

THE VERTEX-RAINBOW INDEX OF A GRAPH

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Abstract

The k -rainbow index $rx_k(G)$ of a connected graph G was introduced by Chartrand, Okamoto and Zhang in 2010. As a natural counterpart of the k -rainbow index, we introduce the concept of k -vertex-rainbow index $rvx_k(G)$ in this paper. In this paper, sharp upper and lower bounds of $rvx_k(G)$ are given for a connected graph G of order n , that is, $0 \leq rvx_k(G) \leq n - 2$. We obtain Nordhaus-Gaddum results for 3-vertex-rainbow index of a graph G of order n , and show that $rvx_3(G) + rvx_3(\bar{G}) = 4$ for $n = 4$ and $2 \leq rvx_3(G) + rvx_3(\bar{G}) \leq n - 1$ for $n \geq 5$. Let $t(n, k, \ell)$ denote the minimal size of a connected graph G of order n with $rvx_k(G) \leq \ell$, where $2 \leq \ell \leq n - 2$ and $2 \leq k \leq n$. Upper and lower bounds on $t(n, k, \ell)$ are also obtained.

Keywords: vertex-coloring, connectivity, vertex-rainbow S -tree, vertex-rainbow index, Nordhaus-Gaddum type.

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1. INTRODUCTION

The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [6]. Readers can see [6, 7, 9] for details. Consider an edge-coloring (not necessarily proper) of a graph $G = (V, E)$. We say that a path of G is *rainbow*, if no two edges on the path have the same color. An edge-colored graph G is *rainbow connected* if every two vertices are connected by a rainbow path. The minimum number of colors required to rainbow color a graph G is called *the rainbow connection*

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number, denoted by $rc(G)$. In [15], Krivelevich and Yuster proposed a similar concept, the concept of vertex-rainbow connection. A vertex-colored graph G is *vertex-rainbow connected* if every two vertices are connected by a path whose internal vertices have distinct colors, and such a path is called a *vertex-rainbow path*. The *vertex-rainbow connection number* of a connected graph G , denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G vertex-rainbow connected. For more results on the rainbow connection and vertex-rainbow connection, we refer to the survey paper [20] of Li, Shi and Sun and a new book [21] of Li and Sun. All graphs considered in this paper are finite, undirected and simple. We follow the notation and terminology of Bondy and Murty [2], unless otherwise stated.

For a nontrivial graph G and a set $S \subseteq V(G)$ of at least two vertices, a *Steiner tree connecting S* (also called an *S -Steiner tree* or simply *S -tree*) is a subtree T of G such that $S \subseteq V(T)$. Let there be given a vertex coloring c of G that may or may not be proper. An S -tree is a *vertex-rainbow S -tree* if no two vertices in $V(T) \setminus S$ are assigned the same color. For a fixed integer k satisfying $2 \leq k \leq |V(G)|$, the coloring c is called a *k -vertex-rainbow coloring* if every k -subset S of $V(G)$ has a vertex-rainbow S -tree. If such c exists, then G is *vertex-rainbow k -tree-connected*. The minimum number of colors that are needed in a k -rainbow coloring of G is called the *k -rainbow index* of G , denoted by $rx_k(G)$. When $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of G . For more details on k -rainbow index, we refer to [3, 4, 8, 12, 17, 18].

Chartrand, Okamoto and Zhang [9] obtained the following result.

Theorem 1.1 [9]. *For every integer $n \geq 6$, $rx_3(K_n) = 3$.*

As a natural counterpart of the k -rainbow index, we introduce the concept of k -vertex-rainbow index $rvx_k(G)$ in this paper. For $S \subseteq V(G)$ and $|S| \geq 2$, an S -Steiner tree T is said to be a *vertex-rainbow S -tree* or a *vertex-rainbow tree connecting S* if the vertices of $V(T) \setminus S$ have distinct colors. For a fixed integer k with $2 \leq k \leq n$, a vertex-coloring c of G is called a *k -vertex-rainbow coloring* if for every k -subset S of $V(G)$ there exists a vertex-rainbow S -tree. In this case, G is called *vertex-rainbow k -tree-connected*. The minimum number of colors that are needed in a k -vertex-rainbow coloring of G is called the *k -vertex-rainbow index* of G , denoted by $rvx_k(G)$. When $k = 2$, $rvx_2(G)$ is nothing new but the vertex-rainbow connection number $rvc(G)$ of G . It follows, for every nontrivial connected graph G of order n , that

