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THE TURÁN NUMBER OF THE GRAPH $2P_5$

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Abstract

We give the Turán number $ex(n, 2P_5)$ for all positive integers n, improving one of the results of Bushaw and Kettle [*Turán numbers of multiple paths and equibipartite forests*, Combininatorics, Probability and Computing, 20 (2011) 837–853]. In particular we prove that $ex(n, 2P_5) = 3n - 5$ for $n \geq 18$.

Keywords: forest, tree, Turán number.

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1. Introduction

We consider a simple graph $G = (V(G), E(G))$. Let $ex(n, G)$ denote the maximum number of edges in a graph on n vertices which does not contain G as a subgraph. Let P_i denote a path consisting of i vertices and let mP_i denote m disjoint copies of P_i. By C_q we denote a cycle of order q. For two vertex disjoint graphs G and F, by $G \cup F$ we denote the vertex disjoint union of G and F, by and $G + F$ the join of the graphs, i.e., the graph $G + F$ is obtained from $G \cup F$ by joining all vertices of G to the vertices of F. By $\overline{G}$ we denote the complement of the graph G. Moreover, for $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let $E(A, B) = \{e \in E(G) \mid e \cap A \neq \emptyset \neq e \cap B\}$ and let $G|_A$ denote the subgraph of G induced by A. For $x \in V(G)$ we define the neighbourhood of x in the graph G as $N_G(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$, and the degree of x as $deg_G(x) = |N_G(x)|$. For $x, y \in V(G)$ by the symbol $dist_G(x, y)$ we denote the distance between the vertices x and y in the graph G. The basic notions not defined in this paper one can find in [5].

Theorems 1 and 2 presented below are very useful for study the Turán numbers. In this paper we improve the result presented in Theorem 5 for the case $2P_5$.

Theorem 1 (Faudree and Schelp [3]). *If G is a graph with $|V(G)| = kn + r$ $(0 \leq k, 0 \leq r < n)$ and G contains no P_{n+1}, then $|E(G)| \leq kn(n-1)/2 + r(r-1)/2$ with equality if and only if $G = kK_n \cup K_r$ or $G = tK_n \cup (K_{(n-1)/2} + \overline{K}_{(n+1)/2+(k-t-1)n+r})$ for some $0 \leq t < k$, where n is odd, and $k > 0, r = (n \pm 1)/2$.*

Corollary 2. *Let n be a positive integer and $n \equiv r (\mathrm{mod}\ 4)$. Then $ex(n, P_5) = \frac{3n+r(r-4)}{2}$.*

Theorem 3 (Erdős, Gallai [2]). *Suppose that $|V(G)| = n$. If the following inequality*

$$\frac{(n-1)(l-1)}{2} + 1 \leq |E(G)|$$

is satisfied for some $l \in \mathbb{N}$, then there exists a cycle $C_q \subset G$ for some $q \geq l$.

Gorgol [4] studied the Turán number for disjoint copies of the path of order 3. Moreover, Gorgol proved more general results concerning the properties of some extremal Turán graphs for disjoint copies of graphs.

Theorem 4 (Gorgol [4]). *Let n be a positive integer. Then*

$$ex(n, 2P_3) = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1, \text{ for } n \geq 9,$$
$$ex(n, 3P_3) = \left\lfloor \frac{n}{2} \right\rfloor + 2n - 4, \text{ for } n \geq 14.$$

Bushaw and Kettle [1] extended the above result as follows.

Theorem 5 (Bushaw and Kettle [1]). *Let n be a positive integer. Then*

$$ex(n, kP_3) = \left\lfloor \frac{n-k+1}{2} \right\rfloor + (n-k+1)(k-1) + \binom{k-1}{2}, \text{ for } n \geq 7k,$$
$$ex(n, kP_t) = \left(n - k\left\lfloor \frac{t}{2} \right\rfloor + 1\right)\left(k\left\lfloor \frac{t}{2} \right\rfloor - 1\right) + \binom{k\left\lfloor \frac{t}{2} \right\rfloor - 1}{2} + \epsilon,$$

for $n \geq 2t\left(1 + k\left(\left\lceil \frac{t}{2} \right\rceil + 1\right)\binom{t}{\lfloor \frac{t}{2} \rfloor}\right)$, where $\epsilon = 1$ for odd t and $\epsilon = 0$ for even t.

