V(G) - V(T))$.

$Stem(T)$ has k branch vertices. Denote the number of leaves of $Stem(T)$ by l. By Lemma 5, $|Leaf(Stem(T))| = l \geq k + 2$. Let $x_1, x_2, \ldots, x_l$ be the leaves of $Stem(T)$. Since T is not a spanning tree of G, there exist two vertices $v \in V(G) - V(T)$ and $u \in Leaf(T)$ which are adjacent in G.

By the choice (T2), we have the following claim.

Claim 2. $Leaf(Stem(T))$ is an independent set of G.

Proof. Assume that there exists two vertices x_i and x_j of $Leaf(Stem(T))$ adjacent in G. Then add $x_i x_j$ to T. The resulting subgraph of G includes the unique cycle, which contains an edge e_1 of $Stem(T)$ incident with a branch vertex. By removing the edge e_1, we obtain a tree T^* whose stem has at most k branch vertices, $|T^*| = |T|$ and $|Leaf(Stem(T^*))| \leq |Leaf(Stem(T))| - 1$. If $Stem(T^*)$ has $k-1$ branch vertices, then add uv to T^*; we obtain a tree whose stem has at most k branch vertex and the order of the tree is greater than $|T|$, which contradicts the condition (T1). Otherwise, T^* contradicts the condition (T2). Hence $Leaf(Stem(T))$ is an independent set of G. □

Claim 3. For every x_i $(1 \leq i \leq l)$, there exists a vertex $y_i \in Leaf(T)$ adjacent to x_i and $N_G(y_i) \subset Leaf(T) \cup \{x_i\}$.

Proof. It is easy to see that for every leaf y of T adjacent to a leaf of $Stem(T)$ in T, y is not adjacent to any vertex of $V(G) - V(T)$ since otherwise we can add an edge joining y to a vertex of $V(G) - V(T)$ to T.

Suppose that for some $1 \le i \le l$, each leave y_{i_j} of T adjacent to x_i is also adjacent to a vertex $z_{i_j} \in (Stem(T) - \{x_i\})$. Then for every leaf y_{i_j} adjacent to x_i in T, remove the edge $y_{i_j}x_i$ from T and add the edge $y_{i_j}z_{i_j}$. Denote the resulting tree of G by T_1. Then T_1 is a tree whose stem has at most k branch vertices. If x_i is adjacent with a branch of $Stem(T)$, then $Leaf(Stem(T_1)) = Leaf(Stem(T)) - \{x_i\}$, which contradicts the condition (T2). If x_i is not adjacent with a branch of $Stem(T)$, then $Stem(T_1) = Stem(T) - \{x_i\}$, which contradicts the condition (T3). Therefore, the claim holds. □

Claim 4. For any two distinct vertices $y, z \in \{v, y_1, y_2, \dots, y_l\}$, $d_G(y, z) \ge 4$.

Proof. First, we show that $d_G(v, y_i) \ge 4$ for every $1 \le i \le l$. Let P_i be a shortest path connecting v and y_i in G. Then there exists a vertex $s \in V(P_i)$ with $s \in V(Stem(T)) - \{x_i\}$. Otherwise, all vertices of P_i between v and y_i are contained in $Leaf(T) \cup (V(G) - V(T)) \cup \{x_i\}$. Then add P_i to T (if P_i passes through x_i, we just add the segment of P_i between v and x_i) and remove the edges of T joining $V(P_i \cap Leaf(T))$ to $V(Stem(T))$ except the edge y_ix_i. Then resulting tree of G is a tree whose stem has at most k branch vertices and the order of the resulting tree is greater than $|T|$, which contradicts the condition (T1).

Hence, by Claim 3, $d_G(v, s) \ge 2$ and $d_G(s, y_i) \ge 2$. Therefore $d_G(v, y_i) = d_G(v, s) + d_G(s, y_i) \ge 4$.

Next, we show that $d_G(y_i, y_j) \ge 4$ for all $1 \le i < j \le l$. Let P_{ij} be the shortest path connecting y_i and y_j in G. Then there exists a vertex $t \in V(P_{ij})$ with $t \in V(Stem(T)) - \{x_i, x_j\}$. Otherwise, all vertices of P_{ij} between y_i and y_j are contained in $Leaf(T) \cup (V(G) - V(T)) \cup \{x_i, x_j\}$. If P_{ij} passes through x_i (or x_j), then $y_ix_i \in E(P_{ij})$ (or $y_jx_j \in E(P_{ij})$), respectively.

Then add P_{ij} to T and remove the edges of T joining $V(P_{ij} \cap Leaf(T))$ to $V(Stem(T))$ except the edges y_ix_i and y_jx_j. Then the resulting subgraph of G includes the unique cycle, which contains an edge e_2 of $Stem(T)$ incident with a branch vertex. By removing the edge e_2, we obtain a tree T_2 whose stem has at most k branch vertices. If P_{ij} contains a vertex of $V(G) - V(T)$, then the order of T_2 is greater than $|T|$, which contradicts the condition (T1). Otherwise, $|T_2| = |T|$ and $|Leaf(Stem(T_2))| = |Leaf(Stem(T))| - 1$. This contradicts the condition (T2). Hence P_{ij} passes through a vertex s in $Stem(T) - \{x_i, x_j\}$.

Hence, by Claims 1 and 3, $d_G(y_i, s) \ge 2$ and $d_G(s, y_j) \ge 2$. Therefore $d_G(y_i, y_j) = d_G(y_i, s) + d_G(s, y_j) \ge 4$ for $1 \le i < j \le k$. □

By Claim 4, we have $\alpha^4(G) \ge l+1 \ge k+3$, which contradicts the condition (i). Next, by Claim 4, we can obtain Claim 5.

Claim 5. (i) $N_G(v) \cap N_G(y_i) = \emptyset$ for $1 \le i \le l$; and (ii) $N_G(y_i) \cap N_G(y_j) = \emptyset$ for $1 \le i \ne j \le l$.

Claim 6. There exists one vertex $w \in Stem(T)$ with $\deg_{Stem(T)}(w) = 2$.

Proof. Otherwise, all vertices of $Stem(T)$ are leaves or branch vertices of $Stem(T)$. If u is adjacent to a leaf or branch vertex of $Stem(T)$, then we add v to T by adding edge uv; we can get a tree $T + uv$ whose stem has k branch vertices and $|T + uv| = |T| + 1$, which contradicts (T1). □

By Claim 6, we have $|Stem(T)| \geq l + k + 1$.

Denote $Y = \{y_1, y_2, \ldots, y_l\}$. By Claims 1–5, we have

$$N_G(v) \subseteq (V(G) - V(T) - \{v\}) \cup (N_G(v) \cap (Leaf(T) - Y)),$$

$$\bigcup_{i=1}^{k+2} N_G(y_i) \subseteq (Leaf(T) - Y - N_G(v)) \cup \{x_1, \ldots, x_{k+2}\}.$$

Hence by letting $m = |N_G(v) \cap (Leaf(T) - Y)|$, we have

$$\begin{aligned} \deg_G(v) + \sum_{i=1}^{k+2} \deg_G(y_i) &\leq |G| - |T| - 1 + m + |Leaf(T)| - m - l + k + 2 \\ &= |G| - |Stem(T)| - l + k + 1 \\ &\leq |G| - 2l \leq |G| - 2k - 4. \end{aligned}$$

Which contradicts the condition (ii) of theorem.

The theorem follows since we either reach a contradiction to condition (i) or a contradiction to condition (ii). ■

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