$$rvx_2(G) \leq rvx_3(G) \leq \cdots \leq rvx_n(G).$$

Let G be the graph in Figure 1(a). We give a vertex-coloring c of the graph G shown in Figure 1(b). If $S = \{v_1, v_2, v_3\}$ (see Figure 1(c)), then the tree T induced by the edges in $\{v_1u_1, v_2u_1, u_1u_4, u_4v_3\}$ is a vertex-rainbow S -tree. If

$S = \{u_1, u_2, v_3\}$, then the tree T induced by the edges in $\{u_1u_2, u_2u_4, u_4v_3\}$ is a vertex-rainbow S -tree. One can easily check that there is a vertex-rainbow S -tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, the vertex-coloring c of G is a 3-vertex-rainbow coloring. Thus G is vertex-rainbow 3-tree-connected.

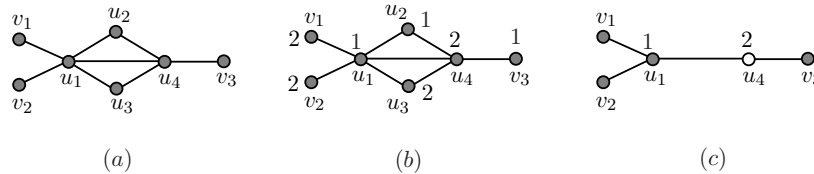


Figure 1. Graphs for the basic definitions.

In some cases $rvx_k(G)$ may be much smaller than $rx_k(G)$. For example, $rvx_k(K_{1,n-1}) = 1$ while $rx_k(K_{1,n-1}) = n - 1$, where $2 \leq k \leq n$. On the other hand, in some other cases, $rx_k(G)$ may be much smaller than $rvx_k(G)$. For $k = 3$, we take n vertex-disjoint cliques of order 4 and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph G has n cut-vertices and hence $rvx_3(G) \geq n$. In fact, $rvx_3(G) = n$ by coloring only the cut-vertices with distinct colors. On the other hand, from Theorem 1.1, it is not difficult to see that $rx_3(G) \leq 9$. Just color the edges of the K_n with, say, color 1, 2, 3 and color the edges of each clique with the colors 4, 5, ..., 9. One can see that the rainbow index and vertex-rainbow index are generalizations of rainbow connection number and vertex-rainbow connection number, respectively.

Steiner tree is used in computer communication networks (see [14]) and optical wireless communication networks (see [13]). As natural combinatorial concepts, the rainbow index and the vertex-rainbow index can also find applications in networking. Suppose we want to route messages in a cellular network in such a way that each link or node on the route between more than two vertices is assigned with a distinct channel. The minimum number of channels that we have to use is exactly the rainbow index and vertex-rainbow index of the underlying graph.

The Steiner distance $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S . Such a tree is an S -Steiner tree. The Steiner k -diameter $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G . Then, it is easy to see the following results.

Proposition 1.2. *Let G be a nontrivial connected graph of order n . Then $rvx_k(G) = 0$ if and only if $sdiam_k(G) = k - 1$.*

Proposition 1.3. *Let G be a nontrivial connected graph of order n ($n \geq 5$), and let k be an integer with $2 \leq k \leq n$. Then*

$$0 \leq rvx_k(G) \leq n - 2.$$

Proof. We only need to show $\text{rvx}_k(G) \leq n-2$. Since G is connected, there exists a spanning tree of G , say T . We give the internal vertices of the tree T different colors. Since T has at least two leaves, we must use at most $n-2$ colors to color all the internal vertices of the tree T . Color the leaves of the tree T with the used colors arbitrarily. Note that such a vertex-coloring makes T vertex-rainbow k -tree-connected. Then $\text{rvx}_k(T) \leq n-2$ and hence $\text{rvx}_k(G) \leq \text{rvx}_k(T) \leq n-2$, as desired. ■

Observation 1.4. Let $K_{s,t}$, K_{n_1,n_2,\dots,n_r} , W_n and P_n denote the complete bipartite graph, complete multipartite graph, wheel and path, respectively. Then