In particular, Bushaw and Kettle [1] counted $ex(n, 2P_5)$ for the case $n \geq 810$. We show that their results can be extended for all positive integers n.

2. Results

We prove the following theorem, extending the result of Bushaw and Kettle for $2P_5$ to all positive integers n.

Theorem 6. *Let n be a positive integer. Then*

$$ex(n, 2P_5) = 3n - 5 \quad for \quad n \geq 18.$$

Moreover,

$$ex(n, 2P_5) = \binom{n}{2} \text{ for } n \leq 9, \quad ex(10, 2P_5) = 36,$$

$$ex(11, 2P_5) = 37, \quad ex(12, 2P_5) = 39, \quad ex(13, 2P_5) = ex(14, 2P_5) = 42,$$

$$ex(15, 2P_5) = 43, \quad ex(16, 2P_5) = 45, \quad ex(17, 2P_5) = 48.$$

Proof. First note that the graph $H = K_3 + (K_2 \cup \overline{K}_{n-5})$ is of order $3n - 5$ and it does not contain $2P_5$ as a subgraph for $n \geq 18$.

The extremal graph K_n gives the lower and upper bounds for $ex(n, 2P_5)$ in the case $n \leq 9$. For $9 < n \leq 17$ the extremal graphs H are gathered in Table 1 in the second column.

The $|E(H)|$ gives the lower bounds on $ex(n, 2P_5)$ for respective n. Therefore, $ex(n, 2P_5) \geq 3n - 5 + \rho$ with $\rho = |E(H)| - (3n - 5)$ for H presented in Table 1 and $\rho = 0$ for $n \geq 18$.

n	H	q	$\|E(G)\| = \|E(H)\| + 1$	ρ	$3n - 4$
10	$K_9 \cup K_1$	9	37	11	26
11	$K_9 \cup K_2$	8, 9	38	9	29
12	$K_9 \cup K_3$	8, 9	40	8	32
13	$K_9 \cup K_4$	8, 9	43	8	35
14	$K_9 \cup K_4 \cup K_1$	7, 8, 9	43	5	38
15	$K_9 \cup K_4 \cup K_2$	7, 8, 9	44	3	41
16	$K_9 \cup K_4 \cup K_3$	7, 8, 9	46	2	44
17	$K_9 \cup 2K_4$	7, 8, 9	49	2	47

Table 1. The lower bound for the cycle C_q. H does not contain $2P_5$ as a subgraph.

Now we would like to prove the upper bound $ex(n, 2P_5) \leq 3n - 5 + \rho$. Let us assume that there exists a graph G such that $|V(G)| = n$, $|E(G)| = 3n - 4 + \rho$ and without a subgraph $2P_5$. Note that

$$\frac{(n-1)(l-1)}{2} + 1 \leq 3n - 4 + \rho$$

gives us

$$l \leq 7 - \frac{4-2\rho}{n-1}.$$

We use Theorem 2 to count the length of the cycle C_q for each of the cases. For $10 \leq n \leq 17$ see the third column in Table 1. For $n \geq 18$ we have $\rho = 0$ and the graph G contains a cycle C_q, $q \geq 6$. Let $F = G - V(C_q)$. Let V_0 be the set of vertices from F which does not create an edge with any vertex of the cycle C_q, i.e., $V_0 = \{f \in V(F) \mid N_G(f) \cap V(C_q) = \emptyset\}$. Let the set of vertices $V_{>0} = V(F) - V_0$. Next, let $V_0 = V_{0+} \cup V_{0-}$, where

$$V_{0+} = \{u \in V_0 \mid N(u) \cap V_{>0} \neq \emptyset\} \text{ and } V_{0-} = V_0 - V_{0+}.$$

Let us consider the following cases.

Case 1. Let $q \geq 10$. Then G contains $2P_5$, a contradiction.

Case 2. Let $q = 9$. Note that $N(f) \cap V(C_9) = \emptyset$ for each $f \in V(F)$, since otherwise we obtain $2P_5$ and we get a contradiction. Thus the minimum number of edges in F is equal to $3n - 4 + \rho - \binom{9}{2} = 3n - 40 + \rho$. By Corollary 2

$$ex(|V(F)|, P_5) = ex(n-9, P_5) = \frac{3(n-9) + r(r-4)}{2}, \text{ where } n - 9 \equiv r (\text{mod } 4).$$

Thus $ex(n-9, P_5) < 3n - 40 + \rho$ for $n > \frac{53+r(r-4)-2\rho}{3}$. So P_5 is a subgraph of F and we get $2P_5$ in G for $n > \frac{53+r(r-4)-2\rho}{3}$, a contradiction (see Table 1).