- (1) for integers s and t with $s \geq 2, t \geq 1$, $\text{rvx}_k(K_{s,t}) = 1$ for $2 \leq k \leq \max\{s, t\}$,
- (2) for $r \geq 2$, $\text{rvx}_k(K_{n_1,n_2,\dots,n_r}) = 1$ for $2 \leq k \leq \max\{n_i \mid 1 \leq i \leq r\}$,
- (3) for $n \geq 5$, $\text{rvx}_k(W_n) = 1$ for $2 \leq k \leq n-3$,
- (4) for $n \geq 4$, $\text{rvx}_k(P_n) = n-2$ for $2 \leq k \leq n-2$; $\text{rvx}_{n-1}(P_n) = 1$; $\text{rvx}_n(P_n) = 0$.

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n and $\mathcal{G}(n, m)$ the subclass of $\mathcal{G}(n)$ containing graphs with n vertices and m edges. Given a graph parameter $f(G)$ and a positive integer n , the Nordhaus-Gaddum (**N-G**) Problem is to determine sharp bounds for: (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [1] by Aouchiche and Hansen.

Chen, Li and Lian [10] gave sharp lower and upper bounds of $\text{rx}_k(G) + \text{rx}_k(\overline{G})$ for $k = 2$. In [11], Chen, Li and Liu obtained sharp lower and upper bounds of $\text{rvx}_k(G) + \text{rvx}_k(\overline{G})$ for $k = 2$. In Section 2, we investigate the case $k = 3$ and prove the following lower and upper bounds on $\text{rvx}_3(G) + \text{rvx}_3(\overline{G})$.

Theorem 1.5. Let G be a graph of order n such that G and \overline{G} are connected graphs. If $n = 4$, then $\text{rvx}_3(G) + \text{rvx}_3(\overline{G}) = 4$. If $n \geq 5$, then we have

$$2 \leq \text{rvx}_3(G) + \text{rvx}_3(\overline{G}) \leq n - 1.$$

Moreover, the bounds are sharp.

Let $s(n, k, \ell)$ denote the minimal size of a connected graph G of order n with $\text{rx}_k(G) \leq \ell$, where $2 \leq \ell \leq n-1$ and $2 \leq k \leq n$. Schiermeyer [24] focused on the case $k = 2$ and gave exact values and upper bounds for $s(n, 2, \ell)$. Later, Li, Li, Sun and Zhao [16] improved Schiermeyer's lower bound of $s(n, 2, 2)$ and get a lower bound of $s(n, 2, \ell)$ for $3 \leq \ell \leq \lceil \frac{n}{2} \rceil$.

In Section 3, we study the vertex case. Let $t(n, k, \ell)$ denote the minimal size of a connected graph G of order n with $\text{rvx}_k(G) \leq \ell$, where $2 \leq \ell \leq n-2$ and $2 \leq k \leq n$. We obtain the following result in Section 3.

Theorem 1.6. *Let k, n, ℓ be three integers with $2 \leq \ell \leq n - 3$ and $2 \leq k \leq n$. If n and ℓ have different parity, then*

$$n - 1 \leq t(n, k, \ell) \leq n - 1 + \frac{n - \ell - 1}{2}.$$

If n and ℓ have the same parity, then

$$n - 1 \leq t(n, k, \ell) \leq n - 1 + \frac{n - \ell}{2}.$$

2. NORDHAUS-GADDUM RESULTS

To begin with, we have the following result.

Proposition 2.1. *Let G be a connected graph of order n . Then the following are equivalent.*

- (1) $\text{rvx}_3(G) = 0$;
- (2) $\text{sdiam}_3(G) = 2$;
- (3) $n - 2 \leq \delta(G) \leq n - 1$.

Proof. From Proposition 1.2, $\text{rvx}_3(G) = 0$ if and only if $\text{sdiam}_3(G) = 2$. So we only need to show the equivalence of (1) and (3). Suppose $n - 2 \leq \delta(G) \leq n - 1$. Clearly, G is a graph obtained from the complete graph of order n by deleting some independent edges. For any $S = \{u, v, w\} \subseteq V(G)$, at least two elements in $\{uv, vw, uw\}$ belong to $E(G)$. Without loss of generality, let $uv, vw \in E(G)$. Then the tree T induced by the edges in $\{uv, vw\}$ is an S -Steiner tree and hence $d_G(S) \leq 2$. From the arbitrariness of S , we have $\text{sdiam}_3(G) \leq 2$ and hence $\text{sdiam}_3(G) = 2$. Therefore, $\text{rvx}_3(G) = 0$.