Case 3. Let $q = 8$. Note that $V_{>0} \neq \emptyset$, since otherwise $|E(G)| \leq \binom{8}{2} + ex(n-8, P_5) < 3n - 4 + \rho$ for $n \geq \frac{40+r(r-4)-2\rho}{3}$, and we get a contradiction for all $n \geq 10$. Note that $N_G(f) \subseteq V(C_8)$ for each $f \in V_{>0}$, otherwise $2P_5$ is a subgraph of G and we get a contradiction.

Case 3.1. Suppose that there exists a vertex f in $V_{>0}$ such that $|N_G(f) \cap V(C_8)| \in \{3, 4\}$. Then without loss of generality we have three subcases illustrated in Figure 1. The first subcase with $N_G(f) = \{0, 2, 4, 6\}$, the second one with $N_G(f) = \{0, 2, 6\}$, and the third with $N_G(f) = \{0, 2, 5\}$.

Note that the dotted lines denote the edges in $E(\overline{G})$, since otherwise we get a cycle longer than C_8. For the first case and the second one we have $|V_{>0}| = 1$. It follows by the assumption that $2P_5$ is not a subgraph of G. Similarly, for the third case we get $N_G(f') \cap V(C_8) = \{5\}$ for each $f' \in V_{>0} - \{f\}$. We have $|V_0| = n - 8 - |V_{>0}|$ and

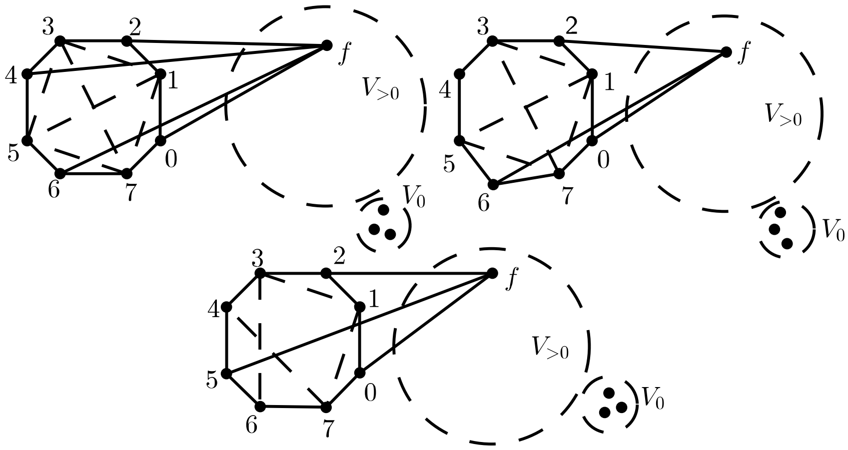

Figure 1. Graphs with the cycle C_8.

$$|E(G|_{V_0})| \geq 3n - 4 + \rho - \binom{8}{2} - 3 - |V_{>0}| + 6 = 3n - 29 + \rho - |V_{>0}|.$$

Let $n - 8 - |V_{>0}| \equiv r(\text{mod } 4)$. Then by Corollary 2, for $n > 12 - \frac{|V_{>0}|+r(4-r)+2\rho}{3}$, we have

$$ex(n - 8 - |V_{>0}|, P_5) = \frac{3(n - 8 - |V_{>0}|) + r(r-4)}{2} < 3n - 30 - |V_{>0}| + \rho.$$

So we get P_5 in $G|_{V_0}$ and P_5 in C_8. Hence for $n \geq 10$ we have $2P_5$ in G, a contradiction.

Case 3.2. Suppose that $|N_G(f) \cap V(C_8)| \in \{1, 2\}$ for each $f \in V(F)$. Then we have the inequality $|E(V(C_8), V_{>0})| \leq 2|V_{>0}|$. Let $n - 8 - |V_{>0}| \equiv r(\text{mod } 4)$. Thus by Corollary 2 and since $|V_{>0}| \leq n - 8$ we get

$$\begin{aligned} |E(G)| &\leq \binom{8}{2} + 2|V_{>0}| + ex(n - 8 - |V_{>0}|, P_5) = 16 + \frac{|V_{>0}| + r(r-4) + 3n}{2} \\ &\leq 2n + 12 + \frac{r(r-4)}{2} < 3n - 4 + \rho \end{aligned}$$

for $n > 16 - \frac{r(4-r)}{2} - \rho$, a contradiction (see Table 1).