Conversely, we assume $\text{rvx}_3(G) = 0$. If $\delta(G) \leq n - 3$, then there exists a vertex $u \in V(G)$ such that $d_G(u) \leq n - 3$. Therefore, there are two vertices, say v, w , such that $uv, uw \notin E(G)$. Choose $S = \{u, v, w\}$. Clearly, any rainbow S -tree must contain at least a vertex in $V(G) \setminus S$, which implies that $\text{rvx}_3(G) \geq 1$, a contradiction. So $n - 2 \leq \delta(G) \leq n - 1$. ■

After the above preparation, we can derive a lower bound of $\text{rvx}_3(G) + \text{rvx}_3(\overline{G})$.

Lemma 2.2. *Let G be a graph of order n such that G and \overline{G} are connected. For $n \geq 5$, we have $\text{rvx}_3(G) + \text{rvx}_3(\overline{G}) \geq 2$. Moreover, the bound is sharp.*

Proof. From Proposition 1.3, we have $\text{rvx}_3(G) \geq 0$ and $\text{rvx}_3(\overline{G}) \geq 0$. If $\text{rvx}_3(G) = 0$, then we have $n - 2 \leq \delta(G) \leq n - 1$ by Proposition 2.1 and hence \overline{G} is disconnected, a contradiction. Similarly, we can get another contradiction for $\text{rvx}_3(\overline{G}) = 0$. Therefore, $\text{rvx}_3(G) \geq 1$ and $\text{rvx}_3(\overline{G}) \geq 1$. So $\text{rvx}_3(G) + \text{rvx}_3(\overline{G}) \geq 2$. ■

To show the sharpness of the above lower bound, we consider the following example.

Example 1. Let H be a graph of order $n - 4$, and let $P = a, b, c, d$ be a path. Let G be the graph obtained from H and the path by adding edges between the vertex a and all vertices of H and adding edges between the vertex d and all vertices of H ; see Figure 2(a). We now show that $rvx_3(G) = rvx_3(\overline{G}) = 1$. Choose $S = \{a, b, d\}$. Then any S -Steiner tree must occupy at least one vertex in $V(G) \setminus S$. Note that the vertices of $V(G) \setminus S$ in the tree must receive different colors. Therefore, $rvx_3(G) \geq 1$.

Conversely, we show that $rvx_3(G) \leq 1$. We give each vertex in G the same color and show that there exists a vertex-rainbow S -tree for any $S \subseteq V(G)$ with $|S| = 3$. Without loss of generality, let $S = \{x, y, z\}$. Suppose first that $|S \cap V(H)| = 3$. Then the tree T induced by the edges in $\{xa, ya, za\}$ is a vertex-rainbow S -tree. Suppose $|S \cap V(H)| = 2$. Without loss of generality, let $x, y \in S \cap V(H)$. If $a \in S$, then the tree T induced by the edges in $\{xa, ya\}$ is a vertex-rainbow S -tree. If $b \in S$, then the tree T induced by the edges in $\{xa, ya, ab\}$ is a vertex-rainbow S -tree. The remaining cases $c \in S$ and $d \in S$ are symmetric.

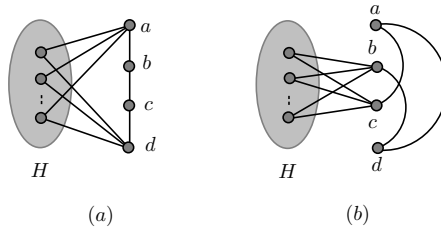


Figure 2. Graphs for Example 1.