Case 4. Let $q = 7$. Note that $V_{>0} \neq \emptyset$, since otherwise $|E(G)| \leq \binom{7}{2} + ex(n-7, P_5) < 3n - 4 + \rho$ for $n \geq 10$, and we get a contradiction.

Case 4.1. Suppose that $\{f, g\} \in E(G|_{V_{>0}})$ for some $f, g \in V_{>0}$ (see Figure 2).

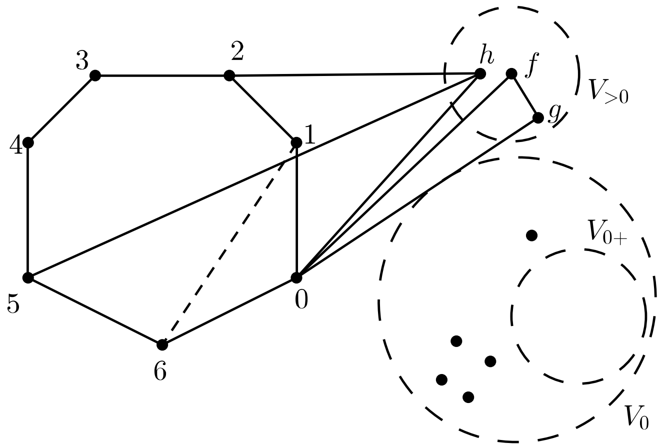

Figure 2. A graph G with the cycle C_7.

Then $V_{0+} = \emptyset$, since otherwise we get $2P_5$ in G. Moreover $|N_G(x) \cap V(C_7)| \leq 2$ for $x = f, g$, and $|N_G(h) \cap V(C_7)| \leq 3$ for all $h \in V_{>0}$, since otherwise we get a longer cycle than C_7. Note that $F|_{V_{>0}}$ contains exactly one edge, otherwise we get $2P_5$ in G. Let $n - 7 - |V_{>0}| \equiv r(\text{mod } 4)$. Suppose that $|N_G(h) \cap V(C_7)| = 3$ for some $h \in V_{>0}$ (see Figure 2). Then by Corollary 2 we obtain the following upper bound on the number of edges in G

$$|E(G)| \leq \binom{7}{2} + 3|V_{>0}| - 8 + ex(|V_0|, P_5) = \frac{1}{2}(21 + 3|V_{>0}| + 3n + r(r-4)) - 8,$$

where the term 8 depends on the number of edges between the cycle C_7 and $V_{>0}$. If $|N_G(h) \cap V(C_7)| < 3$ for all $h \in V_{>0}$, then we get

$$|E(G)| \leq \binom{7}{2} + 2|V_{>0}| - 2 + 1 + ex(|V_0|, P_5) = \frac{1}{2}(21 + |V_{>0}| + 3n + r(r-4)) - 1.$$

Hence for both cases we have $|E(G)| < 3n - 4 + \rho$ for all $n \geq 10$, a contradiction.

Case 4.2. Suppose that $V_{>0}$ is an independent set. We have to consider two subcases.

Case 4.2.1. Let $V_{0+} \neq \emptyset$ and let $|V_0 - V_{0+}| \equiv r(\text{mod } 4)$. Without loss of generality there exist $f \in V_{>0}$ and $f_1 \in V_{0+}$ such that $\{f, 0\}, \{f, f_1\} \in E(G)$ (see Figure 3). Note that V_{0+} is an independent set, since otherwise we get $2P_5$ in G. Similarly, f_1 has a unique neighbour in $V(F)$ and $N_G(g) \cap V(C_7) \subseteq \{0, 2, 5\}$ for all $g \in V_{>0} - \{f\}$. Hence $|E(V_{>0}, V_{0+})| = |V_{0+}|$. Thus by Corollary 2

$$\begin{aligned} |E(G)| &\leq \binom{7}{2} - s + 3|V_{>0}| + |V_{0+}| + ex(|V_0 - V_{0+}|, P_5) \\ &= \binom{7}{2} - s + 3(n-7) - \frac{3|V_0|}{2} - \frac{|V_{0+}| - r(r-4)}{2} \end{aligned}$$

where $s = 2|V_{>0}|$ for $|V_{>0}| \in \{1, 2\}$, and $s = 5$ for $|V_{>0}| \geq 3$. The parameter s depends on the number of edges between the cycle C_7 and $V_{>0}$. Then the below inequality

$$|E(G)| < 3n - 4 + \rho$$

holds for $2|V_{0+}| + \frac{3}{2}|V_0 - V_{0+}| > 4 - s - \rho + \frac{r(r-4)}{2}$ and we obtain a contradiction in this case.