Suppose $|S \cap V(H)| = 1$. Without loss of generality, let $x \in S \cap V(H)$. If $a, b \in S$, then the tree T induced by the edges in $\{xa, ab\}$ is a vertex-rainbow S -tree. If $b, c \in S$, then the tree T induced by the edges in $\{xd, cd, bc\}$ is a vertex-rainbow S -tree. If $a, c \in S$, then the tree T induced by the edges in $\{xa, ab, bc\}$ is a vertex-rainbow S -tree. Note that the remaining cases are symmetric. Suppose $|S \cap V(G')| = 0$. If $a, b, c \in S$, then the tree T induced by the edges in $\{ab, bc\}$ is a vertex-rainbow S -tree. If $a, b, d \in S$, then the tree T induced by the edges in $\{ab, bc, cd\}$ is a vertex-rainbow S -tree. We conclude that $rvx_3(G) \leq 1$. Similarly, one can also check that $rvx_3(\overline{G}) = 1$ (see Figure 2(b)). So $rvx_3(G) + rvx_3(\overline{G}) = 2$.

We are now in a position to give an upper bound of $rvx_3(G) + rvx_3(\overline{G})$. For $n = 4$, we have $G = \overline{G} = P_4$, since we only consider connected graphs. Observe that $rvx_3(G) = rvx_3(\overline{G}) = rvx_3(P_4) = 2$.

Observation 2.3. Let G be a graph of order n ($n = 4$) such that both G and \overline{G} are connected. Then $rvx_3(G) + rvx_3(\overline{G}) = n$.

For $n \geq 5$, we have the following upper bound of $rvx_3(G) + rvx_3(\overline{G})$.

Lemma 2.4. Let G be a graph of order n ($n = 5$) such that both G and \overline{G} are connected. Then $rvx_3(G) + rvx_3(\overline{G}) \leq n - 1$.

Proof. If G is a path of order 5, then $rvx_3(G) = 3$ by Observation 1.4. Observe that $sdiam_3(\overline{G}) = 3$. Then $rvx_3(\overline{G}) \leq 1$ and hence $rvx_3(G) + rvx_3(\overline{G}) \leq 4$, as desired.

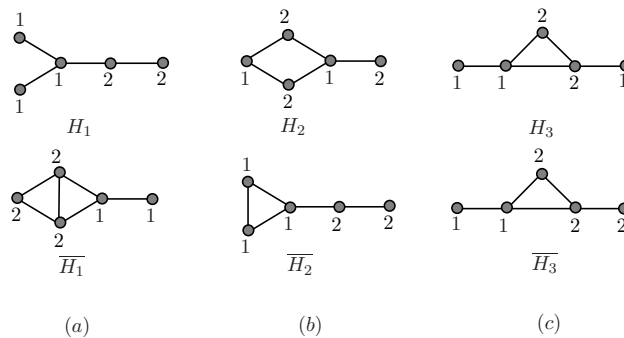


Figure 3. Graphs for Lemma 2.4.

If G is a tree but not a path, then we have $G = H_1$, since \overline{G} is connected (see Figure 3(a)). Clearly, $rvx_3(G) \leq 2$. Furthermore, \overline{G} consists of a K_2 and a K_3 and two edges between them (see Figure 3(a)). So we assign color 1 to the vertices of K_2 and color 2 to the vertices of K_3 , and this vertex-coloring makes the graph G vertex-rainbow 3-tree-connected, that is, $rvx_3(\overline{G}) \leq 2$. Therefore, $rvx_3(G) + rvx_3(\overline{G}) \leq 4$, as desired.

Suppose that both G and \overline{G} are not trees. Then $e(G) \geq 5$ and $e(\overline{G}) \geq 5$. Since $e(G) + e(\overline{G}) = e(K_5) = 10$, it follows that $e(G) = e(\overline{G}) = 5$. If G contains a cycle of length 5, then $G = \overline{G} = C_5$ and hence $rvx_3(G) = rvx_3(\overline{G}) = 2$. If G contains a cycle of length 4, then $G = H_2$ (see Figure 3(b)). Clearly, $rvx_3(G) = rvx_3(\overline{G}) = 2$. If G contains a cycle of length 3, then $G = \overline{G} = H_3$ (see Figure 3(c)). One can check that $rvx_3(G) = rvx_3(\overline{G}) = 2$. Therefore, $rvx_3(G) + rvx_3(\overline{G}) = 4$, as desired. ■

Lemma 2.5. Let G be a nontrivial connected graph of order n , and $rvx_3(G) = \ell$. Let G' be the graph obtained from G by adding a new vertex v to G and making v adjacent to q vertices of G . If $q \geq n - \ell$, then $rvx_3(G') \leq \ell$.

Proof. Let $c : V(G) \rightarrow \{1, 2, \dots, \ell\}$ be a vertex-coloring of G such that G is vertex-rainbow 3-tree-connected. Let $X = \{x_1, x_2, \dots, x_q\}$ be the neighborhood

of v in G . Set $V(G) \setminus X = \{y_1, y_2, \dots, y_{n-q}\}$. We can assume that there exist two vertices y_{j_1}, y_{j_2} such that there is no vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$; otherwise, the result holds obviously.

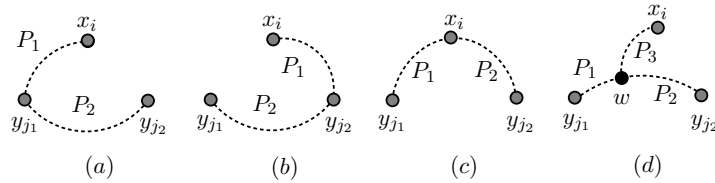


Figure 4. Four type of the Steiner tree T_i .

We define a minimal S -Steiner tree T as a tree connecting S whose subtree obtained by deleting any edge of T does not connect S . Because G is vertex-rainbow 3-tree-connected, there is a minimal vertex-rainbow tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ for each x_i ($i \in \{1, 2, \dots, q\}$). Then the tree T_i is of one of four types shown in Figure 4. For the type shown in (c), the Steiner tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ is a path induced by the edges in $E(P_1) \cup E(P_2)$ and hence the internal vertices of the path T_i must receive different colors. Therefore, the tree induced by the edges in $E(P_1) \cup E(P_2) \cup \{vx_i\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, a contradiction. Clearly, (b) is symmetric to (a). So we only need to consider remaining two cases shown in Figure 4(a), (d). Obviously, $T_i \cap T_j$ may not be empty. Then we have the following claim.

Claim 1. *No other vertex in $\{x_1, x_2, \dots, x_q\}$ different from x_i belongs to T_i for each $1 \leq i \leq q$.*

Proof. Assume, to the contrary, that there exists a vertex $x'_i \in \{x_1, x_2, \dots, x_q\}$ such that $x'_i \neq x_i$ and $x'_i \in V(T_i)$. For the type shown in Figure 4(a), the Steiner tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ is a path induced by the edges in $E(P_1) \cup E(P_2)$ and hence the internal vertices of the path T_i receive different colors. If $x'_i \in V(P_1)$, then the tree induced by the edges in $E(P'_1) \cup E(P_2) \cup \{vx'_i\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, where P'_1 is the path between the vertex x'_i and the vertex y_{j_1} in P_1 , a contradiction. If $x'_i \in V(P_2)$, then the tree induced by the edges in $E(P_2) \cup \{vx'_i\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, a contradiction. For the type shown in Figure 4(d), the Steiner tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ is a tree induced by the edges in $E(P_1) \cup E(P_2) \cup E(P_3)$ and hence the internal vertices of the tree T_i receive different colors. Without loss of generality, let $x'_i \in V(P_1)$. Then the tree induced by the edges in $E(P'_1) \cup E(P_2) \cup E(P_3)$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, where P'_1 is the path between the vertex x'_i and the vertex w in P_1 , a contradiction. \square

From Claim 1, since there is no vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, it follows that there exists a vertex y_{k_i} such that $c(x_i) = c(y_{k_i})$ for each tree T_i ,

which implies that the colors that are assigned to X are among the colors that are assigned to $V(G) \setminus X$. So $\text{rvx}_3(G) = \ell \leq n - q$. Combining this with the hypothesis $q \geq n - \ell$, we have $\text{rvx}_3(G) = n - q$, that is, all vertices in $V(G) \setminus X$ have distinct colors. Now we construct a new graph G' , which is induced by the edges in $E(T_1) \cup E(T_2) \cup \dots \cup E(T_q)$.