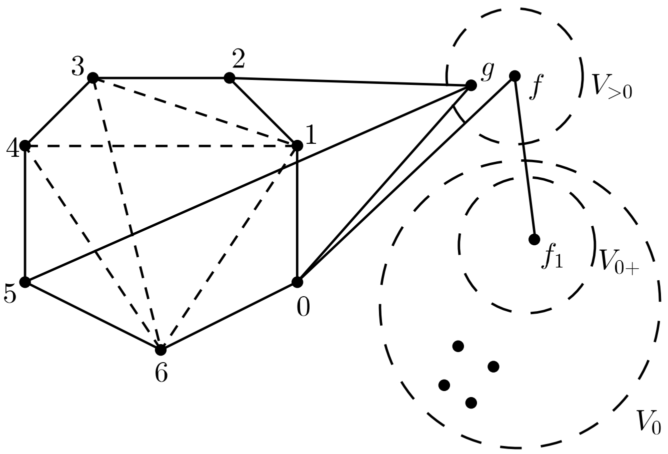

Figure 3. A graph G with the cycle C_7 and $V_{0+} \neq \emptyset$.

Case 4.2.2. Let $V_{0+} = \emptyset$. Without loss of generality we assume that $\{f, 0\} \in E(G)$ for some $f \in V_{>0}$ (see Figure 4). Let $|V_0| \equiv r(\text{mod } 4)$.

If $|N_G(f) \cap V(C_7)| = 3$ then by Corollary 2 we obtain the following inequality

$$\begin{aligned}|E(G)| &\leq \binom{7}{2} - 5 + 3|V_{>0}| + ex(|V_0|, P_5) \\ &= \binom{7}{2} - 5 + 3(n-7) - \frac{3|V_0| - r(r-4)}{2} < 3n - 4 + \rho.\end{aligned}$$

It follows by $|V_0| \geq 0$. So in this case we obtain a contradiction. Similarly, if $|N_G(f) \cap V(C_7)| = 2$ and $|N_G(g) \cap V(C_7)| \leq 2$ for all $g \in V_{>0}$, then for $n \geq 10$ we obtain the following inequality

$$\begin{aligned}|E(G)| &\leq \binom{7}{2} - 2 + 2|V_{>0}| + ex(|V_0|, P_5) \\ &= \binom{7}{2} - 2 + 2(n-7) - \frac{|V_0| - r(r-4)}{2} < 3n - 4 + \rho.\end{aligned}$$

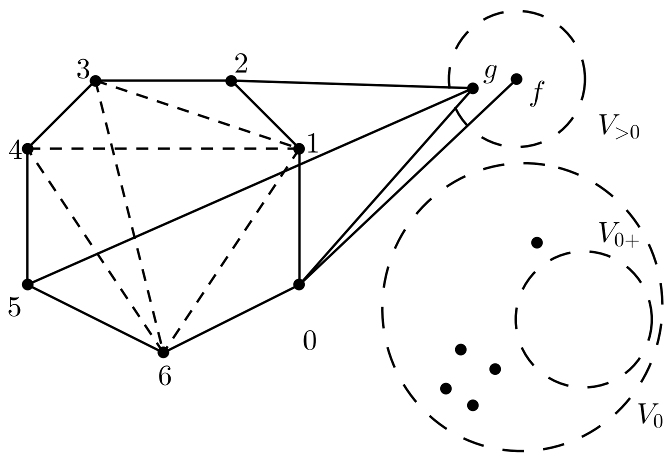

Figure 4. A graph G with the cycle C_7 and $V_{0+} = \emptyset$.

So in this case we get a contradiction. Finally, if $|N_G(g) \cap V(C_7)| = 1$ for all $g \in V_{>0}$, then for $n \geq 10$ we obtain the following inequality

$$|E(G)| \leq \binom{7}{2} + |V_{>0}| + ex(|V_0|, P_5) = \binom{7}{2} + (n-7) + \frac{|V_0| + r(r-4)}{2} < 3n-4+\rho.$$

Thus we get a contradiction.

Case 5. Let $q = 6$. Note that $V_{>0} \neq \emptyset$, since otherwise, by Corollary 2, $|E(G)| \leq \binom{6}{2} + ex(n-6, P_5) < 3n-4$ for $n \geq 7$, and we get a contradiction. Let $\{f, 0\} \in E(G)$ for some $f \in V_{>0}$.

Case 5.1. Suppose that there exists a path P_3 with the consecutive vertices (f, f_1, f_2) in $F|_{V_{>0}}$ (see Figure 5). By $\{f, 0\} \in E(G)$, the second end of the path, i.e., f_2 can be adjacent only to the vertex 0 of $V(C_6)$, since otherwise we get a cycle longer than C_6. So

$$N_G(f) \cap V(C_6) = N_G(f_2) \cap V(C_6) = \{0\},$$

and

$$N_G(f_1) \cap V(C_6) \subseteq \{0, 3\}.$$

Moreover, $N_G(f_3) = \{3\}$, for all $f_3 \in V_{>0} - \{f, f_1, f_2\}$, otherwise we get $2P_5$ in G. This implies that $V_{0+} = \emptyset$. Hence $E(V_{>0}, V_0) = \emptyset$ and

$$|E(G)| \leq \binom{6}{2} + 3 + |V_{>0}| + ex(|V_0|, P_5) < 3n-4 \text{ for } n \geq 9,$$

and we obtain a contradiction in this situation. Thus $G|_{V_{>0}}$ does not contain P_3.

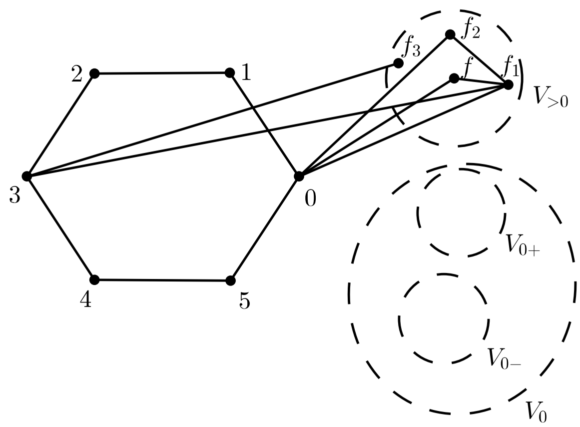

Figure 5. A graph G with the cycle C_6 and a path P_3 in $G|_{V_{>0}}$.

Case 5.2. Assume that there exists an edge $\{f, f_1\}$ in $G|_{V_{>0}}$ (see Figure 6). Note that $|N_G(f_1) \cap V(C_6)| \leq 2$, since otherwise we get a longer cycle. If there exists a second edge $\{f_2, f_3\}$ in the graph $G|_{V_{>0}}$, then $\{f_2, f_3\} \neq \{f, f_1\}$ and without loss of generality we can assume that $N_G(f_2) \cap V(C_6) = N_G(f_3) \cap V(C_6) = \{4\}$, since otherwise we obtain $2P_5$. Similarly, if there exists a third edge $\{f_4, f_5\}$ in $G|_{V_{>0}}$, then $N_G(f_4) \cap V(C_6) = N_G(f_5) \cap V(C_6) = \{2\}$. So there exist at most three independent edges in the graph. Similarly, we note that $|E(V_{>0}, V_0)| \leq |V_{0+}| + 1$, $|E(V_{0+}, V_{0-})| = \emptyset$ and V_{0+} is an independent set, since otherwise we obtain $2P_5$. So by Corollary 2

$$|E(G)| \leq \binom{6}{2} - 2 + 3|V_{>0}| + |V_{0+}| + ex(|V_{0-}|, P_5) < 3n - 4.$$

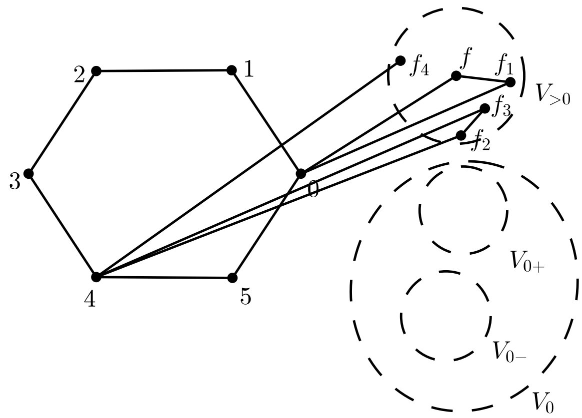

Figure 6. A graph G with the cycle C_6 and an edge in $G|_{V_{>0}}$.