Claim 2. *For every y_t not in G' , there exists a vertex $y_s \in G'$ such that $y_t y_s \in E(G)$.*

Proof. Assume, to the contrary, that $N(y_t) \subseteq \{x_1, x_2, \dots, x_q\}$. Since G is vertex-rainbow 3-tree-connected, there is a vertex-rainbow tree T connecting $\{y_t, y_{j_1}, y_{j_2}\}$. Let x_r be the vertex in the tree T such that $x_r \in N_T(y_t)$. Then the tree induced by the edges in $(E(T) \setminus \{y_t x_r\}) \cup \{v x_r\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, a contradiction. \square

From Claim 2, $G[y_1, y_2, \dots, y_{n-q}]$ is connected. Clearly, $G[y_1, y_2, \dots, y_{n-q}]$ has a spanning tree T . Because the tree T has at least two pendant vertices, there must exist a pendant vertex whose color is different from x_1 , and we assign its color to x_1 . One can easily check that G is still vertex-rainbow 3-tree-connected, and there is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$. If there still exist two vertices y_{j_3}, y_{j_4} such that there is no vertex-rainbow tree connecting $\{v, y_{j_3}, y_{j_4}\}$, then we do the same operation until there is a vertex-rainbow tree connecting $\{v, y_{j_r}, y_{j_s}\}$ for each pair $y_{j_r}, y_{j_s} \in \{1, 2, \dots, n - q\}$. Thus G' is vertex-rainbow 3-tree-connected. So $\text{rvx}_3(G') \leq \ell$. \blacksquare

Proof of Theorem 1.5. We prove this theorem by induction on n . By Lemma 2.4, the result is evident for $n = 5$. We assume that $\text{rvx}_3(G) + \text{rvx}_3(\overline{G}) \leq n - 1$ holds for graphs on n vertices. Observe that the union of a connected graph G and its complement \overline{G} is a complete graph of order n , that is, $G \cup \overline{G} = K_n$. We add a new vertex v to G and add q edges between v and $V(G)$. Denote by G' the resulting graph. Clearly, $\overline{G'}$ is a graph of order $n + 1$ obtained from \overline{G} by adding a new vertex v to \overline{G} and adding $n - q$ edges between v and $V(\overline{G})$.

Claim 3. $\text{rvx}_3(G') \leq \text{rvx}_3(G) + 1$ and $\text{rvx}_3(\overline{G'}) \leq \text{rvx}_3(\overline{G}) + 1$.

Proof. Let c be a $\text{rvx}_3(G)$ -vertex-coloring of G such that G is vertex-rainbow 3-tree-connected. Pick up a vertex $u \in N_G(v)$ and give it a new color. It suffices to show that for any $S \subseteq V(G')$ with $|S| = 3$, there exists a vertex-rainbow S -tree. If $S \subseteq V(G)$, then there exists a vertex-rainbow S -tree, since G is vertex-rainbow 3-tree-connected. Suppose $S \not\subseteq V(G)$. Then $v \in S$. Without loss of generality, let $S = \{v, x, y\}$. Since G is vertex-rainbow 3-tree-connected, there exists a vertex-rainbow tree T' connecting $\{u, x, y\}$. Then the tree T induced by the edges in $E(T') \cup \{uv\}$ is a vertex-rainbow S -tree. Therefore, $\text{rvx}_3(G') \leq \text{rvx}_3(G) + 1$. Similarly, $\text{rvx}_3(\overline{G'}) \leq \text{rvx}_3(\overline{G}) + 1$. \square

From Claim 3, we have $rvx_3(G') + rvx_3(\overline{G}') \leq rvx_3(G) + 1 + rvx_3(\overline{G}) + 1 \leq n + 1$. Clearly, $rvx_3(G') + rvx_3(\overline{G}') \leq n$ except possibly when $rvx_3(G') = rvx_3(G) + 1$ and $rvx_3(\overline{G}') = rvx_3(\overline{G}) + 1$. In this case, by Lemma 2.5, we have $q \leq n - rvx_3(G) - 1$ and $n - q \leq n - rvx_3(\overline{G}) - 1$. Thus, $rvx_3(G) + rvx_3(\overline{G}) \leq (n - 1 - q) + (q - 1) = n - 2$ and hence $rvx_3(G') + rvx_3(\overline{G}') \leq n$, as desired. This completes the induction. ■

To show sharpness of the above bound, we consider the following example.