The term 2 depends on the number of edges between the cycle C_6 and $V_{>0}$. We obtain a contradiction in this case.

Case 5.3. So now assume that $V_{>0}$ is an independent set. Let $V_{0-} = V_{0-+} \cup V_{0--}$, where $V_{0-+} = \{b \in V_{0-} \mid N_G(b) \cap V_{0+} \neq \emptyset\}$ and $V_{0--} = V_{0-} - V_{0-+}$.

Case 5.3.1. Let $V_{0-+} \neq \emptyset$. This situation is presented in Figure 7. Let (f, a, b) be a path in F such that $a \in V_{0+}$ and $b \in V_{0-+}$. Note that $dist_G(f, b) = 2$.

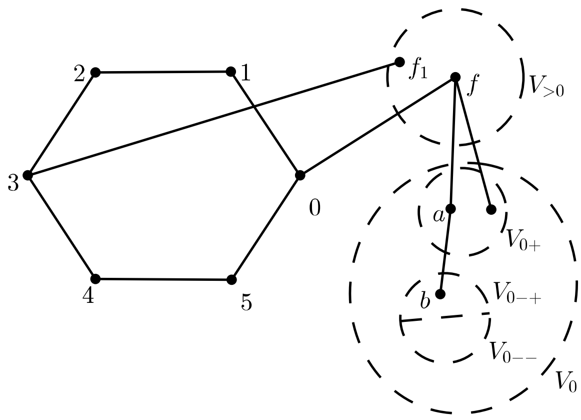

Figure 7. A graph G with the cycle C_6 and the independent set $V_{>0}$.

By the partition of the set $V_{0-} = V_{0--} \cup V_{0-+}$ we get $V_{0+} \neq \emptyset$ and $V_{0-} \neq \emptyset$. Note that V_{0+} is an independent set and $deg_G(x) = 1$ for each $x \in V_{0+} - \{a\}$, since otherwise we get $2P_5$. Recall that $V_{>0}$ is an independent set. Moreover, $N_G(f_1) = \{3\}$ and so $deg_G(f_1) = 1$ for all $f_1 \in V_{>0} - \{f\}$. Thus $N_G(x) = \{f\}$ for all $x \in V_{0+}$ and $|E(V_{>0}, V_0)| = |V_{0+}|$. Similarly, $|E(V_{0+}, V_{0-+})| = |V_{0-+}|$ and $E(V_{0-+}, V_{0--}) = \emptyset$. By Corollary 2 we obtain

$$\begin{aligned} |E(G)| &\leq \binom{6}{2} + |V_{>0}| + |V_{0+}| + 1 + |V_{0-+}| + ex(|V_{0--}|, P_5) \\ &\leq \binom{6}{2} + (n-6) + 1 + \frac{1}{2}|V_{0--}| \leq \frac{3}{2}n + 7 < 3n - 4, \text{ for } n > 7. \end{aligned}$$

So we get a contradiction.

Case 5.3.2. Let $V_{0-+} = \emptyset$. We have three subcases.

Case 5.3.2.1. Let $G|_{V_{0+}}$ contain an edge $\{a, b\}$ (see Figure 8). Thus we have $N_G(f_1) \cap V_{0+} = \emptyset$ for all $f_1 \in V_{>0} - \{f\}$, since otherwise we get $2P_5$ or a longer cycle. Moreover, $G|_{V_{0+}}$ contains at most one edge and $N_G(f_1) \cap V(C_6) = \{3\}$ for all $f_1 \in V_{>0} - \{f\}$. Note that $|V_{0-}| \leq n - 9$. Then by Corollary 2 we obtain

$$\begin{aligned} |E(G)| &\leq \binom{6}{2} + |V_{>0}| + |V_{0+}| + 2 + ex(|V_{0-}|, P_5) \\ &\leq \binom{6}{2} + (n-6) + 2 + \frac{|V_{0-}|}{2} < 3n - 4 \text{ for } n \geq 8. \end{aligned}$$