Example 2. Let G be a path of order n . Then $rvx_3(G) = n - 2$. Observe that $sdiam_3(\overline{G}) = 3$. Then $rvx_3(\overline{G}) = 1$, and so we have $rvx_3(G) + rvx_3(\overline{G}) = (n - 2) + 1 = n - 1$.

3. THE MINIMAL SIZE OF GRAPHS WITH GIVEN VERTEX-RAINBOW INDEX

Recall that $t(n, k, \ell)$ is the minimal size of a connected graph G of order n with $rvx_k(G) \leq \ell$, where $2 \leq \ell \leq n - 2$ and $2 \leq k \leq n$. Let G be a path of order n . Then $rvx_k(G) \leq n - 2$ and hence $t(n, k, n - 2) \leq n - 1$. Since we only consider connected graphs, it follows that $t(n, k, n - 2) \geq n - 1$. Therefore, the following result is immediate.

Observation 3.1. *Let k be an integer with $2 \leq k \leq n$. Then*

$$t(n, k, n - 2) = n - 1.$$

A rose graph R_p with p petals (or p -rose graph) is a graph obtained by taking p cycles with just a vertex in common. The common vertex is called the center of R_p . If the length of each cycle is exactly q , then this rose graph with p petals is called a (p, q) -rose graph, denoted by $R_{p,q}$. Now we are able to prove Theorem 1.6.

Proof of Theorem 1.6. From Observation 3.1, we have $t(n, k, \ell) \geq n - 1$. It suffices to show the upper bound holds. Suppose that n and ℓ have different parity. Then $n - \ell - 1$ is even. Let G be the graph obtained from an $(\frac{n-\ell-1}{2}, 3)$ -rose graph $R_{\frac{n-\ell-1}{2},3}$ and a path $P_{\ell+1}$ by identifying the center of the rose graph and one endpoint of the path. Let w_0 be the center of $R_{\frac{n-\ell-1}{2},3}$, and let $C_i = w_0v_iu_iw_0$ ($1 \leq i \leq \frac{n-\ell-1}{2}$) be the cycle of $R_{\frac{n-\ell-1}{2},3}$. Let $P_{\ell+1} = w_0w_1 \cdots w_\ell$ be the path of order $\ell + 1$. To show that $rvx_k(G) \leq \ell$, we define a vertex-coloring $c : V(G) \rightarrow \{0, 1, 2, \dots, \ell - 1\}$ of G by

$$c(v) = \begin{cases} i, & \text{if } v = w_i \text{ } (0 \leq i \leq \ell - 1); \\ 1, & \text{if } v = u_i \text{ or } v = v_i \text{ } (1 \leq i \leq \frac{n-\ell-1}{2}); \\ 1, & \text{if } v = w_\ell. \end{cases}$$

One can easily see that there exists a vertex-rainbow S -tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $\text{rvx}_k(G) \leq \ell$ and $t(n, k, \ell) \leq n - 1 + \frac{n-\ell-1}{2}$.

Suppose that n and ℓ have the same parity. Then $n - \ell$ is even. Let G be the graph obtained from an $(\frac{n-\ell}{2}, 3)$ -rose graph $R_{\frac{n-\ell}{2}, 3}$ and a path P_ℓ by identifying the center of the rose graph and one endpoint of the path. Let w_0 be the center of $R_{\frac{n-\ell}{2}, 3}$, and let $C_i = w_0 v_i u_i w_0$ ($1 \leq i \leq \frac{n-\ell}{2}$) be the cycle of $R_{\frac{n-\ell}{2}, 3}$. Let $P_\ell = w_0 w_1 \cdots w_{\ell-1}$ be the path of order ℓ . To show that $\text{rvx}_k(G) \leq \ell$, we define a vertex-coloring $c : V(G) \rightarrow \{0, 1, 2, \dots, \ell - 1\}$ of G by

$$c(v) = \begin{cases} i, & \text{if } v = w_i \text{ } (0 \leq i \leq \ell - 1); \\ 1, & \text{if } v = u_i \text{ or } v = v_i \text{ } (1 \leq i \leq \frac{n-\ell}{2}). \end{cases}$$

One can easily see that there exists a vertex-rainbow S -tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $\text{rvx}_k(G) \leq \ell$ and $t(n, k, \ell) \leq n - 1 + \frac{n-\ell}{2}$. ■

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