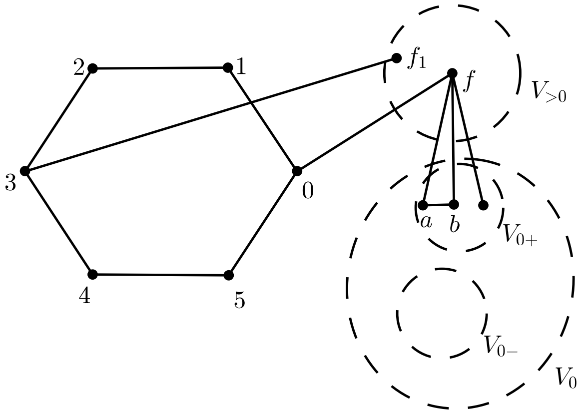

Figure 8. A graph G with the cycle C_6, the independent set $V_{>0}$ and not independent set V_{0+}.

So we obtain a contradiction.

Case 5.3.2.2. Let $V_{0+} \neq \emptyset$ be an independent set (see Figure 9). Note that either for each vertex $g \in V_{0+}$ we have $|N_G(g) \cap V_{>0}| = 1$ or $|N_G(g) \cap V_{>0}| = 2$ and $|V_{0+}| = 1$, since otherwise we obtain a longer cycle or $2P_5$.

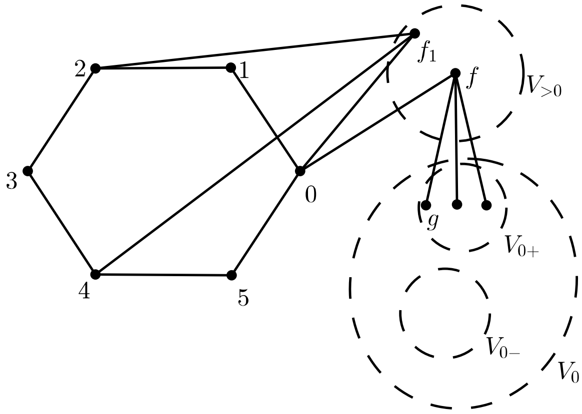

Figure 9. A graph G with the cycle C_6 and the independent sets $V_{>0},\ V_{0+}$.

So by Corollary 2, in both cases, the below inequality

$$|E(G)| \leq \binom{6}{2} + 3|V_{>0}| + |V_{0+}| + ex(|V_{0-}|, P_5) \leq 3n - 3 - 2|V_{0+}| - \frac{3}{2}|V_{0-}| < 3n - 4$$

leads to

$$2|V_{0+}| + \frac{3}{2}|V_0| > 1$$

which is true with assumption $V_{0+} \neq \emptyset$, and we obtain a contradiction.

Case 5.3.2.3. Let $V_{0+} = \emptyset$ (see Figure 10). Let us recall that $V_{>0}$ is an independent set. By calculation of the total number of edges in a graph G we obtain that

$$|E(G)| \leq \binom{6}{2} - 2 + 3|V_{>0}| + ex(|V_0|, P_5) \leq 13 + 3|V_{>0}| + \frac{3}{2}|V_0| < 3n - 4.$$

The term 2 depends on the number of edges between the cycle C_6 and $V_{>0}$. So again we obtain a contradiction.

In summary we see that for all cases we get $2P_5$ as a subgraph of G. So our proof is completed. ■

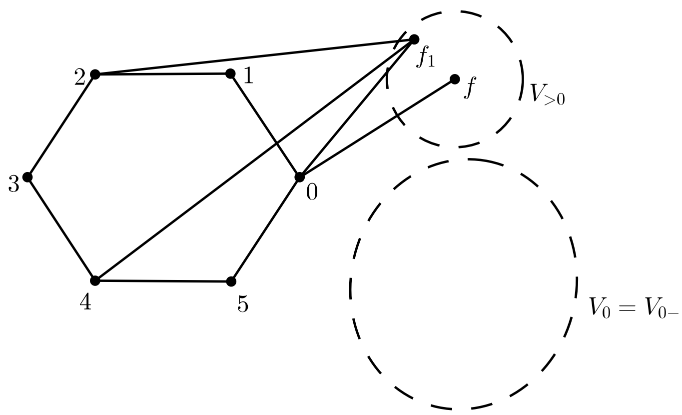

Figure 10. A graph G with the cycle C_6, the independent set $V_{>0}$, and $V_{0+} = \emptyset$.

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